

UPDATE ON QUANTUM COHOMOLOGY OF $\mathrm{IG}(2, 2n)$

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ABSTRACT. We give another proof of the generic semisimplicity of the big quantum cohomology of the symplectic isotropic Grassmannians $\mathrm{IG}(2, 2n)$.

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INTRODUCTION

In papers [6, 12] jointly with Sergey Galkin we have shown that the big quantum cohomology of the symplectic isotropic Grassmannians $\mathrm{IG}(2, 2n)$ is generically semisimple. In [6] only the case of $\mathrm{IG}(2, 6)$ is considered and the proof is computer-assisted. The proof in [12] is not computer-assisted and works for all $\mathrm{IG}(2, 2n)$, but it needs many lengthy computations. The purpose of this paper is to give a pure thought argument which works uniformly for all $\mathrm{IG}(2, 2n)$, is conceptually different from the ones in *loc.cit.*, and might be generalizable to all $\mathrm{IG}(m, 2n)$.

The strategy of our proof is the following. We view the big quantum cohomology as a deformation family of zero-dimensional schemes over a base (see Sec. 1.2). The special fibre of this family is (the spectrum of) the small quantum cohomology. Generic semisimplicity of the big quantum cohomology translates into the smoothness of the generic fibre¹ of the deformation family. Similarly, generic semisimplicity of the small quantum cohomology translates into the smoothness of the special fibre. So it may very well happen that the generic fibre is smooth, whereas the special one is not. This is exactly what happens in the case of $\mathrm{IG}(2, 2n)$. In fact, we prove that for $\mathrm{IG}(2, 2n)$ the total space of the deformation family (the spectrum of the big quantum cohomology) is a regular scheme over a field of characteristic zero. Therefore, by a version of "generic smoothness" in characteristic zero, one concludes that the generic fibre is smooth. Thus, we obtain the generic semisimplicity of the big quantum cohomology,² whereas the small quantum cohomology of $\mathrm{IG}(2, 2n)$ is known to be non-semisimple by [2, 3].

¹By "generic fiber" we always mean "fiber over the generic point of the base scheme".

²This argument reminded us of Problem 2.8 from [8].

Our original motivation to look at this problem comes from Dubrovin's conjecture. We will not elaborate on this here and rather refer to [5, 6, 8] and references therein. Let us just mention that the Grassmannians $\text{IG}(2, 2n)$ are the simplest explicit examples available in the literature where one has to work with big quantum cohomology to formulate Dubrovin's conjecture.

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1. CONVENTIONS AND NOTATION FOR QUANTUM COHOMOLOGY

Here we briefly recall the definition of the quantum cohomology ring for a smooth projective variety X . To simplify the exposition and avoid introducing unnecessary notation we impose from the beginning the following conditions on X : it is a Fano variety of Picard rank 1 and $H^{\text{odd}}(X, \mathbb{Q}) = 0$. For a thorough introduction we refer to [11].

1.1. Definition. Let us fix a graded basis $\Delta_0, \dots, \Delta_s$ in $H^*(X, \mathbb{Q})$ and dual linear coordinates $t_0, \dots, t_s$. It is customary to choose $\Delta_0 = 1$. Let R be the ring of formal power series $\mathbb{Q}[[q]]$, k its field of fractions, and K an algebraic closure of k .

The genus zero Gromov-Witten potential of X is an element $\Phi \in R[[t_0, \dots, t_s]]$ defined by the formula

$$\Phi = \sum_{(i_0, \dots, i_s)} \langle \Delta_0^{\otimes i_0}, \dots, \Delta_s^{\otimes i_s} \rangle \frac{t_0^{i_0} \dots t_s^{i_s}}{i_0! \dots i_s!}, \quad (1.1)$$

where $\langle \Delta_0^{\otimes i_0}, \dots, \Delta_s^{\otimes i_s} \rangle := \sum_{d=0}^{\infty} \langle \Delta_0^{\otimes i_0}, \dots, \Delta_s^{\otimes i_s} \rangle_d q^d$, and the rational numbers $\langle \Delta_0^{\otimes i_0}, \dots, \Delta_s^{\otimes i_s} \rangle_d$ are Gromov-Witten invariants of X of degree d .

Using (1.1) one defines the quantum cohomology ring of X . Namely, on the basis elements we put

$$\Delta_a \star \Delta_b = \sum_c \Phi_{abc} \Delta^c, \quad (1.2)$$

where $\Phi_{abc} = \frac{\partial^3 \Phi}{\partial t_a \partial t_b \partial t_c}$, and $\Delta^0, \dots, \Delta^s$ is the basis dual to $\Delta_0, \dots, \Delta_s$ with respect to the Poincaré pairing. Expression (1.2) is naturally interpreted as an element of $H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} K[[t_0, \dots, t_s]]$.

