The wave equation on the 2-regular Bethe lattice

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Outline

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The Bethe lattice

A z-regular Bethe lattice (a particular kind of Cayley graph, introduced by Hans Bethe in 1935), is an infinite connected cycle-free graph where each node is connected to $z+1$ neighbours. The z-regular Bethe lattice $\mathcal{B}_z$ is thus a 1-dimensional Riemannian manifold with singularities (the nodes). The canonical metric is the obvious one: each edge is identified with $]0,1[$ and has length 1. Given two points p, q in $\mathcal{B}_z$, there exists a *unique* geodesic connecting p and q. The number of nodes at distance $k > 0$ of a given node is

$$N_k = (z+1)z^{k-1}, \quad k > 0.$$

Let $\mathcal{N}_z$ the set of nodes and $\mathcal{A}_z$ the set of edges. Then $\mathcal{B}_z \setminus \mathcal{N}_z$ is the disjoint union of the edges $A \in \mathcal{A}_z$.

Wave equation

Each edge A is identified with the interval $]0,1[$. By definition, a wave on $\mathcal{B}_z$, with speed 1, is a collection of distributions $(u_A(x,t))_{A\in\mathcal{A}_z}$ defined on $]0,1[\times\mathbb{R}_t$ wich satisfy the usual 1-D wave equation on each edge

$$\frac{\partial^2 u_A}{\partial t^2}(x,t) - \frac{\partial^2 u_A}{\partial x^2}(x,t) = 0, \quad 0 < x < 1,\ t \in \mathbb{R},\ A \in \mathcal{A}_z \tag{1.1}$$

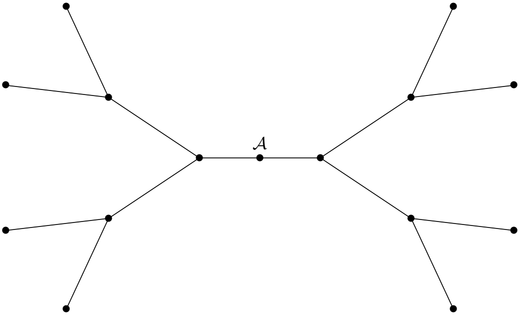
and the Kirchhoff boundary conditions at the nodes $N \in \mathcal{N}_z$. If one denotes by A_i the set of edges adjacent to N, this means

$$u_{A_i}(x = N, t) = u_{A_j}(x = N, t), \quad \forall i, j \tag{1.2}$$

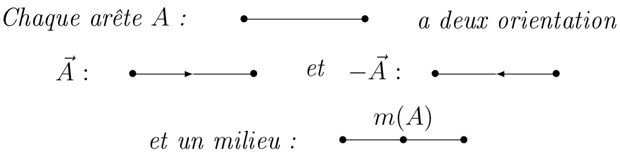
$$\sum_i \partial_n u_{A_i}(x = N, t) = 0 \tag{1.3}$$

The 2-regular Bethe lattice

For simplicity, we restrict ourselves to the case $z = 2$ and we set $\mathcal{B} = \mathcal{B}_2$, $\mathcal{N}_z = \mathcal{N}$ and $\mathcal{A}_z = \mathcal{A}$.



Observe that each edge A has two orientations and a middle point $m(A)$, defined by $x = 1/2$ in the identification $A =]0, 1[$. Observe also that the set of middle points of the Bethe lattice $\mathcal{B}_z$ defines canonically a Bethe lattice $\mathcal{B}_{2z-1}$. In particular, the set of middle points of $\mathcal{B} = \mathcal{B}_2$ defines the Bethe lattice $\mathcal{B}_3$.



Let $\vec{A}$ be a given oriented edge. We denote by $W(t, p, \vec{A})$ the unique wave solution of (1.1), (1.2) and (1.3) and such that

$$\begin{aligned} u_A(x,t) &= \delta_{x=1/2+t} \quad \forall t \in]-1/2, 1/2[\\ u_B(x,t) &= 0 \quad \forall t \in]-1/2, 1/2[, \quad \forall B \neq A \end{aligned} \tag{1.4}$$

It is not difficult to see by finit speed of propagation and elementary properties of a 1-d waves, that the knowledge of $W(t, p, \vec{A})$ allows to solve the Cauchy problem associated to (1.1), (1.2) and (1.3) for any Cauchy data. Moreover, it is not difficult to verify that $W(t, p, \vec{A})$ is at any time a finite sum of Dirac masses. For $t \in \mathbb{Z}$, these Dirac masses are located at the middle points of $\mathcal{B}$, and they propagate according to one of the two orientations on each edge. Therefore, one has

$$W(t, ., \vec{A}) = \sum_{\vec{B}, B \in \mathcal{A}} G(t, \vec{B}, \vec{A}) \delta_{x=1/2 \pm t}, \quad \forall t \in \mathbb{R} \setminus 1/2 + \mathbb{Z} \tag{1.5}$$

where the $\pm$ sign depends on the choice of the orientation of the edge B. Our purpose is to give an explicit formula for the coefficients

$$G(k, \vec{B}, \vec{A}) = ..., \quad k \in \mathbb{N}$$

(the case $k \in -\mathbb{N}$ follows by time symmetry and reverse orientation). One has obviously $G(t, \vec{B}, \vec{A}) = G(k, \vec{B}, \vec{A})$ for $k \in \mathbb{N}$ and $|t - k| < 1/2$.

