Boundary exact controllability to the constant trajectories of a one-dimensional phase-field system with one control force

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Journée "Contrôle et EDP". Versailles, Octobre 2017

General objective:

Controllability properties of a one-dimensional phase-field system of Caginalp type with one control force.

Phase-field system: it is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying a domain.

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Fix $T > 0$. Notation: $Q_T := (0,\pi) \times (0,T)$

$$(1) \quad \begin{cases} \tilde\theta_t - \xi\tilde\theta_{xx} + \frac{1}{2}\rho\xi\tilde\phi_{xx} + \frac{\rho}{\tau}\tilde\theta = f(\tilde\phi) & \text{in } Q_T, \\ \tilde\phi_t - \xi\tilde\phi_{xx} - \frac{2}{\tau}\tilde\theta = -\frac{2}{\rho}f(\tilde\phi) & \text{in } Q_T, \\ \tilde\theta(0,\cdot) = v,\ \tilde\phi(0,\cdot) = c,\ \tilde\theta(\pi,\cdot) = 0,\ \tilde\phi(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde\theta(\cdot,0) = \tilde\theta_0,\ \tilde\phi(\cdot,0) = \tilde\phi_0 & \text{in } (0,\pi). \end{cases}$$

$\tilde\theta = \tilde\theta(x,t)$: the temperature of the material;

$\tilde\phi = \tilde\phi(x,t)$: phase-field function used to identify the solidification level of the material; $c \in \{-1,0,1\}$;

f: nonlinear term which comes from the derivative of the classical regular double-well potential W: $f(\tilde\phi) = -\frac{\rho}{4\tau}\left(\tilde\phi - \tilde\phi^3\right)$.

$\rho > 0, \tau > 0, \xi > 0$: latent heat, relaxation time; thermal diffusivity.

$v \in L^2(0,T)$: control. $\tilde\theta_0, \tilde\phi_0$: initial data.

1. Introduction. Statement of the problem

The phase function $\tilde{\phi}$ describes the phase transition of the material (solid or liquid): $\boxed{\tilde{\phi} = 1}$ solid state of the material; $\boxed{\tilde{\phi} = -1}$ liquid state.

G. CAGINALP, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1986), no. 3, 205–245.

1. Introduction. Statement of the problem

Objective

Null controllability of system (1) at time $T > 0$: for any $\tilde{\theta}_0$ there exists a control $v \in L^2(0,T)$ such that system (1) has a solution $(\tilde{\theta}, \tilde{\phi})$ satisfying

$$\tilde{\theta}(\cdot, T) = 0 \quad \text{in} \quad \Omega.$$

Remark

- The temperature $\tilde{\theta}$ of the material could be zero with the material in solid ($\tilde{\phi} = 1$) or liquid phase ($\tilde{\phi} = -1$). Thus, null controllability of the temperature with $\tilde{\phi}(\cdot, T) = 1$ or $\tilde{\phi}(\cdot, T) = -1$ in $(0, \pi)$.
- The phase-field variable $\tilde{\phi}$ does not have a direct physical meaning. This is the reason way we control the temperature $\tilde{\theta}$ which, in fact, is the unique variable with physical meaning.

1. Introduction. Statement of the problem

Remark

Three main difficulties:

1. Only **one control force** $v \in L^2(0,T)$ and two variables to be controlled $(\tilde{\theta}, \tilde{\phi})$ ($\tilde{\phi}$ is indirectly controlled).
2. **Boundary control**: the control v is exerted at point $x = 0$ by means of the boundary Dirichlet condition for the temperature $\tilde{\theta}$.
3. **Non-linear problem**: Only local controllability results (positive controllability result when the initial data $(\tilde{\theta}_0, \tilde{\phi}_0)$ is near to the desired final state $(0, c)$).

Approximate controllability

Approximate controllability of system (1) at time $T > 0$: for any $(\tilde{\theta}_0, \tilde{\phi}_0)$, $(\tilde{\theta}_d, \tilde{\phi}_d)$ and $\varepsilon > 0$, there exists a control $v \in L^2(0,T)$ such that system (1) has a solution $(\tilde{\theta}, \tilde{\phi})$ satisfying

$$\|(\tilde{\theta}, \tilde{\phi})(\cdot, T) - (\tilde{\theta}_d, \tilde{\phi}_d)\| \leq \varepsilon.$$

Distributed controls, N-dimensional case: $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a **bounded domain**, and $\boxed{\omega \subset \Omega}$, an open subset. **Notation**: $Q_T = \Omega \times (0,T)$; $\Sigma_T = \partial\Omega \times (0,T)$.

