Homogeneity applied to the controllability of a system of parabolic equations

Lionel Rosier
École des Mines, Paris

Journée Contrôle et EDP
Versailles, 12 octobre 2017

This talk is based on joint work with:

- Jean-Michel Coron (LJJL, UPMC, Paris)
- Sergio Guerrero (LJJL, UPMC, Paris)

Let $\omega \subset \Omega \subset \mathbb{R}^N$ be bounded smooth open sets, and let $T > 0$.

The null controllability problem for the heat equation with distributed control reads as follow: given u_0, can we find a function $h = h(t,x)$ s.t. the solution $u = u(t,x)$ of

$$\begin{aligned} u_t &= \Delta u + 1_\omega h, && \text{in } (0,T)\times\Omega, \\ u &= 0, && \text{on } (0,T)\times\partial\Omega, \\ u(0,x) &= u_0(x), && \text{in } \Omega \end{aligned}$$

satisfies

$$u(T,x) = 0 \qquad \text{in } \Omega\ ?$$

Positive answers were given by

1. Fattorini-Russell '71(1D, using moment method)
2. Lebeau-Robbiano '95, Imanuvilov-Fursikov 96' (ND, $\forall(\Omega,\omega,T)$), using Carleman estimates)

- Fernandez Cara, de Tereza [2004]
- Ammar-Khodja, Benabdallah, Dupaix, Kostine [2005]
- Leiva [2005]
- Fernandez-Cara, Gonzalez-Burgos, de Tereza [2006,2009]
- Gonzalez-Burgos, Perez-Garcia [2006]
- Guerrero [2007]
- Ammar-Khodja, Benabdallah, Dupaix, Gonzalez-Burgos [2007,2009]
- ...

F. Ammar-Khodja *et al.* '07 proved that the null controllability of the linear system of heat equations

$$\begin{aligned} u_t = (D\Delta + A)u + 1_\omega Bh, &\qquad \text{in } (0,T)\times\Omega \\ u = 0, &\qquad \text{on } (0,T)\times\partial\Omega \end{aligned}$$

was equivalent to the extended Kalman condition

$$\mathrm{rank}\,[L_p|B] = n \quad \forall p \geq 1,$$

where

- $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix
- $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given matrices
- $h \in L^2(0,T,\mathbb{R}^m)$ is the control input
- $[A|B] := [B, AB, A^2B, ..., A^{n-1}B]$ is Kalman matrix
- $L_p := -\lambda_p D + A$
- $(\lambda_p)_{p\geq 1}$ is the sequence of eigenvalues of $-\Delta$ with Dirichlet b.c.

Consider the system

$$\begin{aligned} y_t - \Delta y &= 1_\omega h, && \text{in } (0,T)\times\Omega, \\ z_t - \Delta z &= |a(x)|^2 y, && \text{in } (0,T)\times\Omega, \\ y(t,x) = z(t,x) &= 0, && \text{in } (0,T)\times\partial\Omega. \end{aligned}$$

where

- $\Omega \subset \mathbb{R}^N$, $\emptyset \neq \omega \subset \Omega$ are open sets
- $a \in L^\infty(\Omega)$ satisfies $|a(x)| > \varepsilon > 0$ a.e. on $\mathcal{O}$, a nonempty open set in Ω

The system is null controllable if

- $\omega \cap \mathcal{O} \neq \emptyset$;
- $\omega \cap \mathcal{O} = \emptyset$ and $N = 1$ (L. de Teresa-LR 2011)
- $\omega \cap \mathcal{O} = \emptyset$, $N \geq 1$, and the GCC is satisfied for (Ω, ω) and $(\Omega, \mathcal{O})$ (Alabau-Boussouira-Léautaud 2013)

Open Question: is it still true without the GCC?

- When the linearized system (around 0) is null controllable, a classical fixed-point approach yields the (local) null controllability of the nonlinear system
- However, some systems exhibit only **nonlinear** coupling terms

- **P. Érdi and J. Tóth, Mathematical Models of Chemical Reactions, 1989**

$$
\begin{aligned}
u_t &= d_1 \Delta u - \alpha(u^k - v^m) \\
v_t &= d_2 \Delta v + \beta(u^k - v^m) \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \qquad \text{on} \quad (0, T) \times \partial\Omega
\end{aligned}
$$

- $d_1, d_2, \alpha, \beta > 0$, $k, m \in \mathbb{N}^*$.
- Underlying reversible chemical reaction: $kA \rightleftharpoons mB$.

