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Small-amplitude solutions for multidimensional hamiltonian PDEs under periodic boundary conditions

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1 Small oscillations in nonlinear hamiltonian PDEs

EXAMPLE: Consider the NLS equation:

(NLS) $u_t + i\Delta u - imu - ig(x, |u|^2)u = 0, \qquad u = u(t,x), \quad x \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d;$

g – real analytic function. This equation can be written in the hamiltonian form:

$$u_t = i\nabla H(u), \quad H(u(x)) = \frac{1}{2}\int_{T^d} \left(|\nabla u|^2 + m|u|^2 + G(x, |u(x)|^2\right) dx\,,$$

where $G'_y(x,y) = g(x,y)$ and $\nabla H(u)$ is the variational derivative of the functional H (i.e., its L_2-gradient). I will regard a solution $u(t,x)$ as a curve

$t \mapsto u(t) \in \langle$ function space $\rangle$

I wish to study global in time **small** solutions and analyse their linear stability. NOTE THAT if $d > 3$, then even for small and smooth initial data it is unknown if a global solution exists. A-priori, a solution $u(t,x)$ exists only for a non-trivial finite time-interval.

WHAT IS KNOWN about long-time behaviour of small solutions: i) sufficient conditions in terms of the function g and the dimension d so that for a smooth initial data a solution exists for all values of time. For example, for equation

$$u_t + i\Delta u - i|u|^{2q}u = 0, \quad q \in \mathbb{N}, \qquad x \in \mathbb{T}^d,$$

this is true if $d \leq 2$ and q is any, or $d = 3$ and $q \leq 2$.

In this case non-trivial UPPER bounds on the growth of the Sobolev norms of solutions as $t \to \infty$ are obtained by J. Bourgain and others.

ii) For the cubic NLS equation $u_t + i\Delta u - i|u|^2 u = 0$ in $\mathbb{T}^2$, Colliander -Keel - Staffilani - Takaoka - Tao obtained LOWER bounds for growth of SOME solutions on long (not infinite) time intervals.

iii) The heuristic theory of wave turbulence studies behaviour of small solutions for NLS equations when $t \gg 1$ and the space-period is not 2π, but L, where $L \gg 1$.

Naturally, to study small solutions, eq. (NLS) should be regarded as a perturbation of the linear Schrödinger equation

$$\text{(S)} \qquad u_t + i\Delta u - imu = 0, \qquad x \in \mathbb{T}^d\,.$$

Solutions for (S) may be written by the Fourier method:

$$u(t,x) = \sum_{s\in\mathbb{Z}^d} u_s e^{i\lambda_s t} e^{is\cdot x}, \qquad \lambda_s = \lambda_s(m) = |s|^2 + m\,.$$

This is a superposition of standing waves with integer *wave-vectors* s. The function $\mathbb{Z}^d \ni s \mapsto \lambda_s$ is called the dispersion relation.

An important special case is given by superpositions of **finitely**-many standing waves: Let $\mathcal{A} \subset \mathbb{Z}^d$, $|\mathcal{A}| = n < \infty$. Consider a superposition of n standing waves with the wave-vectors $s \in \mathcal{A}$:

$$u_{\mathcal{A}}(t,x) = \sum_{s\in\mathcal{A}} u_s e^{i\lambda_s(m)t} e^{is\cdot x}, \qquad |\mathcal{A}| = n < \infty.$$

I will call $\mathcal{A}$ the set of linearly excited modes. Solutions $u_{\mathcal{A}}(t,x)$ with various finite sets $\mathcal{A}$ are dense among all solutions of the linear equation and are time-quasiperiodic (QP).

In particular, if $\mathcal{A}$ is a one-point set, $\mathcal{A} = \{a\}$, then

$$u_{\{a\}}(t,x) = u_a e^{i(|a|^2+m)t} e^{ia\cdot x}$$

is a time-periodic solution. The periodic solutions and their perturbations are significantly easier to study, but there are too few of them – periodic solutions are very exceptional.

In difference with finite dimension, no global approach to prove existence of time-periodic solutions of Hamiltonian PDEs is known.

The goal of my talk is to show that small-amplitude QP solutions also are important for the theory of non-linear equations, and that their role in that theory seriously depends on whether $d = 1$ or $d > 1$.