$$\begin{cases} \tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \frac{1}{2}\rho\xi\Delta\tilde{\phi} + \frac{\rho}{\tau}\tilde{\theta} = f(\tilde{\phi}) + v1_\omega & \text{in } Q_T, \\ \tilde{\phi}_t - \xi\Delta\tilde{\phi} - \frac{2}{\tau}\tilde{\theta} = -\frac{2}{\rho}f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta} = 0, \quad \tilde{\phi} = c \text{ on } \Sigma_T, \quad \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } \Omega. \end{cases}$$

$\boxed{1_\omega}$ is the characteristic function on ω; $\boxed{\kappa}$ is the thermal diffusivity and, in general, $\kappa \neq \xi$; $\boxed{v \in L^2(Q_T)}$ is a **distributed control**

$$\begin{cases} \tilde{\theta}_t - \kappa\Delta\tilde{\theta} + \dfrac{1}{2}\rho\xi\Delta\tilde{\phi} + \dfrac{\rho}{\tau}\tilde{\theta} = f(\tilde{\phi}) + v1_{\omega} & \text{in } Q_T, \\ \tilde{\phi}_t - \xi\Delta\tilde{\phi} - \dfrac{2}{\tau}\tilde{\theta} = -\dfrac{2}{\rho}f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta} = 0, \quad \tilde{\phi} = c \text{ on } \Sigma_T, \quad \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } \Omega. \end{cases}$$

Existing results

1. AMMAR KHODJA, BENABDALLAH, DUPAIX, KOSTIN, *Controllability to the trajectories of phase-field models by one control force*, SIAM J. Control Optim. **42** (2003), no. 5, 1661–1680. $\boxed{N \leq 5}$, local controllability results. Technique: Carleman inequalities.
2. G.-B., PÉREZ-GARCÍA, *Controllability results for some nonlinear coupled parabolic systems by one control force*, Asymptot. Anal. **46** (2006), no. 2, 123–162. General $\boxed{N \geq 1}$, local controllability results. Technique: Carleman inequalities and fictitious controls.

Few results, only for **linear problems** and, in general, in the **one-dimensional** case:

$$\begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D, A \in \mathcal{L}(\mathbb{R}^n)$ ($n \geq 2$ is the number of equations), $B \in \mathcal{L}(\mathbb{R}^m;\mathbb{R}^n)$ ($m \geq 1$ is the number of controls)

$$\begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$\boxed{D = Id}$$

Existing results

1. Fernández-Cara, G-B, de Teresa, *Boundary controllability of parabolic coupled equations*, J. Funct. Anal. **259** (2010), no. 7, 1720–1758. $n=2$, $m=1$ (one control).
   Null controllability **if and only if** rank $[B\,|\,AB] = 2$ (Kalman condition) and $\boxed{\mu_1 - \mu_2 \neq j^2 - k^2}$, $\forall j \neq k$ (μ_i eigenvalues of A).
2. Ammar Khodja, Benabdallah, G.-B., de Teresa, *The Kalman condition for the boundary controllability of coupled parabolic systems. Bounds on biorthogonal families to complex matrix exponentials*, J. Math. Pures Appl. (9) **96** (2011), no. 6, 555–590. Generalization of the previous result to general $\boxed{n \geq 2}$ and $\boxed{m \geq 1}$. **Important**: $D = Id$.

$$\begin{cases} y_t - D\Delta y + Ay = 0 & \text{in } Q_T := \Omega \times (0,T), \\ y(0,\cdot) = Bv1_\gamma & \text{on } \Sigma_T = \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega. \end{cases}$$

$\boxed{D = Id}$, N-dimensional case

Existing results

3. ALABAU-BOUSSOUIRA, LÉAUTAUD, *Indirect controllability of locally coupled wave-type systems and applications*, J. Math. Pures Appl. (9) **99** (2013), no. 5, 544–576.
4. BENABDALLAH, BOYER, G-B, OLIVE, *Sharp estimates of the one-dimensional boundary control cost for parabolic systems and application to the N-dimensional boundary null controllability in cylindrical domains*, SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001.

$$\begin{cases} y_t - D\Delta y + Ay = 0 & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$\boxed{n=2}, \quad \boxed{m=1}, \quad \boxed{D = \operatorname{diag}(1,d)}, \quad d>0, \quad d\neq 1.$$

Existing results

5. Ammar Khodja, Benabdallah, G-B, de Teresa, *Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences*, J. Funct. Anal. **267** (2014), no. 7, 2077–2151.
   - Approximate controllability: if and only if $\boxed{\sqrt{d}\notin\mathbb{Q}}$ (**Fernández-Cara, G-B, de Teresa**).
   - Null controllability: $\boxed{\sqrt{d}\notin\mathbb{Q}}$ and existence of $\boxed{T_0\in[0,\infty]}$ s.t.
     1. if $T>T_0$ the system is null controllable at time T;
     2. if $T<T_0$ the system is not null controllable at time T.

