Rigidity of the conservation laws for the Nonlinear Schrödinger equation

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The linear periodic Shrödinger equation

• In dimension 1, the linear periodic Schrödinger equation reads

\[ \partial_t u = i \partial_x^2 u, \quad u(0, x) = u_0(x), \] (1)

where \( i = \sqrt{-1}, \ t \in \mathbb{R}, \ x \in \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \) and \( u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}. \)

• For \( u_0 \in \mathcal{C}^\infty(\mathbb{T}) \) the solution of (1) is given by the exponential sum

\[ u(t, x) = \sum_{n \in \mathbb{Z}} e^{-itn^2} e^{inx} \hat{u}_0(n), \]

where \( \hat{u}_0(n) \) is the \( n \)'th Fourier coefficient of \( u_0(x) \), i.e.

\[ \hat{u}_0(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} u_0(x) \, dx. \]

• Observe that \( u(t, x) \) is \( 2\pi \)-periodic in time : \( u(t + 2\pi, x) = u(t, x). \)
Conservation of the Sobolev norms

• If a function $u : \mathbb{T} \to \mathbb{C}$ has a Fourier expansion
  \[ u(x) = \sum_{n \in \mathbb{Z}} e^{inx} \hat{u}(n) \]
  then for $s \in \mathbb{R}$, the Sobolev norm $H^s$ of $u$ is defined by
  \[ \|u\|_{H^s}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\hat{u}(n)|^2. \]
  For $s = 0$, we recover an equivalent to the $L^2$ norm and moreover
  \[ \|u\|_{H^1} \approx \|u\|_{L^2} + \|u'\|_{L^2}, \quad \|u\|_{H^2} \approx \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2}, \quad \text{etc.} \]

• It is now clear that the above solution of the linear periodic Schrödinger equation satisfies
  \[ \|u(t, \cdot)\|_{H^s} = \|u_0\|_{H^s}, \quad \forall t \in \mathbb{R}. \quad (2) \]

• We can therefore uniquely extend the solution map $u_0(x) \mapsto u(t, x)$ to a continuous map from $H^s(\mathbb{T})$ to $C(\mathbb{R}; H^s(\mathbb{T}))$, $s \in \mathbb{R}$. Moreover the $H^s(\mathbb{T})$ norm is preserved, i.e. we have (2).
• Let \( u(t, x) \) be the above defined solution of the linear Schrödinger equation with \( u_0 \in C^\infty(\mathbb{R}) \). As we do for Gauss sums, we compute

\[
 u^2(t, x) = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} e^{-it(n_1^2 + n_2^2)} e^{i(n_1 + n_2)x} \hat{u}_0(n_1) \hat{u}_0(n_2).
\]

• We now reorganise the double sum as follows:

\[
 u^2(t, x) = \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} e^{-im_1 t} e^{im_2 x} \left( \sum_{(n_1,n_2) \in A(m_1,m_2)} \hat{u}_0(n_1) \hat{u}_0(n_2) \right),
\]

where

\[
 A(m_1,m_2) \equiv \{(n_1,n_2) \in \mathbb{Z}^2 : n_1^2 + n_2^2 = m_1, n_1 + n_2 = m_2\}.
\]

For fixed \((m_1,m_2) \in \mathbb{Z}^2\), the set \( A(m_1,m_2) \) does not contain more than 2 elements. Therefore

\[
 \left| \sum_{(n_1,n_2) \in A(m_1,m_2)} \hat{u}_0(n_1) \hat{u}_0(n_2) \right|^2 \leq 2 \sum_{(n_1,n_2) \in A(m_1,m_2)} |\hat{u}_0(n_1) \hat{u}_0(n_2)|^2
\]
The Zygmund argument (sequel)

Therefore, we arrive at the bound
\[ \|u^2(t, x)\|_{L_t^2}^2 \lesssim \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{(n_1, n_2) \in A(m_1, m_2)} |\widehat{u}_0(n_1) \widehat{u}_0(n_2)|^2. \]

