Concentration sets for multiple equal depth wells potentials in the 2D elliptic case

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The equations

The limiting behavior as $\varepsilon \to 0$ of solutions to the reaction-diffusion equations of the type

$$\frac{\partial u}{\partial t} - \Delta u_\varepsilon = -\varepsilon^{-2} \nabla V_u(u_\varepsilon)$$

is a source of *active research* in the last decades. The function u_ε takes values in $\mathbb{R}^k$ and V denotes a potential $V : \mathbb{R}^k \to \mathbb{R}$. Of interest are also the stationary solutions we will discuss later on

$$-\Delta u_\varepsilon = -\varepsilon^{-2} \nabla V_u(u_\varepsilon).$$

The equation is the L^2 gradient-flow of the energy $\mathscr{E}$ defined by

$$\mathrm{E}_\varepsilon(u) = \int_\Omega e_\varepsilon(u) = \int_{\mathbb{R}} \varepsilon \frac{|\nabla u|^2}{2} + \frac{V(u)}{\varepsilon}, \text{ for } u : \mathbb{R} \mapsto \mathbb{R}^k.$$

$\Omega \subset \mathbb{R}^N$ being the domain. The properties of the flow (RDG) strongly depend on the potential V . Throughout we assume that

- V is smooth from $\mathbb{R}^k$ to $\mathbb{R}$,
- V tends to infinity at infinity, so that it is bounded below

$$V \geq 0.$$

An intuitive guess is that the flow drives to mimimizers of the potential :

- if V is strictly convex, the solution should tend to the unique minimizer of the potential V.
- Here we consider the case where there are several mimimizers for the potential V $\rightsquigarrow$ Transitions between minimizers

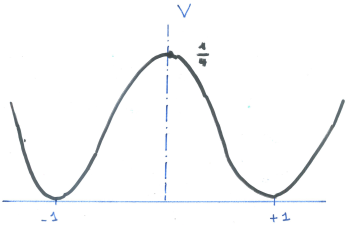
Multiple-well potentials

We assume in this talk that V is has a finite number of and at least two distinct minimizers.

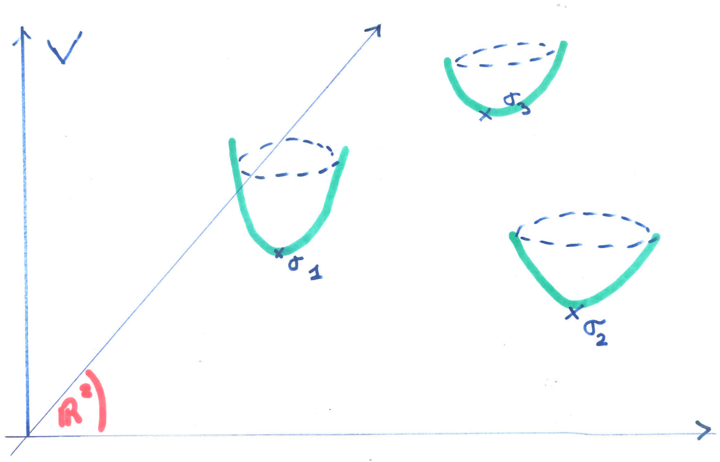
A classical example in the scalar case (Allen-Cahn) $k = 1$

$$V(u) = \frac{(1-u^2)^2}{4}, \qquad (AC)$$

whose minimizers are $+1$ and -1.



The picture for systems



Assumptions on V

(H_1) inf $V = 0$ and the set of minimizers $\Sigma \equiv \{y \in \mathbb{R}^k, V(y) = 0\}$

is a finite set, with **at least two** distinct elements, that is

$$\Sigma = \{\sigma_1, ..., \sigma_q\}, q \geq 2, \ \sigma_i \in \mathbb{R}^k, \forall i = 1, ..., q.$$