1. Introduction. Statement of the problem

$$(1)\quad \begin{cases} \tilde\theta_t - \xi\tilde\theta_{xx} + \frac{1}{2}\rho\xi\tilde\phi_{xx} + \frac{\rho}{\tau}\tilde\theta = f(\tilde\phi) & \text{in } Q_T, \\ \tilde\phi_t - \xi\tilde\phi_{xx} - \frac{2}{\tau}\tilde\theta = -\frac{2}{\rho} f(\tilde\phi) & \text{in } Q_T, \\ \tilde\theta(0,\cdot) = v,\ \tilde\phi(0,\cdot) = c,\ \tilde\theta(\pi,\cdot) = 0,\ \tilde\phi(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde\theta(\cdot,0) = \tilde\theta_0,\ \tilde\phi(\cdot,0) = \tilde\phi_0 & \text{in } (0,\pi). \end{cases}$$

Performing the change of variable $(\theta,\phi) = (\tilde\theta, \tilde\phi - c)$, system (1) becomes

$$(2)\quad \begin{cases} \theta_t - \xi\theta_{xx} + \frac{1}{2}\rho\xi\phi_{xx} - \frac{\rho}{2\tau}\phi + \frac{\rho}{\tau}\theta = g(\phi) & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \frac{1}{\tau}\phi - \frac{2}{\tau}\theta = -\frac{2}{\rho} g(\phi) & \text{in } Q_T, \\ \theta(0,\cdot) = v, \quad \phi(0,\cdot) = \theta(\pi,\cdot) = \phi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,0) = \theta_0, \quad \phi(\cdot,0) = \phi_0 & \text{in } (0,\pi), \end{cases}$$

where $(\theta_0,\phi_0) = (\tilde\theta_0, \tilde\phi_0 - c)$ and $g(\phi) = \pm\frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3$.

1. Introduction. Statement of the problem

System can be written:

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y_0 = (\theta_0, \phi_0)$, $y = (\theta, \phi)$, $F(y) = \begin{pmatrix} g(y_2) \\ -\frac{2}{\rho} g(y_2) \end{pmatrix}$, and

$$(3) \qquad D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \dfrac{\rho}{\tau} & -\dfrac{\rho}{2\tau} \\ -\dfrac{2}{\tau} & \dfrac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Objective

Local null controllability of system (2) at time $T > 0$: There exists $\varepsilon > 0$ s.t. for any y_0 with $\|y_0\| \le \varepsilon$, there exists a control $v \in L^2(0,T)$ such that system (2) has a solution $y = (\theta, \phi)$ satisfying

$$\boxed{y(\cdot,T) = 0} \quad \text{in} \quad (0,\pi).$$

1. Introduction. Statement of the problem

$$(2) \quad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Strategy

1. Null controllability at time $T > 0$ of a homogenous linearized version of system (2) with an **explicit expression of the control cost** with respect to T.
2. Null controllability at time T of a non-homogenous linearized version of system (2).
3. A Fixed-Point argument will imply the local null controllability result at time T for system (2).

2. Controllability of the homogenous linear system

2. Controllability of the homogenous linear system

Let us consider the system

$$(4) \quad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T := (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$,

$$(3) \quad D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proposition

Assume $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$ *and* $v \in L^2(0,T)$. *Then, system* (4) *admits a unique solution* $y = (\theta,\phi) \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1})$ *which depends continuously on the data.*

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Theorem (Approximate controllability)

Fix $T > 0$. Then, system (4) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if

$$(5)\qquad \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k,\ell \geq 1, \quad \ell > k.$$

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Remark

Null controllability implies approximate controllability. Thus, (5) is a necessary condition for the null controllability of this system at time $T > 0$. Condition

$$(5)\qquad \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k.$$

is also equivalent to the property: *"The eigenvalues of the vectorial operators*

$$(6)\qquad L = -D\partial_{xx} + A \quad \textit{and} \quad L^* = -D^*\partial_{xx} + A^*,$$

have geometric multiplicity equal to one". Thus, condition (5) is a Fattorini-Hautus criterium for the boundary approximate controllability of (4).

$$
(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Null controllability with a bound of the control cost?? Yes, but we need a gap condition for the eigenvalues of the vectorial operator $\boxed{L = -D\partial_{xx} + A}$

Strategy

Apply the **moment method** to system (4). We will use:

- the eigenvalues of L and L^*: $\boxed{\lambda_k^{(1)},\ \lambda_k^{(2)}}$;
- the eigenfunctions of L: $\boxed{\Psi_k^{(1)},\ \Psi_k^{(2)}}$;
- the eigenfunctions of L^*: $\boxed{\Phi_k^{(1)},\ \Phi_k^{(2)}}$.