It is well known that (1.2) makes $H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} K[[t_0, \dots, t_s]]$ into a commutative, associative, graded $K[[t_0, \dots, t_s]]$ -algebra with the identity element Δ_0 . We will denote this algebra $\text{BQH}(X)$. For convenience we recall the definition of the grading:

$$\deg(\Delta_i) = |\Delta_i|, \quad \deg(q) = \text{index}(X), \quad \deg(t_i) = 1 - |\Delta_i|,$$

where $|\Delta_i|$ is the Chow degree of Δ_i and $\text{index}(X)$ is the largest integer n such that $-K_X = nH$ for some ample divisor H on X .

The algebra $\text{BQH}(X)$ is called the *big quantum cohomology algebra* of X to distinguish it from a simpler object called the *small quantum cohomology algebra* which is the quotient of $\text{BQH}(X)$ with respect to the ideal $(t_0, \dots, t_s)$. We will denote the latter $\text{QH}(X)$ and use $\star_0$ instead of $\star$ for the product in this algebra. It is a finite dimensional K -algebra. Equivalently one can say that $\text{QH}(X) = H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} K = H^*(X, K)$ as a vector space, and the K -algebra structure is defined by putting $\Delta_a \star_0 \Delta_b = \sum_c \langle \Delta_a, \Delta_b, \Delta_c \rangle \Delta_c$.

1.1.1. Remark. We are using a somewhat non-standard notation $\text{BQH}(X)$ for the big quantum cohomology and $\text{QH}(X)$ for the small quantum cohomology to stress the difference between the two. Note that this notation is different from the one used in [6] and is closer to the notation of [12].

1.1.2. Remark. The above definitions look slightly different from the ones given in [11]. The differences are of two types. The first one is that $\text{QH}(X)$ and $\text{BQH}(X)$ are in fact defined already over the ring R and not only over K . We pass to K from the beginning, since in this paper we are only interested in generic semisimplicity of quantum cohomology. The second difference is that in some papers on quantum cohomology one unifies the coordinate q with those coordinates t_i which are dual to $H^2(X, \mathbb{Q})$, but the resulting structures are equivalent.

1.2. Deformation picture. The small quantum cohomology, if considered over the ring R (cf. Remark 1.1.2), is a deformation of the ordinary cohomology algebra, i.e. if you put $q = 0$, then the quantum product becomes the ordinary cup-product. Similarly, the big quantum cohomology is an even bigger deformation family of algebras. Since we work not over R but over K , we lose the point of classical limit but still retain the fact that $\text{BQH}(X)$ is a deformation family of algebras with the special fiber being $\text{QH}(X)$.

Throughout this paper we view $\text{Spec}(\text{BQH}(X))$ as a deformation family of zero-dimensional schemes over $\text{Spec}(K[[t_0, \dots, t_s]])$. In the base of the deformation we consider the following two points: the origin (the closed point given by the maximal ideal $(t_0, \dots, t_n)$) and the generic point η . The fiber of this family over the origin is the spectrum of the small quantum cohomology $\text{Spec}(\text{QH}(X))$. The fiber over the generic point will be denoted by $\text{Spec}(\text{BQH}(X)_{\eta})$. It is convenient to summarize this setup in the diagram

$$\begin{array}{ccccc}
 \text{Spec}(\text{QH}(X)) & \longrightarrow & \text{Spec}(\text{BQH}(X)) & \longleftarrow & \text{Spec}(\text{BQH}(X)_{\eta}) \\
 \downarrow & & \downarrow \pi & & \downarrow \pi_{\eta} \\
 \text{Spec}(K) & \longrightarrow & \text{Spec}(K[[t_0, \dots, t_s]]) & \longleftarrow & \eta
 \end{array} \tag{1.3}$$

where both squares are Cartesian.

By construction $\text{BQH}(X)$ is a free module of finite rank over $K[[t_0, \dots, t_s]]$. Therefore, it is a noetherian semi-local K -algebra which is flat and finite over $K[[t_0, \dots, t_s]]$. Note that neither $K[[t_0, \dots, t_s]]$ nor $\text{BQH}(X)$ are finitely generated over the ground field K . Therefore, some extra care is required in the standard commutative algebra (or algebraic geometry) constructions. For example, the notion of smoothness is one of such concepts.

1.3. Semisimplicity. Let A be a finite dimensional algebra over a field F of characteristic zero. It is called *semisimple* if it is a product of fields. Equivalently, the algebra A is semisimple iff the scheme $\text{Spec}(A)$ is reduced. Another equivalent condition is to require the morphism $\text{Spec}(A) \rightarrow \text{Spec}(F)$ to be smooth.

Definition: one says that $\text{BQH}(X)$ is *generically semisimple* iff $\text{BQH}(X)_\eta$ is a semisimple algebra.

As already described in the introduction, we will prove the generic semisimplicity of $\text{BQH}(\text{IG}(2, 2n))$ by a version of generic smoothness in characteristic zero.