(H_∞) There exists constant $\alpha_0 > 0$ and $R_0 > 0$ such that

$$y \cdot \nabla V(y) \geq \alpha_0 |y|^2, \text{ if } |y| > R_0.$$

The scalar case

Important efforts have ben devoted so far to the study of solutions of the *Allen-Cahn* equations, i.e. for the special choice of potential

$$V(u) = \frac{(1-u^2)^2}{4}, \tag{2}$$

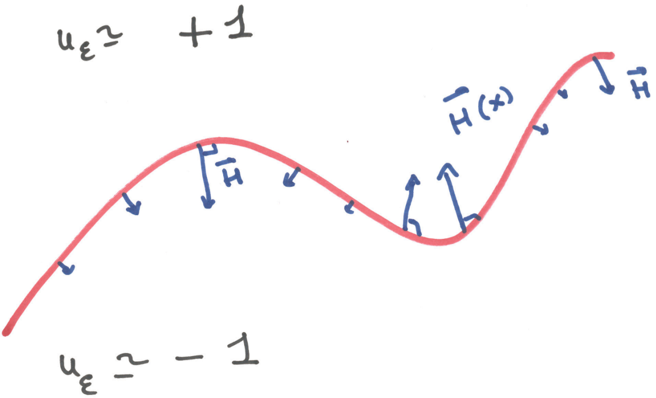
whose infimum equals 0 and whose minimizers are $+1$ and -1, so that $\Sigma = \{+1, -1\}$. It is an elementary model for phase transitions for materials with two equally preferred states, the minimizers $+1$ and -1 of V.
The mathematical theory for this question is now well advanced and may be considered as **quite satisfactory**.

Results for the scalar case

They provide a sound foundation to the intuitive idea that the domain Ω decomposes into regions where the solution takes values close to $+1$ or close to -1, separated by interfaces of width of order ε.

- The interfaces converge to codimension 1 hypersurfaces.
- They are generalized minimal surfaces in the stationary case, or moved by mean curvature for the parabolic case.
- Arguments rely on integral methods and energy estimates
- In the parabolic case Ilmanen proved convergence past possible singularities, to motion by mean curvature in the weak sense of Brakke, a notion phrased in the language of geometric measure theory.
- In the elliptic case convergence to minimal surfaces was established by Modica and Mortola for minimizers, Hutchinson and Tonegawa established related results for non-minimizing solutions.
- The fact that the solutions are **scalar** is crucial in the proofs.

motion by mean curvature



Monotonicity Formula and Clearing-out

Concentration on $N-1$ dimensional sets is deduced from two ingredients

- Monotonicity formulas
- Clearing-out Lemmas

The following inequality (used in Ginzburg-Landau theory)

$$\frac{d}{dr}\left(\frac{1}{r^{N-2}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega,$$

is valid for arbitrary vectorial potentials. It is however not sufficient to establish concentration on $N-1$- dimensional sets where one wishes to have

$$\frac{d}{dr}\left(\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega, \tag{3}$$

Such a formula was derived in the Allen-Cahn scalar case thanks to the maximum principle.

The discrepancy

The proof of the $N-1$ monotonicity in the scalar case relies the positivity of the discrepancy term

$$\xi_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon} V(u_\varepsilon) - \varepsilon \frac{|\nabla u|^2}{2}.$$

Notice that for $N=1$ for $-\varepsilon^2 \dfrac{d^2 u}{dx} = -\nabla_u V(u)$ one has

$$\frac{d}{dx}\left(\frac{1}{\varepsilon} V(u) - \varepsilon \frac{|\dot{u}|^2}{2}\right) = 0,$$

In higher dimensions, the positivity of ξ_ε for scalar solutions was observed first by Payne, Sperb, L. Modica,.. for entire solutions. He proved the remarkable inequality (for $\varepsilon = 1$)

$$-|\nabla u|^2 \Delta \xi \geq 2\frac{1}{2}|\nabla \xi|^2 + 2V(u)\nabla u \cdot \nabla \xi_\varepsilon.$$

Clearing-out lemmas

Clearing-out Lemmas have more or less the following flavour : There exists some constant $\eta_0 > 0$ such that

$$\frac{1}{r^{N-1}} \mathrm{E}_\varepsilon \left(u_\varepsilon, \mathbb{B}^N(x_0, r) \right) \leq \eta_0 \implies u_\varepsilon(x) \simeq \sigma \text{ on } \mathbb{B}^N\left(x_0, \frac{r}{2}\right).$$

where $\sigma \in \Sigma$, the set of minimizers of the potential.
Such a statement is rather easy to prove when monotonicity is established.
Indeed, by monotonicity

$$\frac{1}{\varepsilon^N} \int_{\mathbb{B}^N(\varepsilon)} V(u_\varepsilon) \leq \eta_0.$$

and then the (easy) bound $|\nabla u_\varepsilon| \leq C\varepsilon^{-1}$ allows to conclude.