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Lagrange formula (variations of constants): This formula provides an explicit formula for y:

$$y(\cdot,T) = \boxed{\cdots}\, y_0 + \int_0^T \boxed{\cdots}\, v(T-t)\,dt.$$

Thus, $v \in L^2(0,T)$ is s.t. $y(\cdot,T) = 0$ if and only if

$$\left\langle y(\cdot,T), \Phi_k^{(1)} \right\rangle = 0, \quad \left\langle y(\cdot,T), \Phi_k^{(2)} \right\rangle = 0, \quad \forall k \geq 1.$$

$$
(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

$$
\left\langle y(\cdot,T), \Phi_k^{(1)} \right\rangle = 0, \quad \left\langle y(\cdot,T), \Phi_k^{(2)} \right\rangle = 0, \quad \forall k \geq 1, \iff
$$

$$
\begin{cases} b_k^{(1)} \displaystyle\int_0^T e^{-\lambda_k^{(1)} t} v(T-t)\, dt = -e^{-\lambda_k^{(1)} T} a_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle, & \forall k \geq 1, \\ b_k^{(2)} \displaystyle\int_0^T e^{-\lambda_k^{(2)} t} v(T-t)\, dt = -e^{-\lambda_k^{(2)} T} a_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle, & \forall k \geq 1. \end{cases}
$$

Necessary conditions (approximate controllability)

In order to have a compatible system: $\boxed{b_k^{(1)} \neq 0, b_k^{(2)} \neq 0}$, for any $k \geq 1$.
Also (formula (5)):

$$
\boxed{\lambda_k^{(1)} \neq \lambda_j^{(2)}}, \quad \forall k, j \geq 1.
$$

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

The moment problem

Null controllability of system (4) is equivalent to:
Find $v \in L^2(0,T)$ such that

$$(7) \qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^{(1)} t} v(T-t)\, dt = e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle, & \forall k \geq 1, \\ \displaystyle\int_0^T e^{-\lambda_k^{(2)} t} v(T-t)\, dt = e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle, & \forall k \geq 1, \end{cases}$$

where $c_k^{(j)}$ is a bounded sequence.

The moment problem

Find $v \in L^2(0,T)$ such that

$$(7)\qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^{(1)} t} v(T-t)\,dt = e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle, & \forall k \geq 1, \\ \displaystyle\int_0^T e^{-\lambda_k^{(2)} t} v(T-t)\,dt = e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle, & \forall k \geq 1, \end{cases}$$

where $c_k^{(j)}$ is a bounded sequence.

Two questions

1. Is the system compatible? The set $\left\{ e^{-\lambda_k^{(1)} t}, e^{-\lambda_k^{(2)} t} \right\}_{k \geq 1} \subset L^2(0,T)$ must be **minimal** (linearly independent).

2. If yes, could we find a solution v in $L^2(0,T)$?

Biorthogonal Families: ([FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971)).

Two ingredients:

1. $\boxed{\sum_{k\geq 1}\left(\frac{1}{\lambda_k^{(1)}}+\frac{1}{\lambda_k^{(2)}}\right)<\infty} \Longrightarrow$ the family $\left\{e^{-\lambda_k^{(1)}t}, e^{-\lambda_k^{(2)}t}\right\}_{k\geq 1}$ is minimal in $L^2(0,T) \iff \exists$ a family (not unique) $\left\{q_k^{(1)}, q_k^{(2)}\right\}_{k\geq 1} \subset L^2(0,T)$ s.t. $\int_0^T e^{-\lambda_k^{(i)}t} q_\ell^{(j)}\,dt = \delta_{k\ell}\delta_{ij}, \forall k,\ell\geq 1,$ $i,j=1,2$ (**biorthogonality**).

Important: $\boxed{\lambda_k^{(i)} \sim \xi k^2 + \frac{\rho+1}{2\tau} + O_i(k) > 0}$. OK.

Biorthogonal Families: ([FATTORINI, RUSSELL] Arch. Rat. Mech. Anal. (1971)).

Two ingredients:

1. $\boxed{\sum_{k\geq 1}\left(\frac{1}{\lambda_k^{(1)}}+\frac{1}{\lambda_k^{(2)}}\right)<\infty} \Longrightarrow$ the family $\left\{e^{-\lambda_k^{(1)}t}, e^{-\lambda_k^{(2)}t}\right\}_{k\geq 1}$ is minimal in $L^2(0,T) \iff \exists$ a family (not unique) $\left\{q_k^{(1)}, q_k^{(2)}\right\}_{k\geq 1} \subset L^2(0,T)$ s.t. $\int_0^T e^{-\lambda_k^{(i)}t} q_\ell^{(j)}\,dt = \delta_{k\ell}\delta_{ij}, \forall k,\ell \geq 1$, $i,j = 1,2$ (**biorthogonality**).