An important quantity, related to the energy propagation, is $\mathcal{E}(k,d)$, with $k,d \in \mathbb{N}$. It is defined by

$$\mathcal{E}(k,d) = \sum_{dist(m(B),m(A))=d} |G(k,\vec{B},\vec{A})|^2 \tag{1.6}$$

By finit speed of propagation, one has obviously

$$\mathcal{E}(k,d) = 0 \quad \forall d > k$$

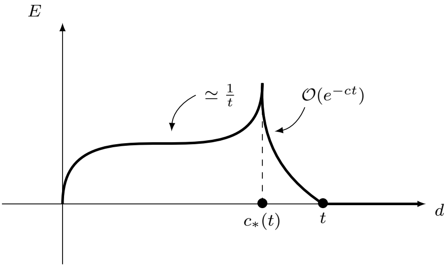
The main result is the following theorem, where $c_* = 2\sqrt{2}/3 < 1$ is an " effective speed of propagation" .

Theorem

1. *For* $\gamma = \frac{d}{k} > c_*$, $\exists\, c = c_\gamma > 0$ *s.t* $\mathcal{E}(k,d) \in \mathcal{O}(e^{-ck})$.
2. *For* $\gamma = \frac{d}{k} \in]0, c_*[$, $\exists\, d_j = d_{j,\gamma}, c_j = c_{j,\gamma}$ $j = 1,2,3,4$

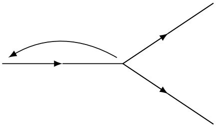
$$\mathcal{E}(k,d) = k^{-1} |\sum_j c_j e^{i d_j k} (1 + O(k^{-1}))|^2.$$

Propagation of energy



Outline

Propagation of waves at nodes



If the incoming wave is $\delta_{x=t}$ for $t < 0$, the outgoing wave is equal to

$$u(x,t) = \left(\frac{-\delta_{x=-t}}{3}, \frac{2\delta_{x=t}}{3}, \frac{2\delta_{x=t}}{3} \right) \quad \text{for} \quad t > 0.$$

Observe that the transfert coefficients satisfy:

$$-1/3 + 2/3 + 2/3 = 1 \quad \text{conservation of charge}$$

$$(-1/3)^2 + (2/3)^2 + (2/3)^2 = 1 \quad \text{conservation of energy}$$

Let $X(0) = \sum_{\vec{A}} x_{\vec{A}} \delta_{\vec{A}}$ the Cauchy data at $t = 0$, i.e :

$$\begin{cases} f_{|t=0, s\in A} = \left(x_{\vec{A}} + x_{-\vec{A}}\right) \delta_{s=1/2}, \\ \frac{\partial f}{\partial t}_{|t=0, s\in A} = \left(-x_{\vec{A}} + x_{-\vec{A}}\right) \delta'_{s=1/2} \end{cases} \tag{2.1}$$

then for $t \in \mathbb{N}$, we get by definition of $G(t, \vec{B}, \vec{A})$

$$X(t) = \sum_{\vec{B}} y_{\vec{B}} \delta_{\vec{B}}, \quad y_{\vec{B}} = \sum_{\vec{A}} G(t, \vec{B}, \vec{A}) x_{\vec{A}}$$

Lemma

The matrix

$$\mathbb{G}(t), \quad \mathbb{G}(t)_{\vec{B}, \vec{A}} = G(t, \vec{B}, \vec{A})$$

is unitary and satisfy the group law $\mathbb{G}(t+s) = \mathbb{G}(t)\mathbb{G}(s)$. *In particular:*

$$\mathbb{G}(t) = \mathbb{G}(1)^t, \quad \forall t \in \mathbb{N}$$

$\vec{A} \equiv \vec{B}$ iff $\mathbb{G}(1, \vec{B}, \vec{A}) \neq 0$.