Consider again

$$u_{\mathcal{A}}(t,x) = \sum_{s\in\mathcal{A}} u_s e^{i(|s|^2+m)t} e^{is\cdot x} ,$$

and denote

$$\rho_s := \tfrac{1}{2}\,|u_s(t)|^2 = \text{const}\,, \qquad s \in \mathcal{A}\,.$$

The vector $\rho = (\rho_s, s \in \mathcal{A}) \subset \mathbb{R}^n_+$ is the *vector of actions* of the solution $u_{\mathcal{A}}$. Clearly

$$u_{\mathcal{A}}(t) = u_{\mathcal{A}}(t;\rho) \in T^n_{\rho,\mathcal{A}} = \{\sum u_s e^{is\cdot x},\ \tfrac{1}{2}|u_s|^2 = \rho_s \text{ if } s \in \mathcal{A};\ u_s = 0 \text{ otherwise}\}\,.$$

This set $T^n_{\rho,\mathcal{A}}$ is an n-torus in the function space. It is invariant for the linear Schrödinger equation (S) and is filled in with its time - QP solutions $u_{\mathcal{A}}(t)$. Abusing language, I will call $T^n_{\rho,\mathcal{A}}$ *a linear torus* and $u_{\mathcal{A}}(t) \in T^n_{\rho,\mathcal{A}}$ – *a linear solution*.

THE PROBLEM. Study how a small QP "linear" solution $u_{\mathcal{A}}(t,x) = \sum_{s\in\mathcal{A}} u_s e^{i\lambda_s(m)t} e^{is\cdot x}$, and the corresponding small invariant "linear" torus $T^n_{\rho,\mathcal{A}}$ are perturbed in (NLS). That is, study if (NLS) has a time-quasiperiodic solution close to $u_{\mathcal{A}}(t,x)$ and an invariant n-torus, close to $T^n_{\rho,\mathcal{A}}$.
There are two main difficulties here – the infinite-dimensionality and the resonances between the frequencies $\{\lambda_s\}$.

For $d = 1$ this problem was resolved in
[SK, Pöschel] Ann. Math. 143 (1996). This was done in two steps:
STEP 1. Put the equation to a normal form in the vicinity of a "linear" torus $T^n_{\rho,\mathcal{A}}$. – That is, in the vicinity of that torus in a function space, choose a special system of coordinates, depending on ρ, such that in the new coordinates the equation we study becomes easier.
STEP 2. After a proper scaling near $T^n_{\rho,\mathcal{A}}$, the obtained normal form equation becomes a small perturbation of a linear hamiltonian system, which depends on the vector-parameter ρ in a non-degenerate way. Next apply to this perturbed equation a theory of perturbations of parameter-depending linear hamiltonian systems (which has been developed before that).

In the 1d case, the proof in $\left[\text{SK, Pöschel}\right]$ works for any finite set $\mathcal{A}$. It allows to construct many time-quasiperiodic solutions $\tilde{u}_{\mathcal{A}}(t;\rho)$ for the 1d (NLS). The closures of these solutions in the function space are smooth "non-linear" tori $\tilde{T}^n_{\rho,\mathcal{A}}$, invariant for (NLS). For $d=1$ the obtained solutions $\tilde{u}_{\mathcal{A}}(t;\rho)$ **always** are linearly stable. Jointly they are "asymptotically dense" near the origin of the phase space, and **are well observed numerically.**

But for $d>1$ the task turned out to be much more complicated since:
$\star$ at the Step 2 the required theorem for perturbations of space-multidimensional parameter-depending linear equations is significantly more complicated than its 1d analogy. – This is the analytic difficulty. It is serious, but – in a sense – technical.