Consider the system

$$u_t - \Delta u + F(u,v) = h 1_\omega \qquad \text{in } (0,T)\times\Omega, \tag{0.1}$$

$$v_t - \Delta v + G(u,v) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{0.2}$$

$$u = v = 0 \qquad \text{on } (0,T)\times\partial\Omega. \tag{0.3}$$

- State: $(u,v) \in L^2(\Omega)^2$; control: $h \in L^2(0,T,L^2(\Omega))$ present in (0.1) only
- if $\frac{\partial G}{\partial u}(0,0) \neq 0$, then the null controllability of the system can be proved by linearization.
- if $\frac{\partial G}{\partial u}(0,0) = 0$, then the linearized system is not null controllable, for its second equation reads

  $$v_t - \Delta v + av = 0 \qquad \text{in } (0,T)\times\Omega.$$

  Thus the classical linearization approach fails.

  Question: is it nevertheless possible to get the null controllability of the full system?

Theorem

The system

$$(S)\begin{cases} u_t - \Delta u = 1_\omega h & \text{in } (0,T)\times\Omega \\ v_t - \Delta v = u^3 & \text{in } (0,T)\times\Omega \\ u = v = 0 & \text{on } (0,T)\times\partial\Omega \end{cases}$$

is locally null controllable in $L^\infty(\Omega)^2$ with controls in $L^\infty((0,T)\times\Omega)$.

The proof rests on the **return method** introduced by **Jean-Michel Coron** for Euler equations of incompressible perfect fluids: we take a linearization along a **smooth, nontrivial** trajectory $(\bar{u},\bar{v})$ such that

$$(\bar{u},\bar{v})_{|t=0} = (0,0) = (\bar{u},\bar{v})_{|t=T}$$

- Step 1 : Construction of the reference trajectory
- Step 2 : Controllability of the linearized system along the trajectory (combining a Carleman estimate with local energy estimates, following **de Teresa** *et al.*)
- Step 3 : Fixed-point argument (Kakutani theorem)

Proposition

Let $\rho > 0$. There exists a function $\bar{v} = \bar{v}(t,x)$, $\bar{v} \neq 0$ such that

$$\bar{v} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^n), \; \bar{v}(t,x) = 0 \text{ for } |t| \geq \rho \text{ or } |x| \geq \rho$$
$$\bar{v}_t = \Delta \bar{v} + \bar{u}^3 \qquad \text{with } \bar{u} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$$

- The corresponding control reads $\bar{h} = \bar{u}_t - \Delta \bar{u} \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$
- ρ arbitrarily small. Shift to get supp $(\bar{h}) \subset (0, T) \times \omega$.
- Main difficulty: **smoothness** of $\bar{u} = (\bar{v}_t - \Delta \bar{v})^{1/3}$.

- By a scaling argument, on may assume $\rho = 1$
- Let $r = |x|$. Look for $\bar{v} = \bar{v}(t, r)$ with $\bar{v} \in C^\infty(\mathbb{R}_t \times \mathbb{R}_r^+)$, $\bar{v}(t, r) = 0$ for $|t| \geq 1$ or $r \geq 1$, and

$$\bar{u} = (\bar{v}_t - \bar{v}_{rr} - \frac{n-1}{r}\bar{v}_r)^{\frac{1}{3}} \in C^\infty(\mathbb{R}_t \times \mathbb{R}_r^+).$$