2. GEOMETRY OF $\text{IG}(2, 2n)$

Let V be a complex vector space endowed with a symplectic form ω . In this case the dimension of V has to be even and we denote it by $2n$. For any $1 \leq m \leq n$ there exists an algebraic variety $\text{IG}_\omega(m, V)$ that parametrizes m -dimensional isotropic subspaces of V . For $m = 2$, and this is the case we are considering in this paper, it has the following explicit description. Consider the ordinary Grassmannian $G(2, V)$ with its Plücker embedding into $\mathbb{P}(\Lambda^2 V)$. The symplectic form ω defines a hyperplane $H_\omega \subset \mathbb{P}(\Lambda^2 V)$ and the intersection of $G(2, V)$ with H_ω is exactly $\text{IG}_\omega(2, V)$. Thus, we have inclusions

$$\text{IG}_\omega(2, V) \subset G(2, V) \subset \mathbb{P}(\Lambda^2 V). \quad (2.1)$$

For different ω 's the varieties $\text{IG}_\omega(m, V)$ are isomorphic, therefore we will often simply write $\text{IG}(m, 2n)$.

2.1. Special cohomology classes. As for ordinary Grassmannians, one considers the short exact sequence of vector bundles on $X = \text{IG}_\omega(2, V)$

$$0 \rightarrow \mathcal{U} \rightarrow \mathcal{V} \rightarrow \mathcal{V}/\mathcal{U} \rightarrow 0, \quad (2.2)$$

where $\mathcal{V}$ is the trivial vector bundle with fiber V , $\mathcal{U}$ is the subbundle of isotropic subspaces, and $\mathcal{V}/\mathcal{U}$ is the quotient bundle. Usually one refers to $\mathcal{U}$ and $\mathcal{V}/\mathcal{U}$ as *tautological subbundle* and *tautological quotient bundle* respectively. Apart from (2.2) one can also consider the short exact sequence

$$0 \rightarrow \mathcal{U}^\perp/\mathcal{U} \rightarrow \mathcal{V}/\mathcal{U} \rightarrow \mathcal{V}/\mathcal{U}^\perp \rightarrow 0. \quad (2.3)$$

By taking Chern classes of vector bundles in these sequences we obtain two sets of cohomology classes which generate the cohomology ring:

i) Special Schubert classes. These are the Chern classes of $\mathcal{V}/\mathcal{U}$. They are indexed by integers $k \in [0, 2n - 2]$ and can be explicitly described as follows. Let

$$Z(E_{2n-k-1}) = \{V_2 \in X \mid \dim(V_2 \cap E_{2n-k-1}) \geq 1\}.$$

Then $\sigma_k = [Z(E_{2n-k-1})] \in H^{2k}(X, \mathbb{Z})$. In the above, E_{2n-k-1} is a subspace of dimension $2n - k - 1$ such that the rank of $\omega|_{E_{2n-k-1}}$ is minimal. Note that $\sigma_0 = 1$ is the fundamental class of X .

ii) These are the Chern classes of $\mathcal{U}$ and $\mathcal{U}^\perp/\mathcal{U}$. Vector bundle $\mathcal{U}$ is of rank 2, so it only has two non-vanishing Chern classes $a_i = c_i(\mathcal{U})$ for $i = 1, 2$. Vector bundle $\mathcal{U}^\perp/\mathcal{U}$ is self-dual of rank $2n - 4$, therefore it has only $n - 2$ non-vanishing Chern classes $b_i = c_{2i}(\mathcal{U}^\perp/\mathcal{U})$ for $i \in [1, n - 2]$.

2.2. Cohomology ring of $\text{IG}(2, 2n)$. The cohomology ring of X can be neatly described in terms of generators and relations. We will give two presentations using the two sets of special cohomology classes defined above.

Proposition 2.1. *The cohomology ring $H^*(X, \mathbb{Q})$ is isomorphic to the quotient of the ring $\mathbb{Q}[\sigma_1, \dots, \sigma_{2n-2}]$ by the ideal generated by the elements*

$$\det(\sigma_{1+j-i})_{1 \leq i, j \leq r}, \quad \text{with } r \in [3, 2n-2] \quad (2.4)$$

and the two elements

$$\sigma_{n-1}^2 + 2 \sum_{i=1}^{n-1} (-1)^i \sigma_{n-1+i} \sigma_{n-1-i} \quad \text{and} \quad \sigma_n^2 + 2 \sum_{i=1}^{n-2} (-1)^i \sigma_{n+i} \sigma_{n-i}. \quad (2.5)$$

The dimension of $H^*(X, \mathbb{Q})$ is equal to $2^2 \binom{n}{2} = 2n(n-1)$.

Proof. This is the statement of [1, Theorem 1.2]. $\square$

Proposition 2.2. *The cohomology ring $H^*(X, \mathbb{Q})$ is isomorphic to the quotient of the ring $\mathbb{Q}[a_1, a_2, b_1, \dots, b_{n-2}]$ by the ideal generated by*

$$(1 - (2a_2 - a_1^2)x^2 + a_2^2x^4)(1 + b_1x^2 + \dots + b_{n-2}x^{2n-4}) = 1 \quad (2.6)$$

The last equality is viewed as an equality of polynomials in the variable x and gives a concise way to write a system of equations in the variables a_i, b_i .