$\star$ But at Step 1 we arrive at a very serious algebraical difficulty. To explain it, consider the cubic NLS $u_t + i\Delta u - imu - i|u|^2 u = 0, \quad x \in \mathbb{T}^d,\ d \geq 2\,.$

The nonlinear part of its Hamiltonian, written in Fourier, is

$$H_4 = \frac{1}{4} \sum_{s_1+s_2=s_3+s_4} u_{s_1} u_{s_2} \bar{u}_{s_3} \bar{u}_{s_4}\,.$$

The normal form transformation deletes all terms of H_4 except the *resonant terms*:

$$(*) \qquad u_{s_1} u_{s_2} \bar{u}_{s_3} \bar{u}_{s_4} \quad \text{such that} \quad s_1 + s_2 = s_3 + s_4 \ \text{ and } \ \lambda_{s_1} + \lambda_{s_2} = \lambda_{s_3} + \lambda_{s_4}\,.$$

In the 1d-case there are just a few terms $(*)$. But if $d \geq 2$, there are many of them since there are plenty of resonances between the frequencies $\lambda_s = |s|^2 + m$, $s \in \mathbb{Z}^d$. Now too many nonlinear terms remain in the normal form, and it is possible to analyse it only if $n \leq 2$. That is, to treat perturbations only of very special solutions $u_{\mathcal{A}}(t)$). This fact was first discussed by Bourgain for $d = 2$. For its detailed analysis for any d see Cl. Procesi & M. Procesi, A KAM–algorithm for the resonant nonlinear Schrödinger equation. *Adv. Math.* 272 (2015), 399-470.

For the moment there is no hope to handle the algebraical difficulty. We encounter the same problems when study other space-multidimensional H PDEs.

I will present a way to overcome the two difficulties, suggested in our work [EGK].

The idea to handle the crucial algebraical difficulty is the following. Consider any HPDE with cubic nonlinearity (e.g. – the cubic NLS). The resonant part its Hamiltonian, which stays in the NF, is formed by the resonant monomials, corresponding to the indices s_1, s_2, s_3, s_4 such that

$$(*) \qquad s_1 + s_2 = s_3 + s_4 \;\text{ and }\; \lambda_{s_1} + \lambda_{s_2} - \lambda_{s_3} - \lambda_{s_4} = 0\,.$$

Let us assume that the H PDE involves a mass-parameter m. Then $\lambda_s = \lambda_s(m)$ and $\lambda_{s_1} + \lambda_{s_2} - \lambda_{s_3} - \lambda_{s_4}$ is an analytic function of m. If the dispersion relation $s \mapsto \lambda_s$ is "non-degenerate", this function of m vanishes identically only for very special quadruples s_1, s_2, s_3, s_4. Then for each "regular" (i.e. "not very special") quadruple there is a zero-measure set $M(s_1, s_2, s_3, s_4)$ such that $\lambda_{s_1} + \lambda_{s_2} - \lambda_{s_3} - \lambda_{s_4} \neq 0$ for all m outside this set. So for m outside the zero-measure set $M = \cup M(s_1, s_2, s_3, s_4)$ the relation $(*)$ may hold only for very special quadruples s_1, s_2, s_3, s_4. Then the normal form contains just "a few" terms and may be analysed. Accordingly, the H PDE also may be analysed for all $m \notin M$.

For the (NLS) the dispersion function is $\lambda_s = |s|^2 + m$. It depends on m linearly, i.e. in a very degenerate way, and for (NLS) the idea does not work.

MAIN EXAMPLE (and our main goal): the Klein-Gordon equation

$$(KG) \qquad u_{tt} - \Delta u + mu + g(x,u) = 0, \quad x \in \mathbb{T}^d, \; m \in [1,2]\,.$$

It is well known that (KG) may be rewritten as an abstract NLS equation. So the preceeding discussion applies to it. The dispersion relation for (KG) is $\lambda_s = \sqrt{|s|^2 + m}$. This is a perfect nonlinear function of m. But asymptotically $\lambda_s \sim |s| + O(s^{-1})$. So if $d > 1$, then asymptotically $\lambda_s = |s|$ is a $\sqrt{\text{integer}}$, and has rather complicated Diophantine properties. This fact makes the problem to study small solutions of (KG) more difficult. In [EGK] we decided to start the realisation of our program with an easier equation (and now we are working on (KG)).

Recent works on the KAM for (KG):

W.-M. Wang, Preprint (2016);

M. Berti, Ph. Bolle, Preprint (2016).

2 Beam equation.

Following J.Geng $\&$ J. You (Nanjing), let us consider the nonlinear beam equation:

(Beam)
$$u_{tt} + \Delta^2 u + mu + g(x,u) = 0,\ x \in \mathbb{T}^d\,,$$
$$g(x,u) = u^3 + O(u^4) =: G'_u(x,u)\,,\ m \in [1,2].$$

Now $\lambda_s = \lambda_s(m) = \sqrt{|s|^4 + m} \sim |s|^2 + O(|s|^{-2})$. This is better than in the case of (KG) equation since $|s|^2 \in \mathbb{Z}$.