Definition

A path γ of length $\ell(\gamma) = k$ connecting $\vec{A}$ to $\vec{B}$ is a $k+1$-uplet

$$\gamma = \left(\vec{A}, \vec{C}_1, ..., \vec{C}_{k-1}, \vec{B}\right) = \left(\vec{C}_0, ..., \vec{C}_k\right), \; \vec{A} = \vec{C}_0, ..., \vec{B} = \vec{C}_k$$

with $\vec{C}_j \equiv \vec{C}_{j+1} \, \forall \, j \in \{0, ..., k-1\}$.
One says that $\left(\vec{C}_j, \vec{C}_{j+1}\right)$ is an inversion if $\vec{C}_{j+1} = -\vec{C}_j$ and we denote by $r(\gamma)$ the total number of inversion. Thus one has $0 \leq r(\gamma) \leq k = \ell(\gamma)$.

We denote by $\mathcal{C}_{t,r}(\vec{A}, \vec{B})$ the set of path of length $t \geq 1$, with r inversion, connecting $\vec{A}$ to $\vec{B}$ and $\mathcal{C}_t = \cup_{r \geq 0} \mathcal{C}_{t,r}(\vec{A}, \vec{B})$.

By definition of the product of 2 matrices one has with $\alpha = -1/3, \beta = 2/3$

$$G(t, \vec{B}, \vec{A}) = \sum_{\gamma \in \mathcal{C}_t(\vec{A}, \vec{B})} \alpha^{r(\gamma)} \beta^{t - r(\gamma)}, \ \forall\, t \geq 1,$$

$$G(t, \vec{B}, \vec{A}) = \beta^t \sum_{r=0}^{t} \left(\frac{\alpha}{\beta}\right)^r \left|\mathcal{C}_{t,r}(\vec{A}, \vec{B})\right|, \ \forall\, t \geq 1. \tag{2.2}$$

Thus we have to compute $\left|\mathcal{C}_{t,r}(\vec{A}, \vec{B})\right| =$ **the number of path of length** t **with** r **inversions connecting** $\vec{A}$ **à** $\vec{B}$**.**

Observe that $\left|\mathcal{C}_{t,r}(\vec{A},\vec{B})\right|$ depends only on the distance d between $m(\vec{A})$ and $m(\vec{B})$ and on the 4 possible orientations of $\vec{A}$ à $\vec{B}$.

We denote by $\Gamma_{t,r}(d,j)$, $j \in \{1,2,3,4\}$ the number of path connecting $\vec{A}$ to $\vec{B}$, of length t, with r inversions, $dist(m(\vec{A}), m(\vec{B})) = d$ and the orientation $(\vec{A},\vec{B})$ of type $j \in \{1,2,3,4\}$.

Then we introduce the generating function:

$$\mathbb{F}_j(X,Y,Z) = \sum_{t\geq 0, d\geq 0, r\geq 0} \Gamma_{t,r}(d,j) X^t Y^r Z^d. \tag{2.3}$$

The computation of $\Gamma_{t,r}(d,j)$ is done by induction on $t \in \mathbb{N}$. This gives the following proposition

Proposition

$$\begin{cases} \mathbb{F}_1 = 1 + XZ\mathbb{F}_1 + XY\mathbb{F}_2 + X\left[\mathbb{F}_2 - \mathbb{F}_{2|Z=0}\right] + 2X\partial_Z\mathbb{F}_{2|Z=0} \\ \mathbb{F}_2 = XY\mathbb{F}_1 + \frac{2X}{Z}\left[\mathbb{F}_2 - \mathbb{F}_{2|Z=0}\right] \\ \mathbb{F}_3 = 1 + XZ\mathbb{F}_4 + \frac{2X}{Z}\left[\mathbb{F}_3 - \mathbb{F}_{3|Z=0}\right] \\ \mathbb{F}_4 = XY\mathbb{F}_3 + X\left[\mathbb{F}_3 - \mathbb{F}_{3|Z=0}\right] + XZ\mathbb{F}_4 + 2X\partial_Z\mathbb{F}_{2|Z=0}. \end{cases} \tag{2.4}$$

By construction, when the orientation of $(\vec{A}, \vec{B})$ is $j \in \{1, 2, 3, 4\}$, and $dist(m(\vec{A}), m(\vec{B})) = d$, one has

$$G(t, \vec{B}, \vec{A}) = \left(\frac{2}{3}\right)^t \frac{1}{t!}\,\partial_X^t \frac{1}{d!}\partial_Z^d \mathbb{F}_j|_{X=Z=0, Y=-1/2}. \tag{2.5}$$

Computation of $\mathbb{F}_j$

Observe that $Y = -1/2$ is fixed. We set $X = \sqrt{2}\,x$, $Z = \sqrt{2}\,z$, and $\mathbb{F}_1 = \Theta(f_1), \mathbb{F}_4 = \Theta(f_4)$, with $\Theta(f) = f - \frac{2x}{z}\left(f - f_{|z=0}\right)$. Then f_1, f_4 satisfy:

$$\left\{\begin{array}{l} \begin{bmatrix} p & -q \\ -q & p \end{bmatrix} \begin{bmatrix} f_1 \\ f_4 \end{bmatrix} = \frac{1}{1+\frac{9x^2}{2}} \begin{bmatrix} 1 \\ -\frac{x}{\sqrt{2}} \end{bmatrix} \\ p(f) = \left[1 - \mu\left(z + \frac{1}{z}\right)\right] f + \left(\frac{\mu}{z} - 5\delta\right) f_{|z=0} \\ q(f) = -\sqrt{2}\,\delta\partial_z f_{|z=0} \end{array}\right. \qquad (2.6)$$