Proof. This result is well known to specialists but we include a short proof for the convenience of the reader.

Let us start by checking that (2.6) holds in the cohomology ring. Define $P(x) = 1 + a_1x + a_2x^2$ and $Q(x) = 1 + b_1x^2 + \dots + b_{n-2}x^{2n-4}$ and rewrite (2.6) as $P(x)P(-x)Q(x) = 1$. We interpret the polynomial $P(x)$ as the total Chern class of $\mathcal{U}$ and $Q(x)$ as the total Chern class of $\mathcal{U}^\perp/\mathcal{U}$. Now by using basic properties of Chern classes and short exact sequences (2.2) and (2.3) it is easy to see that the above relation does hold.

The above discussion shows that we have a natural homomorphism of $\mathbb{Q}$ -algebras

$$\psi: \mathbb{Q}[a_1, a_2, b_1, \dots, b_{n-2}]/(P(x)P(-x)Q(x) - 1) \rightarrow H^*(X, \mathbb{Q}) \quad (2.7)$$

sending a_i 's to a_i 's and b_i 's to b_i 's. To prove the proposition it is enough to establish two facts: i) σ_i 's can be expressed in terms of a_i 's and b_i 's, ii) the dimensions of both algebras in (2.7) are equal. To prove the first fact one can use again simple properties of Chern classes. The proof of the second fact is given below.

First, we need to show that $\mathbb{Q}[a_1, a_2, b_1, \dots, b_{n-2}]/(P(x)P(-x)Q(x) - 1)$ is finite dimensional. This can be done similarly to the finite-dimensionality part of the proof of [1, Theorem 1.2]: essentially it is an application of [1, Lemma 1.2]. Second, we need to compute the dimension of this algebra. For this we can proceed similarly to the proof of [1, Lemma 1.1], which is based on [13], and get the desired $2n(2n-1)$. $\square$

2.3. Lines on $\text{IG}(2, 2n)$. Here we recall the description of lines on X (see [10] for example). Since we have an inclusion $X \subset \text{G}(2, 2n)$, a line on X is also a line on the ordinary Grassmannian $\text{G}(2, 2n)$. In particular it is given by a pair (W_1, W_3) of nested subspaces of dimensions 1 and 3 respectively. The corresponding line being the set

$$\ell(W_1, W_3) = \{V_2 \in X \mid W_1 \subset V_2 \subset W_3\}.$$

In X , there are two different types of lines corresponding to two Sp_{2n} -orbits in the variety Y of all lines on X :

- If W_3 is not isotropic, then W_1 is the kernel of the symplectic form ω on W_3 , therefore W_1 is determined by W_3 . We shall denote by $\ell(W_3)$ a line of this type. The set of these lines is the open Sp_{2n} -orbit in Y which we will denote by $\dot{Y}$.
- If W_3 is isotropic, then W_1 is any one-dimensional subspace in W_3 . This is the closed Sp_{2n} -orbit in Y .

3. SMALL QUANTUM COHOMOLOGY OF $\text{IG}(2, 2n)$

3.1. Two presentations. As was already mentioned in Section 2.2, the ordinary cohomology ring of the Grassmannian $\text{IG}(2, 2n)$ is generated by the special Schubert classes $\sigma_1, \dots, \sigma_{2n-2}$ with relations (2.4) and (2.5). Since $\deg(q) = 2n - 1$, the only relation that needs to be modified is the second equation in (2.5). Moreover, up to a constant factor, this modification is unique for degree reasons. The complete answer for arbitrary Grassmannians $\text{IG}(m, 2n)$ was given in [1, Theorem 1.5] which we reproduce here in the special case of $m = 2$.

Theorem 3.1 (Buch-Kresch-Tamvakis). *The small quantum cohomology ring $\text{QH}(X)$ is isomorphic to the quotient of the ring $K[\sigma_1, \dots, \sigma_{2n-2}]$ by the ideal generated by the elements*

$$\det(\sigma_{1+j-i})_{1 \leq i, j \leq r}, \quad \text{with } r \in [3, 2n - 2]$$

and the two elements

$$\sigma_{n-1}^2 + 2 \sum_{i=1}^{n-1} (-1)^i \sigma_{n-1+i} \sigma_{n-1-i} \quad \text{and} \quad \sigma_n^2 + 2 \sum_{i=1}^{n-2} (-1)^i \sigma_{n+i} \sigma_{n-i} + (-1)^{n+1} q \sigma_1.$$

Combining the above theorem with Proposition 2.2 we arrive at the following statement.

Corollary 3.2. *The small quantum cohomology ring $\text{QH}(X)$ is isomorphic to the quotient of the ring $K[a_1, a_2, b_1, \dots, b_{n-2}]$ by the ideal generated by*

$$(1 - (2a_2 - a_1)x^2 + a_2^2x^4)(1 + b_1x^2 + \dots + b_{n-2}x^{2n-4}) = 1 + qa_1x^{2n}. \quad (3.1)$$

3.2. Structure of $\text{QH}(X)$. In this paragraph we will study the decomposition of the K -algebra $\text{QH}(X)$ into the direct product, or, equivalently, the decomposition of the K -scheme $\text{Spec}(\text{QH}(X))$ into connected components.