Preliminary transformation of the equation: denote

$$\Lambda = \sqrt{\Delta^2 + m}, \quad \psi = 2^{-1/2}(\Lambda^{1/2}u + i\Lambda^{-1/2}\dot{u})\,.$$

In terms of the complex function $\psi(t) = \psi(t,x)$ the equation reads

$$\dot{\psi} = i\Big(\Lambda\psi - 2^{-1/2}\Lambda^{-1/2}g(x, 2^{-1/2}\Lambda^{-1/2}(\psi + \bar{\psi}))\Big)\,.$$

The ψ-equation is a hamiltonian system. It may be written in the hamiltonian form:

$$\dot{\psi} = i\nabla_{\psi} H_{beam}\,, \quad H_{beam} = \int \Big[(\Lambda\psi)\bar{\psi} + G(x, \Lambda^{-1/2}\Big(\frac{\psi+\bar{\psi}}{\sqrt{2}}\Big)) \Big] dx\,.$$

This equation is similar to NLS. Let us write $\psi(x)$ as Fourier series

$$\psi(x) = \sum_{s\in\mathbb{Z}^d} \xi_s e^{is\cdot x}\,, \qquad \bar{\psi}(x) = \sum_{s\in\mathbb{Z}^d} \bar{\xi}_s e^{-is\cdot x}\,.$$

Denote $\eta_s = \bar{\xi}_s$. Then $\bar{\psi}(x) = \sum_s \eta_s e^{-is\cdot x}$, and we may write the beam equation as the Hamiltonian system for the pair of infinite **complex** vectors $\xi = (\xi_s, s \in \mathbb{Z}^d)$, $\eta = (\eta_s, s \in \mathbb{Z}^d)$:

(Beam) $$\dot{\xi}_s = i\frac{\partial H}{\partial \eta_s}, \qquad \dot{\eta}_s = -i\frac{\partial H}{\partial \xi_s}, \quad s \in \mathbb{Z}^d\,,$$

where

$$H_{beam} = \sum \lambda_s \xi_s \eta_s + \int G\Big(x, \sum_s \frac{\xi_s e^{is\cdot x} + \eta_s e^{-is\cdot x}}{\sqrt{2\lambda_s}}\Big).$$

In the complex (ξ,η)-variables, the "linear" QP solutions $\psi_{\mathcal{A}}(t,x)$ of the linear beam equation which we wish to perturb reed

$$(\xi_s(t),\eta_s(t))_{\mathcal{A}} = \begin{cases} (\xi_{s0}e^{i\lambda_s t}, \eta_{s0}e^{-i\lambda_s t}), & s \in \mathcal{A}\,, \\ 0 & s \notin \mathcal{A}\,. \end{cases}$$

A "linear" solution is real if $\xi_s(t) = \bar{\eta}_s(t)$. It stays on the real "linear" n-torus

$$T^n_{\rho,\mathcal{A}} = \{(\xi,\eta) : \xi_a = \bar{\eta}_a, \tfrac{1}{2}|\xi_a|^2 = \tfrac{1}{2}|\eta_a|^2 = \rho_a \text{ if } \ a \in \mathcal{A};\ \xi_s = \eta_s = 0 \text{ if } \ s \notin \mathcal{A}\}\,,$$

which is invariant for the linear beam equation.

MY GOAL: *Study solutions of (Beam) near the torus* $T^n_{\rho,\mathcal{A}}$.

The problem depends on 3 different parameters:

$$m \in [1,2], \qquad \rho \in \mathbb{R}^n_+,\ |\rho| \ll 1, \qquad \mathcal{A} \subset \mathbb{Z}^d,\ |\mathcal{A}| = n\,.$$

All the three parameters are crucially important and play very different roles.

The problem simplifies significantly if $n = 1$ or $d = 1$.