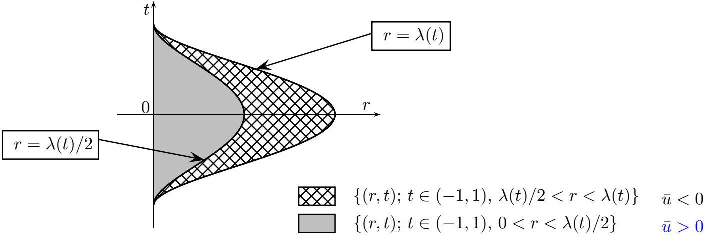
- Search for $\bar{v}$ in the form

$$\bar{v}(t, r) = \sum_{i=0}^{3} f_i(t) g_i(z)$$

with $z = r/\lambda(t)$, where

$$\begin{aligned} \lambda(t) &= \varepsilon(1 - t^2)^2 \\ f_0(t) &= 1_{(-1,1)}(t) e^{-\frac{1}{1-t^2}} \\ \text{supp}(f_i) &\subset [-1, 1], \qquad 1 \leq i \leq 3 \\ \text{supp}(g_i) &\subset [\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} + \frac{\delta}{2}], \qquad 1 \leq i \leq 3 \end{aligned}$$

Support of $(\bar{u}, \bar{v})$ for $\lambda = \varepsilon(1 - t^2)^2$



- Let $Q = -\lambda^2\bar{u}^3 = \lambda^2[\bar{v}_{rr} + \frac{n-1}{r}\bar{v}_r - \bar{v}_t]$.
  One aims to "control" the behavior of Q when it changes of sign.
- We will define f_1, f_2, f_3 and g_1, g_2, g_3 in such a way that

$$Q = Q_z = Q_{zz} = 0 \qquad \text{and } Q_{zzz} \geq const.f_0 \qquad \text{for } \; z = 1/2$$

  (hence $Q \sim (z - \frac{1}{2})^3 f_0$)
- We obtain

$$Q = \sum_{i=0}^{3}[f_i(g_i^{(2)} + \frac{n-1}{z}g_i^{(1)}) + z\lambda\dot{\lambda}f_i g_i^{(1)} - \lambda^2\dot{f}_i g_i]$$

Lemma

There exists a fonction $G \in C^\infty(0,+\infty)$ *such that*

$$\begin{aligned} G(z) &= (z-\frac{1}{2})^3 && \text{for } \frac{1}{2}-\delta < z < \frac{1}{2}+\delta \\ (z-\frac{1}{2})G(z) &> 0 && \text{for } 0 < z < 1,\ z \neq \frac{1}{2} \end{aligned}$$

and such that the solution g_0 *of the Cauchy problem*

$$\begin{aligned} & g_0''(z) + \frac{n-1}{z} g_0'(z) = G(z), \quad z > 0 \\ & g_0(1) = g_0'(1) = 0 \end{aligned}$$

satisfies

$$g_0(z) = \begin{cases} 1 - z^2 & \text{si } 0 < z < \delta \\ e^{-\frac{1}{1-z^2}} & \text{si } 1-\delta < z < 1 \\ 0 & \text{si } z \geq 1. \end{cases}$$

Construction of f_1

$$Q(.,\frac{1}{2}) = \frac{1}{2}\lambda\dot{\lambda}f_0 g_0^{(1)}(\frac{1}{2}) - \lambda^2 \dot{f}_0 g_0(\frac{1}{2})$$
$$+\sum_{i=1}^{3}\{f_i(t)(g_i^{(2)}(\frac{1}{2}) + 2(n-1)g_i^{(1)}(\frac{1}{2}))$$
$$+\frac{1}{2}\lambda\dot{\lambda}f_i g_i^{(1)}(\frac{1}{2}) - \lambda^2 \dot{f}_i g_i(\frac{1}{2})\}$$

- For $1 \leq i \leq 3$ and $0 \leq j \leq 2$ we impose

$$g_i^{(j)}(\frac{1}{2}) = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2 \\ 0 & \text{if not} \end{cases}$$

- We define $f_1(t)$ in such a way that $Q(.,\frac{1}{2}) = 0$

For $0 \leq k \leq 3$

$$\partial_z^k Q = \sum_{i=0}^{3} \{ f_i(t) (g_i^{(k+2)} + (n-1) \partial_z^k (g_i^{(1)} / z))$$
$$+ \lambda \dot{\lambda} f_i (z g_i^{(k+1)} + k \, g_i^{(k)}) - \lambda^2 \dot{f}_i g_i^{(k)} \}$$

- For $1 \leq i \leq 3$ and $0 \leq j \leq 4$ we impose

$$g_i^{(j)}(\frac{1}{2}) = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{if not} \end{cases}$$