We now reorganise the sum again and we get \( \|u_0\|^2_{L_2^2} \times \|u_0\|^2_{L_2^2}. \) Thus
\[ \|u(t, x)\|_{L_t^4}^4 = \|u^2(t, x)\|_{L_t^2}^2 \lesssim \|u_0\|_{L_2^4}^4. \]

By a density argument, we get the following remarkable property: If \( u(t, x) \) solves the linear periodic Schrödinger equation with initial data
\[ u_0 \in L^2(\mathbb{T}) \]
then
\[ \|u(t, \cdot)\|_{L^4} < \infty, \text{ a.s in } t \in \mathbb{R}. \]
This property can only hold only a.s. because of the strict inclusion
\[ L^4(\mathbb{T}) \subset L^2(\mathbb{T}). \]
The linear Schrödinger equation on the line

• Consider now the 1d linear Schrödinger equation on the real line

\[ \partial_t u = i \partial_x^2 u, \quad u(0, x) = u_0(x) \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \]  

If \( u_0 \) is in the Schwartz class then the solution is given by

\[ u(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi^2} e^{ix\xi} \hat{u}_0(\xi) d\xi, \]

where \( \hat{u}_0(\xi) \), \( \xi \in \mathbb{R} \) is the Fourier transform of \( u_0 \), defined by

\[ \hat{u}_0(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u_0(x) dx. \]

• The Sobolev norm \( H^s \) of functions on \( \mathbb{R} \) is now defined by

\[ \|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi. \]

Since

\[ \hat{u}(t, \cdot)(\xi) = e^{-it\xi^2} \hat{u}_0(\xi) \implies |\hat{u}(t, \cdot)(\xi)| = |\hat{u}_0(\xi)| \]

the solution of (3) satisfies

\[ \|u(t, \cdot)\|_{H^s} = \|u_0\|_{H^s}. \]
The dispersion

- By applying a stationary phase estimate to

\[ u(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi^2} e^{ix\xi} \hat{u_0}(\xi) d\xi, \]

we obtain that there is \( c \in \mathbb{C} \) and \( C > 0 \) such that for such that for every \( t \geq 1 \), every \( x \in \mathbb{R} \),

\[ \left| u(t, x) - c e^{i\frac{x^2}{4t}} \hat{u_0}(x/2t) \right| \leq C t^{-\frac{3}{4}} \| xu_0 \|_{L^2}. \]

In particular, for \( t \geq 1 \),

\[ |u(t, x)| \leq C(u_0) t^{-\frac{1}{2}}. \]

- Therefore the solution disperses keeping the \( H^s \) norms conserved.
- Another manifestation of the dispersion is the Strichartz estimate

\[ \| u(t, x) \|_{L^6(\mathbb{R} \times \mathbb{R})} \leq C \| u_0(x) \|_{L^2(\mathbb{R})}. \]
A fully non linear model

• Consider the equation

\[
\partial_t u = -i|u|^2 u, \quad u(0, x) = u_0(x).
\]

• For \( u_0 \in L^2 \), the solution is given by:

\[
u(t, x) = e^{-it|u_0(x)|^2} u_0(x).
\]

• Then

\[
\partial_x u(t, x) = e^{-it|u_0(x)|^2} \left( \partial_x u_0(x) - itu_0(x) \partial_x (|u_0(x)|^2) \right).
\]

• Therefore for \( u_0(x) \) such that \( |u_0(x)| \) is not a constant, there exists \( C > 0 \) and \( A \geq 1 \) such that for \( t \geq A \),

\[
\|u(t, \cdot)\|_{H^1} \geq C t,
\]

i.e. the \( H^1 \) norm grows in time! Similarly for \( H^s, s \geq 0 \) initial data

\[
\|u(t, \cdot)\|_{H^s} \geq C t^s.
\]

• A remarkable work by P. Gérard- S. Grellier shows that the growth may become even faster (exponential ?) if one "truncates" \( |u|^2 u \) ...
The 1d Nonlinear Schrödinger equation (NLS)