3 Admissible sets $\mathcal{A}$

In difference with the space-one-dimensional equations, in the multidimensional case we have to assume that the finite set of excited modes $\mathcal{A}$ is in some sense "nondegenerate":

Definition. A finite set of linearly excited modes $\mathcal{A} \subset \mathbb{Z}^d$ is called admissible if

1) $a, b \in \mathcal{A}, a \neq b \;\Rightarrow\; |a| \neq |b|$.

2) if $d \geq 3$ and $|\mathcal{A}| \geq 2$, then another – more involved condition – should hold:

For any two different points $a, b \in \mathcal{A}$, the integer sphere of radius $|b|$ with the center at $a + b$ intersects the integer sphere of radius $|a|$ with the center at the origin at most in two points.

Lemma. *Admissible sets are typical*: take at random n points $a_1, \dots a_n$ in the large cube

$$K^d = \{s \in \mathbb{Z}^d : |s_j| \leq N, j = 1, \dots, d\}.$$

Set $\mathcal{A} = \{a_1, \dots, a_n\}$. Then $\mathbf{P}\{\mathcal{A}$ is admissible $\} = 1 - O(N^{-\gamma})\,, \;\; \gamma > 0\,.$

Everywhere below I assume that $\mathcal{A}$ is admissible.

Recall that we want to perturb the real ”linear” solutions

$$T^n_{\rho,\mathcal{A}} \ni (\xi_s(t;\rho), \eta_s(t;\rho))_{\mathcal{A}} = \begin{cases} (\xi_{s0} e^{i\lambda_s t}, \eta_{s0} e^{-i\lambda_s t}), & s \in \mathcal{A}\,, \\ 0 & s \notin \mathcal{A}\,, \end{cases}$$

$\rho_a = \frac{1}{2}|\xi_a|^2 = \frac{1}{2}|\eta_a|^2$. The linearly excited modes $\{a \in \mathcal{A}\}$ are the most important. Denote by ω the frequency vector, formed by them:

$$\omega = \omega^{\mathcal{A}}(m) = (\omega_a, a \in \mathcal{A}) \in \mathbb{R}^n, \qquad \omega_a = \lambda_a\,.$$

Then $\omega_{a_1} \neq \omega_{a_2}$ if $a_1 \neq a_2$ since the set $\mathcal{A}$ is admissible.

4 The normal form

I recall that in terms of the complex Fourier coefficients $\xi = (\xi_s, s \in \mathbb{Z}^d)$, $\eta = (\eta_s, s \in \mathbb{Z}^d)$, the Hamiltonian of (Beam) is

$$H_{beam} = \sum \lambda_s \xi_s \eta_s + \int G\Big(x, \sum_s \frac{\xi_s e^{is\cdot x} + \eta_s e^{-is\cdot x}}{\sqrt{2\lambda_s}}\Big) =: H^2_{beam} + \dots ,$$

and that we wish to study (Beam) near the small real "linear" n-tori $T^n_{\rho,\mathcal{A}}$, where the vector of actions $\rho = (\rho_a, a \in \mathcal{A})$ is a small parameter of this problem,

$$\rho \in \mathbb{R}^n_+ , \quad |\rho| \ll 1 , \qquad n = |\mathcal{A}| .$$

First, near $T^n_{\rho,\mathcal{A}}$ I make the usual elementary change of coordinate: I keep the infinitely-many coordinates ξ_s, η_s with $s \notin \mathcal{A}$ without change, and for the finitely-many modes ξ_a, η_a with $a \in \mathcal{A}$ pass from them to the action-angles (r, θ), where $r \in \mathbb{R}^n$ is small and $\theta \in \mathbb{T}^n$, using the usual relation:

$$\xi_a = \sqrt{2(\rho_a + r_a)}\, e^{i\theta_a}, \ \ \eta_a = \sqrt{2(\rho_a + r_a)}\, e^{-i\theta_a}, \ \ a \in \mathcal{A}\,.$$

Now the linear torus $T^n_{\rho,\mathcal{A}}$ reeds

$$T^n_{\rho,\mathcal{A}} = \{r = 0, \theta \in \mathbb{T}^n, \xi_s = \eta_s = 0 \ \ \forall\, s \in \mathbb{Z}^d \setminus \mathcal{A}\}\,.$$