- For $1 \leq k \leq 2$ we define $f_{k+1}(t)$ in such a way that $\partial_z^k Q(., \frac{1}{2}) = 0$

- A simple calculation gives $Q_{zzz} = 6f_0 + R$ with
$$|R| \leq const\ \varepsilon^2 f_0 \quad \text{for } |z - \frac{1}{2}| < \frac{\delta}{2}$$
We pick ε **small enough** so that $Q_{zzz} \geq f_0$ when $|z - \frac{1}{2}| < \frac{\delta}{2}$ and $(z - \frac{1}{2})Q > 0$ for all $z \in (0, 1)$.
- It follows that $Q(t, z) = f_0(t)(z - \frac{1}{2})^3 \varphi(t, z)$ with $\varphi \in C^\infty((-1, 1)_t \times (0, 1)_z)$ and $\varphi > 0$
- One can prove that v and u are of class C^∞.

- Result **false** if u^3 is replaced by u^2: if we consider

$$\begin{cases} u_t - \Delta u = 1_\omega h & \text{in } (0,T)\times\Omega \\ v_t - \Delta v = u^2 & \text{in } (0,T)\times\Omega \\ u = v = 0 & \text{on } (0,T)\times\Omega \end{cases}$$

  then we know from **maximum principle** that $v_0 \geq 0$ implies $v(t,.) \geq e^{t\Delta}v_0$ for $t \geq 0$, so that the null controllability fails.

- However, the null controllability can be recovered by taking complex-valued controls $h \in L^\infty((0,T)\times\Omega;\mathbb{C})$.

- Consider now

$$\begin{cases} u_t - \Delta u + F(u,v) = h 1_\omega \\ v_t - \Delta v + G(u,v) = 0 \\ u = v = 0 \qquad\qquad \text{for } x \in \partial\Omega, \end{cases}$$

  where the first integer p such $\partial^p G / \partial u^p(0,0) \neq 0$ is odd

- Finding a reference trajectory (as before) is hopeless in general

Let $\omega \subset \Omega \subset \mathbb{R}^N$ and consider the system

$$u_t - \Delta u + F(u,v) = h 1_\omega \quad \text{in } (0,T)\times\Omega, \tag{0.4}$$

$$v_t - \Delta u + G(u,v) = 0 \quad \text{in } (0,T)\times\Omega, \tag{0.5}$$

$$u = v = 0 \quad \text{on } (0,T)\times\partial\Omega \tag{0.6}$$

where $F, G \in C^\infty(\mathbb{R}^2,\mathbb{R})$ satisfy $F(0,0) = 0$ and

$$\begin{aligned} \frac{\partial^i G}{\partial u^i}(0,0) &= 0, \quad \forall i \in \{0,...,2k\}, \\ \frac{\partial^{2k+1} G}{\partial u^{2k+1}}(0,0) &\neq 0 \end{aligned}$$

for some $k \in \mathbb{N}$.

Then the system (0.4)-(0.6) is locally null controllable in time T.

Main result

Theorem

For all $p \in (n+2, \infty)$ *and* $T > 0$*, there exist* $C > 0$ *and* $\varepsilon > 0$ *such that for*

$$|u_0|_{W^{1,\infty}(\Omega)} < \varepsilon, \quad |v_0|_{W^{1,\infty}(\Omega)} < \varepsilon,$$

there exists some control $h \in L^p((0,T) \times \Omega)$ *satisfying*

$$|h|_{L^p((0,T)\times\Omega)} \leq C \left(|u_0|_{W^{1,\infty}(\Omega)} + |v_0|_{W^{1,\infty}(\Omega)}^{\frac{1}{2k+1}} \right)$$

and such that the solution (u,v) *issued from* $(u,v)_{|t=0} = (u_0, v_0)$ *satisfies* $(u,v)_{|t=T} = (0,0)$*.*

- Construction of the reference trajectory $(\bar{u}, \bar{v})$ almost impossible for the full system

$$v_t - \Delta v + G(u, v) = 0 \quad \ldots$$

- ... but possible for the first homogeneous approximation of G, namely $G_0(u, v) = c_1 u^{2k+1} + c_2 v$. For that trajectory $(\bar{u}, \bar{v})$, $\bar{v} \sim \bar{u}^{2k+1}$ as $t \to T$.