$r = \frac{\sqrt{2}}{3}, \mu = \mu(x) = \frac{2x}{1+\frac{x^2}{r^2}}, \delta = \delta(x) = \frac{x^2}{1+\frac{x^2}{r^2}} = \frac{x}{2}\mu.$

The above system is an equation for functions of z, with x as holomorphic parameter in the complex disc $\left\{x \in \mathbb{C}, |x| < r = \frac{\sqrt{2}}{3}\right\}$. For $x = 0$ one as $p = Id$ et $q = 0$.

Let $z_-(\mu) = \frac{1}{2\mu}\left[1 - \sqrt{1-4\mu^2}\right]$.

Lemma

Let $\mu \in U = \mathbb{C} \setminus \{\mu \in \mathbb{C},\ \mu^2 \in [\frac{1}{4}, +\infty[\}$ *and* $(\alpha, \beta) \in \mathbb{C}^2$. *The system*

$$\left\{\begin{array}{l} p(f) - q(g) = \alpha \\ -q(f) + p(g) = \beta \end{array}\right. \tag{2.7}$$

admits a unique solution holomorphic in $|z| < 1$

$$f = \frac{az_-}{\mu}\frac{1}{1-zz_-},\ g = \frac{bz_-}{\mu}\frac{1}{1-zz_-} \tag{2.8}$$

with $(a, b) \in \mathbb{C}^2$ *solution of:*

$$\begin{bmatrix} \left(1 - 5\delta\frac{z_-}{\mu}\right) & \sqrt{2}\delta\frac{z_-^2}{\mu} \\ \frac{\sqrt{2}\delta z_-^2}{\mu} & \left(1 - 5\frac{\delta z_-}{\mu}\right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \tag{2.9}$$

We denote by $(A(x), B(x))^T = (a, b)^T$ the solution with right hand side

$$(\alpha, \beta)^T = \left(\frac{1}{1 + \frac{x^2}{r^2}}, \frac{-x}{\sqrt{2}\left(1 + \frac{x^2}{r^2}\right)} \right).$$

and $g_1(x) = \frac{A(x)z_-(\mu(x))(1-2xz_-(\mu(x)))}{\mu(x)}$. We get

$$\mathcal{E}_1(t, d) = 2^d \, |G_1(t, d)|^2 = \left| \left(\frac{\sqrt{2}}{3} \right)^t \frac{1}{t!} \, \partial_x^t \left(g_1(x) z_- \right)_{x=0} \right|^2. \qquad (2.10)$$

Lemma

The function $g_1(x)$ is holomorphic in the disc $D_r = \{x, \, |x| < r = \sqrt{2}/3\}$. Its boundary value $\theta \mapsto g_1(re^{i\theta})$ is analytic except for $\pm\theta_0, \pm(\theta_0 + \pi)$ with $\cos\theta_0 = 2r$. Near $x_0 \in \{re^{i\theta_0}, \, re^{-i\theta_0}, \, -re^{i\theta_0}, \, -re^{-i\theta_0}\}$ one has $g_1(x) = a(x) + (x - x_0)^{1/2} b(x)$, with a, b holomorphic near x_0.

Set

$$J(t,d) = r^t \, \frac{1}{t!} \, \partial_x^t \left[g_1 z_-^d \right]_{|x=0} . \tag{2.11}$$

Then $\mathcal{E}_1(t,d) = |J(t,d)|^2$.

By Cauchy formula, with $\zeta = re^{i\theta}$, and $z_-(\mu(\zeta)) = e^{i\varphi(\theta)}$, we get

$$J(t,d) = \frac{r^t}{2i\pi} \int_\gamma \frac{g_1(\zeta) z_-(\mu(\zeta))^d}{\zeta^{t+1}} \, d\zeta = \frac{1}{2\pi} \int_0^{2\pi} g_1(re^{i\theta}) \, e^{i(d\varphi - t\theta)} \, d\theta.$$

For $d = \gamma t, \gamma \in [0,1]$, it just remain to apply the phase stationary theorem to the integral with t as large parameter

$$J(t,d) = \frac{1}{2\pi} \int_0^{2\pi} g_1(re^{i\theta}) \, e^{it(\gamma\varphi - \theta)} \, d\theta. \tag{2.12}$$