   Important: $\boxed{\lambda_k^{(i)} \sim \xi k^2 + \frac{\rho+1}{2\tau} + O_i(k) > 0}$. OK.

2. **Separability:** We can construct a biorthogonal family with a good bound for $\|q_k^{(i)}\|_{L^2(0,T)}$ for any $k \geq 1$, $i = 1,2$.

The moment problem

Given y_0 and $\{c_k^{(j)}\}$, a bounded sequence, find $v \in L^2(0,T)$ such that

$$(7)\quad \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^{(1)} t} v(T-t)\,dt = e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle \equiv C_k^{(1)}, & \forall k \geq 1, \\ \displaystyle\int_0^T e^{-\lambda_k^{(2)} t} v(T-t)\,dt = e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle \equiv C_k^{(2)}, & \forall k \geq 1. \end{cases}$$

Formal solution:

$$v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right)$$
$$= \sum_{k\geq 1} \left(e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle q_k^{(1)}(t) + e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle q_k^{(2)}(t) \right)$$

Is this series convergent in $L^2(0,T)$? It depends on $\boxed{\|q_k^{(i)}\|_{L^2(0,T)}}$

Bounds on Biorthogonal Families: ([Benabdallah, Boyer, G-B, Olive], SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001).

Lemma (Separability and bounds)

Consider $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ s. t. $\Lambda_k \neq \Lambda_n$, for any $k \neq n$. Also assume that there exist an integer $M \geq 1$ and positive constants p, δ and α such that

$$(8) \qquad \begin{cases} |\Lambda_k - \Lambda_n| \geq \delta \left| k^2 - n^2 \right|, & \forall k, n \in \mathbb{N},\ |k-n| \geq M, \\ \inf\limits_{k\neq n,\ |k-n|<M} |\Lambda_k - \Lambda_n| > 0, & \\ |p\sqrt{r} - \mathcal{N}(r)| \leq \alpha, & \forall r > 0, \end{cases}$$

($\mathcal{N}(r) := \#\{k : \Lambda_k \leq r\}$ is the counting function associated to $\{\Lambda_k\}_{k\geq 1}$). Then, $\exists \widetilde{T}_0 > 0$ and $C > 0$ s. t., for any $T \in (0, \widetilde{T}_0)$, we can find $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ biorthogonal to $\{e^{-\Lambda_k t}\}_{k\geq 1}$

$$\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\Lambda_k} + \frac{C}{T}}, \quad \forall k \geq 1.$$

Without loss of generality, assume $\boxed{T \in (0, \widetilde{T}_0)}$. We had (approximate controllability):

$$(5) \quad \xi^2 \tau^2 (\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k.$$

Also assume:

$$(9) \qquad \boxed{\xi \neq \frac{1}{j^2}\frac{\rho}{\tau}, \quad \forall j \geq 1}.$$

Conclusion, we can apply the previous lemma to $\left\{\lambda_k^{(1)}, \ \lambda_k^{(2)}\right\}_{k\geq 1}$: $\exists$ a **biorthogonal family** $\left\{q_k^{(1)}, q_k^{(2)}\right\}_{k\geq 1}$ in $L^2(0,T)$ to the exponentials s.t.

$$\|q_k^{(i)}\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k^{(i)}} + \frac{C}{T}}, \quad \forall k \geq 1, i = 1, 2.$$

$C > 0$ is independent of k, i and T.

Recall

The moment problem

Given y_0 and $\{c_k^{(j)}\}$, a bounded sequence, find $v \in L^2(0,T)$ such that

$$(7) \quad \begin{cases} \displaystyle\int_0^T e^{-\lambda_k^{(1)} t} v(T-t)\,dt = e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle \equiv C_k^{(1)}, & \forall k \geq 1, \\ \displaystyle\int_0^T e^{-\lambda_k^{(2)} t} v(T-t)\,dt = e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle \equiv C_k^{(2)}, & \forall k \geq 1. \end{cases}$$

$$v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right)$$
$$= \sum_{k\geq 1} \left(e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle q_k^{(1)}(t) + e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle q_k^{(2)}(t) \right)$$

$$\boxed{\left| C_k^{(i)} \right| \leq C\, k\, e^{-\lambda_k^{(i)} T} \|y_0\|_{H^{-1}}}, \quad \forall k \geq 1, i = 1, 2 \; (C > 0 \text{ independent of } k, i, T).$$

The series of v converges absolutely in $L^2(0,T)$:

$$\begin{cases} v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right), \\ \left| C_k^{(i)} \right| \leq C\, k\, e^{-\lambda_k^{(i)} T} \|y_0\|_{H^{-1}}, \quad \forall k \geq 1, \quad i = 1, 2, \\ \|q_k^{(i)}\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k^{(i)}} + \frac{C}{T}}, \quad \forall k \geq 1, \quad i = 1, 2. \end{cases}$$

and

$$\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}} \|y_0\|_{H^{-1}}$$

for positive constants C_0 and M independent of T.