- So, we write

$$G(u, v) = G_0(u, v) + R(u, v)$$

where $R(u, v)$ is a disturbance to be balanced on $(0, T)$, and we pick a reference trajectory $(\bar{u}, \bar{v})$ for which

$$\bar{v}_t - \Delta \bar{v} + G_0(\bar{u}, \bar{v}) = 0$$

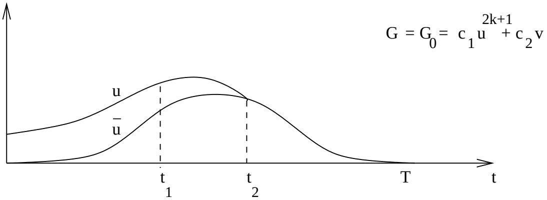
with $G_0(u, v) = c_1 u^{2k+1} + c_2 v$.

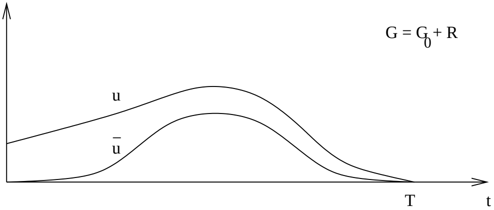
For

$$G(u,v) = u^5 - u^7 - 3v^2 + uv + 2v$$

we have $p = 2k + 1 = 5$ and

$$G_0(u,v) = u^5 + 2v.$$


G = G0 = c1 u2k+1 + c2 v
u
ū
t1
t2
T
t


G = G0 + R
u
ū
T
t

Consider the map (between some appropriate Banach spaces: the functions in $\mathcal{X}$ vanish at $t = T$)

$$\begin{array}{rcl} \mathcal{A}: \mathbb{R} \times W_0^{1,\infty}(\Omega)^2 \times \mathcal{X} & \to & \mathcal{Y} \\ (\varepsilon, U^0, V^0, (U, V, H)) & \mapsto & (\mathcal{A}_1, \mathcal{A}_2, U(0) - U^0, V(0) - V^0) \end{array}$$

defined by

$$\mathcal{A}_1 = \begin{cases} \frac{1}{\varepsilon}(u_t - \Delta u + F(u, v) - h1_\omega) & \text{if } \varepsilon \neq 0, \\ U_t - \Delta U + f_0 U - H1_\omega & \text{if } \varepsilon = 0, \end{cases}$$

$$\mathcal{A}_2 = \begin{cases} \frac{1}{\varepsilon^{2k+1}}(v_t - \Delta v + G(u, v)) & \text{if } \varepsilon \neq 0, \\ V_t - \Delta V + \frac{1}{2k+1} g_0((U + \bar{u})^{2k+1} - \bar{u}^{2k+1}) + g_1 V & \\ & \text{if } \varepsilon = 0, \end{cases}$$

with

$$u := \varepsilon(\bar{u} + U), \quad v := \varepsilon^{2k+1}(\bar{v} + V), \quad h := \varepsilon(\bar{h} + H)$$

and $f_0 := \frac{\partial F}{\partial u}(0, 0)$, $g_0 := \frac{1}{(2k)!} \frac{\partial^{2k+1} G}{\partial u^{2k+1}}(0, 0)$, $g_1 := \frac{\partial G}{\partial v}(0, 0)$.

Denoting

$$\mathcal{L} = \frac{\partial \mathcal{A}}{\partial (U, V, H)}(0)$$

we find that

$$\begin{aligned} \mathcal{L}(U, V, H) &= (U_t - \Delta U + f_0 U - H 1_\omega, \\ & \quad V_t - \Delta V + g_0 \bar{u}^{2k} U + g_1 V, U(0), V(0)) \end{aligned}$$

Using Carleman estimates, one can check that the map $\mathcal{L}$ is **onto**, and we infer the null controllability of the full system by a variant of the implicit function theorem.