• We considered so far the linear model

\[ \partial_t u = i \partial_x^2 u \]

and the fully nonlinear model

\[ \partial_t u = -i |u|^2 u \]

• The 1d NLS is obtained when one takes into account both effects:

\[ \partial_t u = i \partial_x^2 u - i |u|^2 u \]

or equivalently

\[ i \partial_t u + \partial_x^2 u = |u|^2 u . \]

• For the linear model the Sobolev norms $H^s$ of the solutions remain bounded while for the fully nonlinear model they grow as far as $s > 0$.

• The question we discuss today is which effect dominates in the context of NLS.
Global well-posedness and basic conservation laws for NLS

- Thanks to the $1d$ Sobolev embedding $H^s \subset L^\infty$, $s > 1/2$, we can easily solve locally in $H^s$, $s > 1/2$ the problem

$$i\partial_t u + \partial_x^2 u = |u|^2 u. \quad (4)$$

- Multiply (4) with $\bar{u}$, $i\partial_t \bar{u}$, integrate over $x$ and take the imaginary part. It comes:

$$\frac{d}{dt} \|u(t, \cdot)\|_{L^2}^2 = 0, \quad \frac{d}{dt} \left( \|\partial_x u(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|u(t, \cdot)\|_{L^4}^4 \right) = 0.$$

- One can deduce the second conservation law as the Hamiltonian conservation resulting from the Hamiltonian formulation of NLS.

- Therefore, for $s \geq 1$ we can extend globally in time the local solutions. Moreover, the $L^2$ and the $H^1$ norms of the solutions remain bounded in time. Therefore, concerning the $H^1$ norm, the linear effect dominates.

**Question:** What about the $H^s$, $s > 1$ norms?

**Remark:** The question of growing Sobolev norms may be seen as a competition between the kinetic and the potential energies.
Higher order conservation laws for 1d NLS

- Using the Lax representation of the 1d NLS, Zakharov-Shabat (1972) obtained that if $u$ is an $H^s$, $s \geq 2$ solution of

$$i\partial_t u + \partial_x^2 u = |u|^2 u$$

then

$$\frac{d}{dt} \left( \|\partial_x^2 u\|_{L^2}^2 + 2\|\text{Re}(\partial_x u \bar{u})\|_{L^2}^2 + 3\|u \partial_x u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^6}^6 \right) = 0.$$  

Here $x$ can be both in $\mathbb{T}$ or $\mathbb{R}$.
- Therefore the $H^2$ norms of the solutions remain bounded in time.
- Similarly one gets uniform in time bounds for the $H^s$ norms, $s = 3, 4, 5, \ldots$.
- Recent work (2016) by Koch-Tataru extends these bounds for all $s \geq 0$ in the case $x \in \mathbb{R}$ (for $x \in \mathbb{T}$, there is an earlier work by Grebert-Kappeler).
Conclusion of the 1d analysis

• In summary, for the 1d NLS both on $\mathbb{R}$ and $\mathbb{T}$, the linear effect dominates concerning the bounds on the Sobolev norms of the solutions.

• What happens in higher dimensions, i.e. for the equation

$$i \partial_t u + \Delta u = |u|^2 u,$$

where $\Delta$ is the Laplace operator?

• **Remark.** In higher dimensions, even the global well-posedness is a quite nontrivial problem.
The 3d NLS

• Let \((M, g)\) be a smooth 3d riemannian manifold with a Laplace-Beltrami operator \(\Delta\). Consider the Cauchy problem

\[i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times M \to \mathbb{C}.\]  \hspace{1cm} (5)

• As in 1d, in the context of (5), we again have the conserved quantities

\[\|U\|_{L^2(M)}, \quad \|U\|_{H^1(M)}^2 + \frac{1}{2}\|U\|_{L^4(M)}^4.\]