Conclusion

We have solved the moment problem and we had proved the null controllability result for system (4).

$$
\text{(4)} \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Theorem

Let us us fix $T > 0$ and consider ξ, ρ and τ, positive real numbers satisfying

$$
\begin{cases} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \dfrac{1}{j^2}\dfrac{\rho}{\tau}, \quad \forall j \geq 1. \end{cases}
$$

*Then, system (4) is exactly controllable to zero in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$. Moreover, there exist two constants $C_0, M > 0$, only depending on ξ, ρ and τ, s. t. the control v can be constructed satisfying (**control cost**)*

$$
\boxed{\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}}\, \|y_0\|_{H^{-1}}}
$$

(4) $$\begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Remark

1. Condition

(5) $$\xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k.$$

is necessary for the approximate controllability of the system. Therefore, it is also necessary for the null controllability of it.

$$(4) \quad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Remark

2. In the case in which assumption (9) does not hold, i.e., if $\boxed{\xi = \frac{1}{j^2}\frac{\rho}{\tau}}$, for some $j \geq 1$, then, the eigenvalues of L concentrate: $\boxed{\inf_{k,\ell \geq 1, k \neq \ell} |\Lambda_k - \Lambda_\ell| = 0}$. In this case the controllability problem for system (4) has a minimal time $T_0 \in [0,\infty]$ of null controllability which is related to the condensation index of the sequence. Even in the case $T_0 = 0$ (and therefore, system (4) is null controllable for any $T > 0$), without the separability assumption, providing an estimate of the norm of v with respect to $T > 0$ and y_0 is an open problem.

3. Null controllability of the non-homogenous linear system

3. Null controllability of the non-homogenous linear system

Let us now consider the system

$$(10) \qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$, f is a given function (heat source) and

$$(3) \qquad D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proposition

Assume $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ and $f \in L^2(Q_T)$. Then, system (10) *admits a unique solution $y = (\theta, \phi) \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1})$ which depends continuously on the data.*

3. Null controllability of the non-homogenous linear system

$$(10)\qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Objective

Null controllability of system (10) at time $T > 0$.

Important: The source term f must have an exponential decay when $t \to T$:

$$e^{\frac{C}{T-t}} f \in L^2(Q_T),$$

for an appropriate positive constant C.

We follow
[Liu, Takahashi, Tucsnak], ESAIM Control Optim. Calc. Var. **19** (2013), no. 1, 20–42.

3. Null controllability of the non-homogenous linear system

Recall

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Theorem

Let us us fix $T > 0$ and consider ξ, ρ and τ, positive real numbers satisfying

$$\begin{cases} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \dfrac{1}{j^2}\dfrac{\rho}{\tau}, \quad \forall j \geq 1. \end{cases}$$

*Then, system (4) is exactly controllable to zero in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$. Moreover, there exist two constants $C_0, M > 0$, only depending on ξ, ρ and τ, s. t. the control v can be constructed satisfying (**control cost**)*

$$\boxed{\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}}\, \|y_0\|_{H^{-1}}}$$

$$(10)\qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

From the previous expression of the **control cost** for the homogenous problem, we define the functions

$$\boxed{\rho_{\mathcal{F}}(t) := e^{\frac{b^2(a+1)M}{(b-1)(T-t)}}}, \quad \boxed{\rho_0(t) := e^{\frac{aM}{(b-1)(T-t)}}}, \quad \forall t \in \left[T\left(1-\frac{1}{b^2}\right), T\right],$$

extended to $\left[0, T(1-1/b^2)\right]$ in a constant way ($a, b > 1$ are constants that will be chosen later).