Carleman estimate: (like those by Fursikov-Imanuvilov)

$$\iint_{(0,T)\times\Omega} e^{-s\rho(t,x)\eta(t)}\left((s\eta)^3|z|^2 + s\eta|\nabla z|^2 + (s\eta)^{-1}(|\Delta z|^2 + |z_t|^2)\right)$$
$$\leq C\Bigg(\iint_{(0,T)\times\Omega} e^{-s\rho(t,x)\eta(t)}|z_t + \Delta z|^2$$
$$+ \iint_{|x-x_0|<c|t-T|} e^{-s\rho(t,x)\eta(t)} s^3\eta^5|z|^2\Bigg)$$

for a smooth function $\rho(t,x) \geq const > 0$ satisfying

- $|\nabla\rho(t,x)| > const > 0 \qquad$ for $|x - x_0| \geq c|t - T|$
- $\partial\rho/\partial\nu \geq 0 \qquad$ in $(0, T) \times \partial\Omega$,

and $\eta(t) = (T - t)^{-1}$.

- Let $a > 0$, $b > 0$, $p \in \mathbb{N}$ and $q \in \mathbb{N}$, and let $\omega \subset \Omega \subset \mathbb{R}^N$ be bounded smooth open sets.
- Then the system

$$\begin{aligned} u_t = \Delta u - a(u^p - v^q) + h1_\omega \qquad & \text{in } (0, T) \times \Omega \\ v_t = \Delta v + b(u^p - v^q) \qquad & \text{in } (0, T) \times \Omega \\ u = v = 0 \qquad & \text{on } (0, T) \times \partial\Omega \end{aligned}$$

is null controllable iff p is odd

$$
\begin{aligned}
w_t &= (1+i\alpha)\Delta w + Rw - (1+i\beta)|w|^2 w + 1_\omega h \\
w &= 0 \qquad\qquad \text{on} \quad (0,T)\times\partial\Omega
\end{aligned}
$$

- Simple model of turbulence
- $w = w(t,x) \in \mathbb{C}$, $h = h(t,x) \in \mathbb{C}$. Coefficients $R, \alpha, \beta \in \mathbb{R}$.
- Null controllability proved in **LR, B.-Y. Zhang [2009]**; **Fu [2009]**.
- If control h supposed to take **real** values, writing $w = u + iv$, we obtain

$$
\begin{aligned}
u_t &= \Delta u - \alpha\Delta v + Ru - (u^2+v^2)(u-\beta v) + 1_\omega h \\
v_t &= \alpha\Delta u + \Delta v + Rv - (u^2+v^2)(\beta u + v) \\
u = v &= 0 \qquad\qquad \text{on} \quad (0,T)\times\partial\Omega
\end{aligned}
$$

- With $\alpha = 0$ the system reads

$$\begin{aligned} u_t &= \Delta u + Ru - (u^2 + v^2)(u - \beta v) + 1_\omega h \\ v_t &= \Delta v + Rv - (u^2 + v^2)(\beta u + v) \\ u = v &= 0 \qquad \text{on} \quad (0,T) \times \partial\Omega \end{aligned}$$

- If $\beta = 0$, the system **fails to be null controllable** . Indeed, for $v_0 \geq 0$, if $|R - (u^2 + v^2)| \leq C$ for $0 < t < T$, we have

$$v(t,.) \geq e^{t(\Delta - C)} v_0.$$

- If $\beta \neq 0$, the null controllability follows from our main result.

Conclusion

- A system of two parabolic equations with a nonlinear coupling term having an "odd" leading term is null controllable.
- The proof of that result combines the return method, the construction of a reference trajectory for the homogeneous approximating system, Carleman estimates, and a homogeneity argument.
- This is the first instance of a null controllability result for a system of PDE obtained through a homogeneity argument for which the usual linearization procedure fails.
- There are many other systems of PDEs (not necessarily parabolic) involving purely nonlinear coupling terms: e.g.
  1. $\chi^{(2)}$ second-harmonic generation equation (SHG) of type I

$$
\begin{aligned}
iu_t + u_{xx} - cu &= \bar{u}v \\
i\gamma v_t + v_{xx} - dv &= u^2
\end{aligned}
$$

  2. Zakharov system

$$
\begin{aligned}
iu_t + \Delta u &= uv \\
v_{tt} - \Delta v &= \Delta |u|^2
\end{aligned}
$$