**Theorem 1 (Burq-Gérard-Tz. 2001)**

*Suppose that \(M\) is compact without boundary. For \(s \geq 1\) and \(U_0 \in H^s(M)\) there is a unique global solution of (5) in \(C(\mathbb{R}; H^s(M))\). The dependence with respect to the initial data in continuous. The \(L^2\) and the \(H^1\) norms of the solutions are uniformly bounded in time.*

• The result remains true for non compact manifolds with a controlled behaviour at infinity such as \(\mathbb{R}^3, \mathbb{R} \times \mathbb{T}^2, \mathbb{R}^2 \times \mathbb{T}, \mathbb{R} \times S^2\) or a long range perturbation of \(\mathbb{R}^3\) outside a compact set.

**Question** : Do the \(H^s\) norms, \(s \neq 0, 1\) remain bounded ?
Consider the Cauchy problem
\[ i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}. \] (6)

**Theorem 2 (Ginibre-Velo, Bourgain, Dodson)**

For \( s > 5/7 \) the problem (6) is globally well-posed in \( H^s(\mathbb{R}^3) \). Moreover, for every \( U_0 \in H^s \) there is \( C > 0 \) such that for every \( t \in \mathbb{R} \) the solution of (6) satisfies
\[ \| U(t, \cdot) \|_{H^s(\mathbb{R}^3)} \leq C. \] (7)

For \( s \geq 1 \), one may proceed in two steps:

1) Using Morawetz identities (a way of exploiting the good sign of the nonlinearity in a dispersive estimate) one first shows that the \( L^p(\mathbb{R}^3) \), \( p \in (2, 6) \) norms of the solution go to zero as \( t \) tends to infinity.

2) Then by a perturbative analysis one reinforces this information to a control on space-time norms like \( L^{10}(\mathbb{R} \times \mathbb{R}^3) \) of the solutions which in turn implies (7).
Consider the Cauchy problem

\[ i\partial_t U + \Delta U = |U|^2 U, \quad U|_{t=0} = U_0, \quad U : \mathbb{R} \times (\mathbb{R} \times \mathbb{T}^2) \to \mathbb{C}. \]  

(8)

1) The problem (8) is locally well-posed in $H^s(\mathbb{R} \times \mathbb{T}^2)$, $s > 1/2$ (ideas by Bourgain).

2) It is ill-posed for $s \in (0, 1/2)$ (ideas by Lebeau).

3) It is globally-well-posed for $s > 5/6$ (ideas by Tao et al.).

**Theorem 3 (Pausader-Tz. 2017)**

For every $s \in (1/2, \infty)$, $s \neq 1$ there exists $U_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ such that the corresponding solution of (8) is globally defined and

\[ \limsup_{t \to \infty} \|U(t)\|_{H^s(\mathbb{R} \times \mathbb{T}^2)} = +\infty. \]

Recall that for $s \geq 1$, the conservation laws provide an a priori bound on the $H^1(\mathbb{R} \times \mathbb{T}^2)$ norm. We also always have an a priori bound on the $L^2(\mathbb{R} \times \mathbb{T}^2)$ norm. A nonlinear interpolation is therefore impossible.

Previous work by Hani-Pausader-Tz.-Visciglia 2013, obtained this result for $s \geq 30$. 
Reduction of the problem

• Let $U(t)$ be a solution of the cubic defocusing NLS, posed on $\mathbb{R} \times \mathbb{T}^2$. Then $F(t) = e^{-it\Delta}U(t)$ solves

$$i\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)],$$

where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F, G, H] := e^{-it\Delta}(e^{it\Delta}F \cdot e^{-it\Delta}G \cdot e^{it\Delta}H).$$

• Denote by $\hat{F}_p(\xi)$ or $\mathcal{F}(F)(\xi, p)$ the Fourier transform on $\mathbb{R} \times \mathbb{T}^2$ of $F$. Then one computes:

$$\mathcal{F}\mathcal{N}^t[F, G, H](\xi, p) = \sum_{p-p_1+p_2-p_3=0} e^{it[|p|^2-|p_1|^2+|p_2|^2-|p_3|^2]} \int_{\mathbb{R}^2} e^{it2\eta\kappa} \hat{F}_{p_1}(\xi - \eta) \hat{G}_{p_2}(\xi - \eta - \kappa) \hat{H}_{p_3}(\xi - \kappa) d\kappa d\eta.$$
Reduction of the problem (sequel)