In the new variables the quadratic part $\sum_{s\in\mathbb{Z}^d} \lambda_s(m)\xi_s\eta_s$ of the hamiltonian becomes

$$H^2_{beam} = \text{Const} + \omega(m)\cdot r + \sum_{s\in\mathbb{Z}^d\setminus\mathcal{A}} \lambda_s(m)\xi_s\eta_s\,.$$

Vector $\rho = (\rho_1, \ldots, \rho_n)$ is a parameter, and $\big((r,\theta), (\xi_s, \eta_s), s \in \mathbb{Z}^d \setminus \mathcal{A}\big)$ are the coordinates near $T^n_{\rho,\mathcal{A}}$.

DENOTE

$$\mathcal{A}^+ = \{s \in \mathbb{Z}^d \setminus \mathcal{A} : |s| = |a| \text{ for some } a \in \mathcal{A}\}$$

($\mathcal{A}^+$ is the "shade of $\mathcal{A}$"). Note that $\mathcal{A}^+$ is a finite set. Then

$$\mathbb{Z}^d = \mathcal{A} \cup \mathcal{A}^+ \cup \langle \text{the rest} \rangle .$$

Accordingly I write

$$H_{beam} = \text{Const} + \omega(m) \cdot r + \sum_{s \in \mathcal{A}^+} \lambda_s(m) \xi_s \eta_s + \sum_{s \in \mathbb{Z}^d \setminus (\mathcal{A} \cup \mathcal{A}^+)} \lambda_s(m) \xi_s \eta_s + \dots .$$

Assume that $\mathcal{A}$ is admissible. Then for $m \in [1, 2]$ outside certain bad zero-measure set $\mathcal{C}$, in [EGK] we obtain the following

Normal Form Theorem:

THEOREM 1. If the action-vector $\rho \in \mathbb{R}^n_+$ is sufficiently small, then near $T^n_{\rho,\mathcal{A}}$ there exists a canonical transformation from new variables $\big(\tilde r, \tilde\theta, (\tilde\xi_s, \tilde\eta_s, s \in \mathbb{Z}^d \setminus \mathcal{A})\big)$ to the old variables $\big(r, \theta, (\xi_s, \eta_s, s \in \mathbb{Z}^d \setminus \mathcal{A})\big)$, such that in the tilde–variables the transformed Hamiltonian $H_{\text{new}} = H_{\text{new}}(\tilde r, \tilde\theta, \tilde\xi, \tilde\eta;\ \rho)$ reeds:

$$H_{\text{new}} = \Omega(\rho) \cdot \tilde r + \Big\langle K(\rho) \left(\tilde\xi^f, \tilde\eta^f\right), \left(\tilde\xi^f, \tilde\eta^f\right) \Big\rangle + \sum_{a \in \mathbb{Z}^d \setminus (\mathcal{A} \cup \mathcal{A}^+)} \Lambda_a(\rho) \tilde\xi_a \tilde\eta_a + H_3.$$

The H_{new} explicitly depends on the small vector-parameter ρ:

i) the frequency-vector $\Omega(\rho)$ is affine in ρ, $\Omega(\rho) = \omega(m) + L\rho$, where the linear operator L is non-degenerate, so we have the non-degenerate frequency modulation $\rho \mapsto \Omega(\rho)$;

ii) the frequencies $\Lambda_a(\rho) = \lambda_a(m) + \sum_b M_a^b \rho_b$ also are affine in ρ;

iii) $K(\rho)$ is an **explicit** real symmetric matrix of size $2|\mathcal{A}^+|$ quadratic in $\sqrt{\rho_j}$'s;

iv) the Hamiltonian vector-field $iJ\nabla H_3$ is small compare to $|\rho|$.