With the previous functions, we also introduce the weighted Banach spaces

$$\mathcal{F} := \left\{ f \in L^2(Q_T;\mathbb{R}^2) : \rho_{\mathcal{F}} f \in L^2(Q_T;\mathbb{R}^2) \right\},$$
$$\mathcal{V} := \left\{ v \in L^2(0,T) : \rho_0 v \in L^2(0,T) \right\},$$
$$\mathcal{Y}_0 := \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T;\mathbb{R}^2) \right\},$$
$$\mathcal{Y} := \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T) \times C^0(\overline{Q}_T) \right\}.$$

Theorem

Consider ξ, ρ and τ, three positive real numbers satisfying (5) *and* (9), *and* $T > 0$. *Then, for any* $\boxed{(y_0, f) \in H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F}}$ *(resp.,* $\boxed{(y_0, f) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \times \mathcal{F}}$*), there exists* $v \in \mathcal{V}$ *(which depends linearly on the data) such that*

$$\|v\|_{\mathcal{V}} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \, \|(y_0, f)\|_{(H^{-1})^2 \times \mathcal{F}}$$

and the solution y *of* (10) *associated to* (y_0, f) *satisfies* $y \in \mathcal{Y}_0$ *(resp.,* $y \in \mathcal{Y}$*) and*

$$\|y\|_{\mathcal{Y}_0} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \, \|(y_0, f)\|_{(H^{-1})^2 \times \mathcal{F}} ,$$

(resp.,

$$\|y\|_{\mathcal{Y}} \leq C\, e^{C\left(T+\frac{1}{T}\right)} \, \|(y_0, f)\|_{H^{-1} \times H_0^1 \times \mathcal{F}}),$$

for a positive constant C *independent of* T.

$$(10) \qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Proof: The proof is very technical. One uses in a fundamental way the expression of the **control cost** for the homogeneous problem:

$$\boxed{\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}} \|y_0\|_{H^{-1}}}.$$

3. Null controllability of the non-homogenous linear system

$$
(10) \quad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Remark

The previous theorem, in particular, provides a null controllability result at time T for system (10) when $\boxed{(y_0, f) \in H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F}}$ or when $\boxed{(y_0, f) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \times \mathcal{F}}$. Indeed, observe that the solution y associated to $v \in \mathcal{V}$ satisfies $y \in \mathcal{Y}_0$ (resp., $y \in \mathcal{Y}$), with $\boxed{\rho_0(t) := e^{\frac{aM}{(b-1)(T-t)}}}$ and

$$
\mathcal{V} := \left\{ v \in L^2(0,T) : \rho_0 v \in L^2(0,T) \right\},
$$
$$
\mathcal{Y}_0 := \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T;\mathbb{R}^2) \right\},
$$
$$
\mathcal{Y} := \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T) \times C^0(\overline{Q}_T) \right\}.
$$

So, $\boxed{y(\cdot,T) = 0}$ in $(0,\pi)$.

4. Local null controllability for the phase-field system

4. Local null controllability for the phase-field system

Recall we had: $T > 0$ fixed. Phase-field system:

$$(1) \quad \begin{cases} \tilde{\theta}_t - \xi\tilde{\theta}_{xx} + \frac{1}{2}\rho\xi\tilde{\phi}_{xx} + \frac{\rho}{\tau}\tilde{\theta} = f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\phi}_t - \xi\tilde{\phi}_{xx} - \frac{2}{\tau}\tilde{\theta} = -\frac{2}{\rho}f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta}(0,\cdot) = v,\ \tilde{\phi}(0,\cdot) = c,\ \tilde{\theta}(\pi,\cdot) = 0,\ \tilde{\phi}(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } (0,\pi). \end{cases}$$

$\boxed{\tilde{\theta} = \tilde{\theta}(x,t)}$: the temperature of the material;

$\boxed{\tilde{\phi} = \tilde{\phi}(x,t)}$: phase-field function used to identify the solidification level of the material; $\boxed{c \in \{-1,0,1\}}$;

$\boxed{f}$: nonlinear term which comes from the derivative of the classical regular double-well potential W: $f(\tilde{\phi}) = -\frac{\rho}{4\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right)$.

$\boxed{\rho > 0, \tau > 0, \xi > 0}$: latent heat, relaxation time; thermal diffusivity.

$\boxed{v \in L^2(0,T)}$: control. $\boxed{\tilde{\theta}_0, \tilde{\phi}_0}$: initial data.

4. Local null controllability for the phase-field system

Objective

Null controllability of system (1) at time $T > 0$ (maintaining the material solid or liquid at this time): for any $(\tilde{\theta}_0, \tilde{\phi}_0)$ there exists a control $v \in L^2(0,T)$ such that system (1) has a solution $(\tilde{\theta}, \tilde{\phi})$ satisfying

$$\tilde{\theta}(\cdot, T) = 0, \quad \tilde{\phi}(\cdot, T) = c \quad \text{in} \quad \Omega.$$