• Ignoring the time oscillations (normal form reduction) and a stationary phase argument \((t \gg 1)\) suggests to define \(\mathcal{R}\) as

\[
\mathcal{F}\mathcal{R}[F, G, H](\xi, p) := \sum_{p + p_2 = p_1 + p_3} \hat{F}_{p_1}(\xi) \hat{G}_{p_2}(\xi) \hat{H}_{p_3}(\xi)
\]

and one expects that the nonlinearity can be decomposed as follows

\[
\mathcal{N}^t[F, G, H] = \frac{\pi}{t} \mathcal{R}[F, G, H] + \text{better terms}
\]

• We therefore define the resonant system as

\[
i\partial_t G(t) = \mathcal{R}[G(t), G(t), G(t)].
\]

• The dependence on \(\xi\) is merely parametric.

• We prove that given a solution \(G\) of the resonant system, bounded in ”some norm”, there is a solution of the true problem ”close” to \(G(\pi \ln(t))\) for \(t \gg 1\).
How we justify the normal form reduction and the stationary phase?

- This is a long argument, using the following tools:
  1) Variants of the Zygmund argument, we saw in the beginning of the lecture
  2) Variants of the dispersive estimate, we saw in the beginning of the lecture
  3) A new lemma of Christ-Kiselev type
  4) Almost orthogonality arguments, inspired by the work of Bourgain
  5) The Bourgain/Tataru spaces

- A combination if 1) and 2) provides a low regularity in the periodic variable $L^4_{t,x,y}$ dispersive estimate which is very efficient.
Reduction to the resonant system on $\mathbb{T}^2$

- We take initial data of the resonance system of the form

$$G_0(x, y) = \mathcal{F}_\mathbb{R}^{-1}(\varphi)(x)g(y), \quad x \in \mathbb{R}, y \in \mathbb{T}^2,$$

with $\varphi$ real valued. The solution $G(t)$ to the resonance system with initial data $G_0(x, y)$ as above is given in Fourier space by

$$\hat{G}_p(t, \xi) = \varphi(\xi)a_p(\varphi(\xi)^2t), \quad a_p(0) = \mathcal{F}_{\mathbb{T}^d}(g)(p),$$

where the vector $(a_p)_{p \in \mathbb{Z}^2}$ solves the resonant equation

$$i\partial_t a_p(t) = \sum_{p + p_2 = p_1 + p_3} a_{p_1}(t)a_{p_2}(t)a_{p_3}(t).$$

- In particular, if $\varphi = 1$ on an open interval $I$, then $\hat{G}_p(t, \xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. We can therefore apply the following result.
Theorem 4 (growth for the resonant equation)

Let $s > 0$, $s \neq 1$. There exist global solutions to the resonant equation in $C(\mathbb{R}; h^s)$ such that

$$\sup_{t \geq 0} \|a_p(t)\|_{h^s} = \infty$$

but for every $\varepsilon > 0$

$$\sup_{t \geq 0} \|a_p(t)\|_{h^{s-\varepsilon}} < \infty.$$

- **Notation:**
  $$\|a_p\|_{h^s}^2 := \sum_{p \in \mathbb{Z}^2} (1 + |p|^2)^s |a_p|^2.$$

- **Remark.** Unfortunately, we have that, $a_p(t) \notin h^\sigma$ for $\sigma > s$. 


On the analysis of the resonant equation

- The analysis of the resonant equation is inspired by a work of Colliander-Keel-Staffilani-Takaoka-Tao. Two important aspects are:
  1) There are many invariant subspaces for the resonant equation.
  2) There is a superposition principle: for some initial data it "behaves as a linear equation".