We will use the presentation of $\text{QH}(X)$ described in Corollary 3.2. Thus, the K -scheme $\text{Spec}(\text{QH}(X))$ is given as a closed subscheme of the affine space $\mathbb{A}^n =$

$\text{Spec}(K[a_1, a_2, b_1, \dots, b_{n-2}])$ defined by the equations

$$\begin{aligned}
2a_2 - a_1^2 + b_1 &= 0 \\
a_2^2 + b_1(2a_2 - a_1^2) + b_2 &= 0 \\
b_1a_2^2 + b_2(2a_2 - a_1^2) + b_3 &= 0 \\
&\dots \\
b_{i-1}a_2^2 + b_i(2a_2 - a_1^2) + b_{i+1} &= 0 \\
&\dots \\
b_{n-4}a_2^2 + b_{n-3}(2a_2 - a_1^2) + b_{n-2} &= 0 \\
b_{n-3}a_2^2 + b_{n-2}(2a_2 - a_1^2) &= 0 \\
b_{n-2}a_2^2 - qa_1 &= 0.
\end{aligned} \tag{3.2}$$

It is clear that the origin of $\mathbb{A}^n$ is a solution of this system. Moreover, this solution corresponds to a fat point of $\text{Spec}(\text{QH}(X))$. Indeed, it is easy to see that the Zariski tangent space of (3.2) at the origin is one-dimensional. Thus, the origin is a fat point of $\text{Spec}(\text{QH}(X))$. Let A be the corresponding factor of $\text{QH}(X)$. Thus, we have the direct product decomposition

$$\text{QH}(X) = A \times B, \tag{3.3}$$

where B corresponds to components of $\text{Spec}(\text{QH}(X))$ supported outside of the origin.

To determine the structure of A we look for solutions of (3.2) in the ring $K[\varepsilon]/\varepsilon^{n-1}$ that extend the zero solution. It is easy to see that putting $a_1 = 0$, letting a_2 be an arbitrary element of $K[\varepsilon]/\varepsilon^{n-1}$ of the form $O(\varepsilon)$, and recovering b_i 's by elimination process gives rise to a solution of (3.2) with the property $b_i = O(\varepsilon^i)$. Another way to phrase it is to say that we have obtained a surjective algebra homomorphism

$$A \rightarrow K[\varepsilon]/\varepsilon^{n-1}. \tag{3.4}$$

Thus, the dimension of A is at least $n - 1$. Moreover, below we will see that this map is an isomorphism.

Now let us examine the structure of B , i.e. we need to study solutions of (3.2) different from the origin³. It is convenient to rewrite (3.1) as

$$(z^4 - (2a_2 - a_1)z^2 + a_2^2)(z^{2n-4} + b_1z^{2n-6} + \dots + b_{n-2}) = z^{2n} - a_1,$$

where we set $q = -1$ for convenience. By making the substitution $a_1 = z_1 + z_2, a_2 = z_1z_2$, and putting $Q(z) = z^{2n-4} + b_1z^{2n-6} + \dots + b_{n-2}$ we arrive at

$$(z^2 - z_1^2)(z^2 - z_2^2)Q(z) = z^{2n} - (z_1 + z_2). \tag{3.5}$$

In geometric terms this manipulation corresponds to pulling back our system (3.2) with respect to the morphism

$$\text{Spec}(K[z_1, z_2, b_1, \dots, b_{n-2}]) \rightarrow \text{Spec}(K[a_1, a_2, b_1, \dots, b_{n-2}])$$

³Note that from (3.2) it is clear that variables b_i can be eliminated, i.e. they are determined by a_i 's.

defined by

$$a_1 \mapsto z_1 + z_2, \quad a_2 \mapsto z_1 z_2, \quad b_i \mapsto b_i.$$

It is a double cover unramified outside of the locus $z_1 = z_2$.

Let us count solutions of (3.5) for which $z_1 \neq z_2$ and both of them are non-zero. This reduces to counting pairs z_1, z_2 satisfying

$$\begin{aligned} z_1^{2n} &= z_1 + z_2 \\ z_2^{2n} &= z_1 + z_2. \end{aligned}$$

Eliminating z_2 using the first equation we obtain that z_1 must be a solution of

$$(z_1^{2n} - z_1)^{2n} = z_1^{2n}.$$

Now it is straightforward to count that there are $2(n-1)(2n-1)$ distinct pairs of numbers (z_1, z_2) satisfying our conditions that we obtain in this way.

In terms of the original system (3.2) the above computation means that there are at least $(n-1)(2n-1)$ distinct solutions outside of the origin. Note that we have divided the number of solutions by 2, since the initial count was on the double cover. In other words the dimension of B is at least $(n-1)(2n-1)$.