So H_{new} as a function of the parameter $\rho \in \mathbb{R}_+$ is:

$$H_{\text{new}} = \text{Const}(\rho) + \langle \text{ linear in } \rho \text{ and explicit } \rangle + o(\rho).$$

$$H_{\text{new}} = \Omega(\rho) \cdot \tilde{r} + \left\langle K(\rho) \left(\tilde{\xi}^f, \tilde{\eta}^f\right), \left(\tilde{\xi}^f, \tilde{\eta}^f\right) \right\rangle + \sum_{a \in \mathbb{Z}^d \setminus (\mathcal{A} \cup \mathcal{A}^+)} \Lambda_a(\rho) \tilde{\xi}_a \tilde{\eta}_a + H_3.$$

Properties of the matrix $K(\rho)$ and its Hamiltonian operator $iJK(\rho)$ are crucial :

The matrix $K(\rho)$ is of size $2|\mathcal{A}^+| \times 2|\mathcal{A}^+|$ and is explicit. Its Hamiltonian operator $iJK(\rho)$ IS NOT Hermitian or anti-Hermitian. What we know about it:

a) (non-degeneracy) for typical values of ρ, **the operator $iJK(\rho)$ is invertible**;

b) (easy cases) if $d = 1$ or $|\mathcal{A}| = 1$, then all eigenvalues of $iJK(\rho)$ are elliptic;

c) (stability/instability) if $d \geq 2$ and $|\mathcal{A}| \geq 2$, then some of the eigenvalues are elliptic, and

i) for some values of ρ the operator $iJK(\rho)$ has no hyperbolic eigenvalues;

ii) but for some other ρ's part of its eigenvalues may be hyperbolic;

iii) some of the elliptic eigenvalues are multiple identically in ρ. But **for typical ρ all the hyperbolic eigenvalues are simple.**

In [EGK] we proved an abstract KAM-theorem, applicable to Hamiltonians of the form H_{new}. For that theorem both conditions, given above in bold, are crucially important.
Its application gives us:

5 Final theorem

There exists a zero-measure set $\mathcal{C} \subset [1,2]$ such that for **each** $m \notin \mathcal{C}$ **and any admissible set** $\mathcal{A}$ the following holds:

THEOREM 2. There is a closed set $\mathbf{R} \subset \mathbb{R}^n_+ = \{\rho\}$ of the action-vectors ρ which has density one at zero, such that if $\rho \in \mathbf{R}$, then the "linear" time-QP solution $(\xi, \eta)_{\mathcal{A}}(t; \rho)$ of the linear beam equation persists as a time-QP solution $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}(t; \rho)$ of (Beam). The closure of the curve $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}(t; \rho)$ in the equation's function-space is a smooth n-torus, close to the original "linear" torus $T^n_{\rho, \mathcal{A}}$, and invariant for (Beam). A constructed solution $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}(t, \rho)$ is linearly stable if and only if the hamiltonian matrix $iJK(\rho)$ is stable. If $d = 1$ or $|\mathcal{A}| = 1$, then $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}$ always is stable. While if $d \geq 2$, then for each admissible set $\mathcal{A}$ the persisted solutions $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}$ are linearly stable for *some values of* ρ, but for certain admissible sets $\mathcal{A}$ the persisted solutions are unstable for some other values of ρ.

CONJECTURE. The instability and stability of the KAM-solutions $(\xi, \eta)^{\mathrm{n/l}}_{\mathcal{A}}(t; \rho)$ both are typical, if $d \geq 2$.

The conjecture is a question from linear algebra since $JK(\rho)$ is an explicitly defined finite real matrix.

Together the constructed quasiperiodic solutions $(\xi,\eta)^{\mathrm{n/l}}_{\mathcal{A}}(t;\rho)$ of (Beam) with a fixed set $\mathcal{A}$ form in the function space of (Beam) a subset $\mathcal{T}_{\mathcal{A}}$ of the Hausdorff dimension $2n$, $n=|\mathcal{A}|$, invariant for the equation. Now consider $\mathcal{T}=\cup\mathcal{T}_{\mathcal{A}}$, where the union is taken over all admissible sets $\mathcal{A}$. This set is invariant for (Beam) and has infinite Hausdorff dimension. Some time-quasiperiodic solutions of (Beam), lying on $\mathcal{T}$, are linearly stable, while, if $d\geq 2$, then some others are unstable.

If $d\geq 2$, then the constructed linearly unstable KAM-solutions $(\xi,\eta)^{\mathrm{n/l}}_{\mathcal{A}}(t;\rho)$ of (Beam) create around them certain zones of instability. So our theory implies some instability features for small-amplitude solutions for the space-multidimensional beam equations. In the Nonlinear Physics this behaviour is known as the *modulation instability*.