We also had the change $\boxed{(\theta, \phi) = (\tilde{\theta}, \tilde{\phi} - c)}$ and system (1) became

$$(2) \quad \begin{cases} \theta_t - \xi\theta_{xx} + \dfrac{1}{2}\rho\xi\phi_{xx} - \dfrac{\rho}{2\tau}\phi + \dfrac{\rho}{\tau}\theta = g(\phi) & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \dfrac{1}{\tau}\phi - \dfrac{2}{\tau}\theta = -\dfrac{2}{\rho}g(\phi) & \text{in } Q_T, \\ \theta(0,\cdot) = v, \quad \phi(0,\cdot) = \theta(\pi,\cdot) = \phi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,0) = \theta_0, \quad \phi(\cdot,0) = \phi_0 & \text{in } (0,\pi), \end{cases}$$

where $(\theta_0, \phi_0) = (\tilde{\theta}_0, \tilde{\phi}_0 - c)$ and $\boxed{g(\phi) = \pm\dfrac{3\rho}{4\tau}\phi^2 + \dfrac{\rho}{4\tau}\phi^3}$.

4. Local null controllability for the phase-field system

In vectorial form, the system can be written:

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y_0 = (\theta_0, \phi_0)$, $y = (\theta, \phi)$, $F(y) = \begin{pmatrix} g(y_2) \\ -\frac{2}{\rho} g(y_2) \end{pmatrix}$, and

$$(3)\qquad D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \dfrac{\rho}{\tau} & -\dfrac{\rho}{2\tau} \\ -\dfrac{2}{\tau} & \dfrac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Objective

Local null controllability of system (2) at time $T > 0$: There exists $\varepsilon > 0$ s.t. for any y_0 with $\|y_0\| \le \varepsilon$, there exists a control $v \in L^2(0,T)$ such that system (2) has a solution $y = (\theta, \phi)$ satisfying

$$\boxed{y(\cdot, T) = 0} \quad \text{in} \quad (0,\pi).$$

4. Local null controllability for the phase-field system

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

The idea is to apply a **fixed-point argument** to system (2). To this end, we take

$$y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \text{ with } \boxed{\|y_0\|_{H^{-1}\times H_0^1} \le \varepsilon},$$

with $\varepsilon > 0$ to be determined.

Now, take $f \in \mathcal{F} = \left\{ f \in L^2(Q_T;\mathbb{R}^2) : \rho_{\mathcal{F}} f \in L^2(Q_T;\mathbb{R}^2) \right\}$ with

$$\boxed{f \in \overline{B}_{\mathcal{F},\varepsilon} = \{ f \in \mathcal{F} : \|f\|_{\mathcal{F}} \le \varepsilon \}}.$$

and we use the null controllability result for the non-homogenous linear system (Theorem 3.2) for (y_0, f): $\boxed{\exists v_f \in \mathcal{V}, y_f = (\theta_f, \phi_f) \in \mathcal{Y}}$ solutions of the non-homogenous linear system and

$$\|y_f\|_{\mathcal{Y}} + \|v_f\|_{\mathcal{V}} \le C\, e^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}\times H_0^1} + \|f\|_{\mathcal{F}} \right) \le C\, e^{C\left(T+\frac{1}{T}\right)}\, \varepsilon,$$

4. Local null controllability for the phase-field system

$$
(2) \qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Fixed-point operator

$$
\mathcal{N}(f) = F(y_f) = \begin{pmatrix} \pm\dfrac{3\rho}{4\tau}{\phi_f}^2 + \dfrac{\rho}{4\tau}{\phi_f}^3 \\ \mp\dfrac{3}{2\tau}{\phi_f}^2 - \dfrac{1}{2\tau}{\phi_f}^3 \end{pmatrix}.
$$

It is clear that if f is a fixed-point of $\mathcal{N}$, i.e., if $f = \mathcal{N}(f)$, then $y_f \in \mathcal{Y}$ is a solution of system (2) and $y(\cdot,T) = 0$ in $(0,\pi)$.

It is possible to choose parameters $a, b \in (1,\infty)$ and $\varepsilon > 0$ (depending on T) such that:

1. $\mathcal{N}(\overline{B}_{\mathcal{F},\varepsilon}) \subseteq \overline{B}_{\mathcal{F},\varepsilon}$.
2. $\mathcal{N}$ is a contraction mapping

Conclusion

We can apply the Banach Fixed-Point Theorem. This proves that the operator $\mathcal{N}$ has a fixed-point.

We have proved:

Theorem

Fix $T > 0$ and assume

$$\left\{ \begin{array}{c} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \dfrac{1}{j^2}\dfrac{\rho}{\tau}, \quad \forall j \geq 1, \end{array} \right.$$

Then, there exists $\varepsilon > 0$ such that, for any $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1} \times (c + H_0^1)$, with $\boxed{\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0 - c\|_{H_0^1} \leq \varepsilon}$, there exists $v \in L^2(0,T)$ for which system (1) has a unique solution which satisfies $y(\cdot, T) = 0$ in $(0,T)$.

Reference

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Thank you for your attention