Up to now we have shown that $\dim_K(A) \geq n-1$ and $\dim_K(B) \geq (n-1)(2n-1)$. Since $\dim_K(\text{QH}(X)) = 2n(n-1)$, this implies that $\dim_K(A) = n-1$ and $\dim_K(B) = (n-1)(2n-1)$. Hence, (3.4) is an isomorphism.

The above discussion can be summarized in the following statement.

Proposition 3.3. *The scheme $\text{Spec}(\text{QH}(X))$ decomposes into the disjoint union of $(2n-1)(n-1)$ reduced points $\text{Spec}(K)$ and one fat point $\text{Spec}(K[\varepsilon]/\varepsilon^{n-1})$.*

4. FOUR-POINT GROMOV-WITTEN INVARIANTS

In Section 5 we will study the deformation of $\text{QH}(X)$ in the big quantum cohomology $\text{BQH}(X)$ in a very special direction, namely in the direction σ_2 . In particular we shall need to compute the following invariants:

$$I_1(\text{pt}, \sigma_2, \sigma_i, \sigma_j).$$

Note first that for dimension reasons these invariants vanish except if we have $i+j = 2n-2$. We shall also use the following result which simply follows from Kleiman-Bertini's Theorem [9] (see also [4, Lemma 14]).

Lemma 4.1. *The Gromov-Witten invariant $I_1(\text{pt}, \sigma_2, \sigma_i, \sigma_j)$ is the number of lines meeting general representatives of the cohomology classes pt , σ_2 , σ_i and σ_j . Furthermore, given any open dense subset of the set of lines, the above lines can be chosen in this open subset.*

As a consequence, we may only consider lines in $\mathring{Y}$, the open orbit of the variety Y of lines on X in order to compute the above invariants. For $Z \subset X$ a subvariety in X , we define

$$\mathring{Y}(Z) = \{\ell \in \mathring{Y} \mid \ell \cap Z \neq \emptyset\}.$$

Lemma 4.2. *We have the following equalities*

$$\begin{aligned} \mathring{Y}(\{E_2\}) &= \{\ell(W_3) \in \mathring{Y} \mid E_2 \subset W_3\} \simeq \mathbb{P}(\mathbb{C}^{2n}/E_2) \setminus \mathbb{P}(E_2^\perp/E_2) \\ \mathring{Y}(Z(E_{2n-k-1})) &= \{\ell(W_3) \in \mathring{Y} \mid \dim(W_3 \cap E_{2n-k-1}) \geq 1\}. \end{aligned}$$

For E_2 and E_{2n-k-1} in general position, their intersection is isomorphic to

$$\mathbb{P}(E_2 + E_{2n-k-1}/E_2) \setminus \mathbb{P}((E_2 + E_{2n-k-1}) \cap E_2^\perp/E_2) \subset \mathbb{P}(\mathbb{C}^{2n}/E_2) \setminus \mathbb{P}(E_2^\perp/E_2).$$

Proof. The first equality is clear. For the isomorphism we just map W_3 to W_3/E_2 and remark that W_3 is non isotropic if and only if W_3/E_2 is not contained in $E_2^\perp/E_2$.

The second equality is also easy. Let $V_2 \in \ell(W_3) \cap Z(E_{2n-k-1})$. Then $\dim(V_2 \cap E_{2n-k-1}) \geq 1$ and $V_2 \subset W_3$. Conversely, for W_3 satisfying $\dim(W_3 \cap E_{2n-k-1}) \geq 1$, we can find a 2-dimensional isotropic subspace $V_2 \subset W_3$ meeting E_{2n-k-1} non trivially.

For $k = 0$, we have $E_2 + E_{2n-k-1} = \mathbb{C}^{2n}$ (recall that E_2 and E_{2n-k-1} are in general position) thus $\dot{Y}(Z(E_{2n-k-1})) = \dot{Y}$ and the result follows. For $k \geq 1$, we have $E_2 \cap E_{2n-k-1} = 0$ (since they are in general position). This implies for W_3 in the intersection the inclusion $W_3 \subset E_2 + E_{2n-k-1}$ thus $W_3/E_2 \subset E_2 + E_{2n-k-1}/E_2$ proving the result. $\square$

Corollary 4.3. *We have $I_1(\text{pt}, \sigma_2, \sigma_i, \sigma_j) = \delta_{i+j, 2n-2}$.*

Proof. We already explained that the invariant vanishes unless $i + j = 2n - 2$. In that case, $I_1(\text{pt}, \sigma_2, \sigma_i, \sigma_j)$ is the number of lines meeting $\{E_2\}$, $Z(\{E_{2n-3}\})$, $Z(\{E_{2n-i-1}\})$ and $Z(\{E_{2n-j-1}\})$ where the spaces E_k are in general position and such that $\omega|_{E_k}$ has minimal rank. The variety of such lines is the intersection in $\mathbb{P}(\mathbb{C}^{2n}/E_2) \setminus \mathbb{P}(E_2^\perp/E_2)$ of the three linear spaces

$$\mathbb{P}(E_2 + E_{2n-3}/E_2), \quad \mathbb{P}(E_2 + E_{2n-i-1}/E_2) \quad \text{and} \quad \mathbb{P}(E_2 + E_{2n-j-1}/E_2).$$

These linear spaces are in general position and of respective codimension 1, $i - 1$ and $j - 1$. These codimensions add up to the dimension of $\mathbb{P}(\mathbb{C}^{2n}/E_2)$ thus the linear spaces meet in exactly one point. $\square$

5. BIG QUANTUM COHOMOLOGY OF $\text{IG}(2, 2n)$

In [6, 12] it is proved that the big quantum cohomology of $\text{IG}(2, 2n)$ is generically semisimple. In this section we will give an alternative proof of this result. This is the main result of this article.

As before we let $X = \text{IG}(2, 2n)$ and consider the deformation $\text{BQH}_\tau(X)$ of the small quantum cohomology $\text{QH}(X)$ inside the big quantum cohomology $\text{BQH}(X)$ in the direction $\tau = \sigma_2$. Explicitly it means the following. For any cohomology classes $a, b \in H^*(X)$ the product is of the form

$$a \star_\tau b = a \star_0 b + t \sum_{\sigma} \sum_{d \geq 1} q^d I_d(a, b, \sigma, \sigma_2) \sigma^\vee + O(t^2),$$

where $a \star_0 b$ is the small quantum product, σ runs over the basis of the cohomology consisting of Schubert classes, $\sigma^\vee$ is the dual basis, and t is the deformation parameter.

According to the dimension axiom for GW invariants $I_d(\sigma_i, \sigma_j, \sigma, \sigma_2)$ vanishes unless

$$i + j + \deg(\sigma) + 2 = (2n - 1)d + 2(2n - 2).$$

From here one easily deduces that $d = 1$ is the only possibility unless $i = j = 2n - 2$. Therefore, applying Corollary 4.3, we have

$$\sigma_i \star_\tau \sigma_j = \sigma_i \star_0 \sigma_j + \delta_{i+j, 2n-2} qt + O(t^2) \tag{5.1}$$

for $i + j < 4n - 4$.

Consider the following elements in $\text{BQH}_\tau(X)$

$$\Delta_r = \det(\sigma_{1+j-i})_{1 \leq i, j \leq r} \quad \text{for } r \in [3, 2n - 2]$$

$$\Sigma_{2n-2} = \sigma_{n-1} \star_\tau \sigma_{n-1} + 2 \sum_{i=1}^{n-1} (-1)^i \sigma_{n-1+i} \star_\tau \sigma_{n-1-i}$$

$$\Sigma_{2n} = \sigma_n \star_\tau \sigma_n + 2 \sum_{i=1}^{n-2} (-1)^i \sigma_{n+i} \star_\tau \sigma_{n-i} + (-1)^{n+1} q \sigma_1,$$

where all products are taken in $\text{BQH}_\tau(X)$. Note that these are “the same” elements as those defining relations in the presentation of $\text{QH}(X)$ in Theorem 3.1. The lower indices indicate the degree of the respective element in $\text{BQH}_\tau(X)$. The variable t can only appear together with a positive power of q so that $q^d t$ can only occur in elements of degree at least $\deg(q) + \deg(t) = 2n - 2$.

Lemma 5.1. *We have $\Delta_r = O(t^2)$ for all $r \in [3, 2n - 2]$, $\Sigma_{2n-2} = (-1)^n q t + O(t^2)$ and $\Sigma_{2n} = O(t^2)$.*

Proof. (i) Here we prove the statement about Δ_r ’s. From Theorem 3.1 we know that $\Delta_r = O(t^2)$ for $r \in [3, 2n - 3]$. Thus, we only need to consider Δ_{2n-2} . Inductively developing the determinant with respect to the first column we have

$$\Delta_{2n-2} = \sum_{s=1}^{2n-2} (-1)^{s-1} \sigma_s \star_\tau \Delta_{2n-2-s} = O(t^2) - \sigma_{2n-4} \star_\tau \Delta_2 + \sigma_{2n-3} \star_\tau \Delta_1 - \sigma_{2n-2},$$

where we used the fact that $\Delta_r = O(t^2)$ for $r \in [3, 2n - 3]$. Thus, we need to prove that

$$\sigma_{2n-4} \star_\tau \sigma_2 - \sigma_{2n-4} \star_\tau \sigma_1 \star_\tau \sigma_1 + \sigma_{2n-3} \star_\tau \sigma_1 - \sigma_{2n-2} \quad (5.2)$$

is of the form $O(t^2)$. We will see this by reducing everything to products of special Schubert classes. First, let us look at the term $\sigma_{2n-4} \star_\tau \sigma_1 \star_\tau \sigma_1$. By degree reasons (or dimension axiom for GW invariants) we have

$$(\sigma_{2n-4} \star_\tau \sigma_1) \star_\tau \sigma_1 = (\sigma_{2n-3} + \sigma'_{2n-3}) \star_\tau \sigma_1, \quad (5.3)$$

where σ'_{2n-3} is a Schubert class of degree $2n - 3$ different from σ_{2n-3} . Since by (5.1) we have $\sigma_{2n-3} \star_\tau \sigma_1 = \sigma_{2n-3} \star_0 \sigma_1 + q t + O(t^2)$, we only need to take care of the term $\sigma'_{2n-3} \star_\tau \sigma_1$. By degree reasons we have

$$\sigma'_{2n-3} \star_\tau \sigma_1 = \sigma'_{2n-3} \star_0 \sigma_1 + I_1(\sigma'_{2n-3}, \sigma_1, \text{pt}, \sigma_2) q t + O(t^2).$$

Applying the dimension axiom for GW invariants we see that the 4-point invariant $I_1(\sigma'_{2n-3}, \sigma_1, \text{pt}, \sigma_2)$ is equal to the 3-point invariant $I_1(\sigma'_{2n-3}, \text{pt}, \sigma_2)$. The latter can be computed using results in [3] and we get $I_1(\sigma'_{2n-3}, \text{pt}, \sigma_2) = 1$. Therefore, we obtain

$$\sigma'_{2n-3} \star_\tau \sigma_1 = \sigma'_{2n-3} \star_0 \sigma_1 + q t + O(t^2). \quad (5.4)$$

Plugging (5.3) and (5.4) into (5.2) and using that Δ_{2n-2} vanishes modulo t we get the required statement.

(ii) To prove the statements about Σ_{2n-2} and Σ_{2n} we only need to use (5.1) and the fact that these elements vanish modulo t . $\square$

5.1. Structure of $\text{BQH}_\tau(X)$. Recall from Section 3.2 that for $\text{QH}(X)$ we have the direct product decomposition

$$\text{QH}(X) = A \times B = A \times \prod_{i \in I} B_i$$

where $A = K[\varepsilon]/\varepsilon^{n-1}$ each B_i is the ground field K . By Hensel's lemma this decomposition lifts to $\text{BQH}_\tau(X)$ and we have

$$\text{BQH}_\tau(X) = \mathcal{A} \times \prod_{i \in I} \mathcal{B}_i \quad (5.5)$$

where $\mathcal{A}$ and $\mathcal{B}_i$ are algebras over $K[[t]]$.

By construction $K[[t]]$ -algebra $\text{BQH}_\tau(X)$ is a free module over $K[[t]]$ of finite rank equal to $\dim H^*(X)$. Therefore, $\mathcal{A}$ and $\mathcal{B}_i$'s are also free $K[[t]]$ -modules of finite rank. Reducing modulo t one sees that the rank of $\mathcal{A}$ is $n-1$ and the ranks of $\mathcal{B}_i$'s are all equal to 1. Similarly we obtain that the natural algebra homomorphism $K[[t]] \rightarrow \mathcal{B}_i$ is an isomorphism of K -algebras.

Proposition 5.2. *The ring $\text{BQH}(X)$ is regular.*

Proof. Let us first prove that $\text{BQH}_\tau(X)$ is a regular ring, which reduces to showing that $\mathcal{A}$ is regular. For this we use the presentation of $\text{BQH}_\tau(X)$ as the quotient of $(K[[t]])[\sigma_1, \dots, \sigma_{2n-2}]$ with respect to the ideal generated by the elements Δ_r with $r \in [3, 2n-2]$, Σ_{2n-2} and Σ_{2n} . Computing the Jacobian of this system at the origin one sees that it is of maximal possible rank $2n-2$. From here, using Prop. 8.4A and Prop. 8.7 of [7], as usual, one obtains that $\dim_K \mu/\mu^2 = 1$, where μ is the maximal ideal of $\mathcal{A}$. Since the Krull dimension of $\mathcal{A}$ is also equal to one, we obtain the regularity of $\mathcal{A}$.

The regularity of $\text{BQH}(X)$ can be obtained by using an identical argument with the Jacobian. The only difference is the number of the deformation parameters. Note that no additional GW invariants are necessary. $\square$

Theorem 5.3. *$\text{BQH}(X)$ is generically semisimple.*

Proof. This is a simple corollary of the above Proposition 5.2. Let us give a short proof for completeness.

The coordinate ring of the generic fiber

$$\text{BQH}(X)_\eta = \text{BQH}(X) \otimes_{K[[t_0, \dots, t_s]]} K((t_0, \dots, t_s))$$

is a localization of $\text{BQH}(X)$. Therefore, it is also a regular ring, since $\text{BQH}(X)$ was regular. This implies that $\text{BQH}(X)_\eta$ is a product of finite field extensions of $K((t_0, \dots, t_s))$ which was our definition of semisimplicity. $\square$

Remark 5.4. *We view the above theorem as a consequence of "generic smoothness" in characteristic zero. The reason it is not formulated this way lies in the fact that both $\text{BQH}(X)$ and $K[[t_0, \dots, t_s]]$ have modules of Kähler differentials of infinite rank. It might be possible to bypass this obstacle by working with a version of Kähler differentials that uses the topology of these rings but we did not pursue this.*

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