Tools in the scalar Allen-Cahn case

To summarize the methods used in the scale Allen-Cahn case one has

$$\text{sign of discrepancy} \implies \text{monotonicity} \implies \text{clearing} - \text{out}$$

whereas

$$\begin{cases} \text{clearing out} + \text{monotonicity} \implies \text{concentration on } N-1 \text{ dimensional sets} \\ \text{monotonicity} \implies \text{(Preiss)} \ \text{ rectifiability of concentration set} \end{cases}$$

and

$$\text{sign of discrepancy } + \text{stress} - \text{energy tensor}$$
$$\Downarrow$$
$$\text{stationary sets or motion by mean} - \text{curvature}$$

Conclusion: Sign of discrepancy is crucial !

Back to the vectorial case

Main observation

In the **vectorial case**, positivity of the discrepancy as well as the monotonicity formula are known to fail for some solutions, e. g. for the **Ginzburg-Landau system**. Whether they might still hold under additional conditions on the potential or the solution itself is open.

$\Downarrow$

New ideas are required !

I will next present a result where some parts of the program have been carried out in the absence of monotonicity as well as sign of discrepancy. It concerns

- **the two-dimensional case**, i. e we work on a domain $\Omega \subset \mathbb{R}^2$
- **The elliptic system**

The assumptions

Assume we are given a constant $\mathrm{M}_0 > 0$ and a family $(u_\varepsilon)_{0<\varepsilon\leq 1}$ of solutions to the equation

$$-\Delta u_\varepsilon = \nabla V_u(u_\varepsilon) \quad \text{on } \Omega.$$

satisfying the natural energy bound

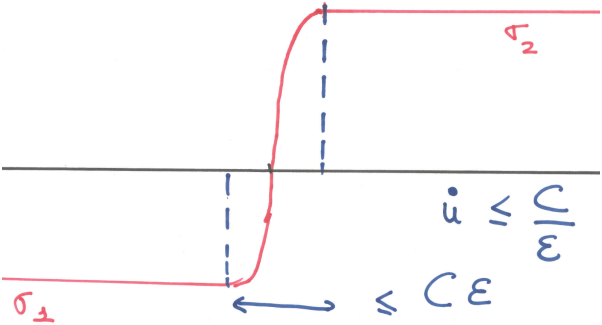
$$\mathrm{E}_\varepsilon(v_\varepsilon) \leq \mathrm{M}_0, \ \forall \varepsilon > 0. \tag{4}$$

Remark : This bound is natural because the energy of one-dimensional transition, i. e. solutions

$$-\ddot{u} = \varepsilon^{-2} \nabla v_u(u)$$

is finite, bounded independently of ε.

Energy of an interface



$$\begin{cases} \displaystyle\int_{\mathbb{R}} |\dot{u}|^2 \simeq C^2 \int_{[-C\varepsilon, C\varepsilon]} \varepsilon^{-2} \simeq C\varepsilon^{-1} \\ \displaystyle\int_{\mathbb{R}} V(u)|^2 \simeq C \int_{[-C\varepsilon, C\varepsilon]} \simeq C\varepsilon \end{cases}$$

Limiting measures

We introduce the family $(\nu_\varepsilon)_{0<\varepsilon\leq 1}$ of measures defined on Ω

$$\nu_\varepsilon \equiv e_\varepsilon(u_\varepsilon)\,dx \text{ on } \Omega. \tag{5}$$

In view of the energy bound, the total mass of the measures is bounded by M_0, that is $\nu_\varepsilon(\Omega) \leq M_0$. By compactness,there exists a decreasing subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to 0 and a limiting measure $\nu_\star$ on Ω such that

$$\nu_{\varepsilon_n} \rightharpoonup \nu_\star \text{ in the sense of measures on } \Omega \text{ as } n \to +\infty. \tag{6}$$

Our main result is the following.

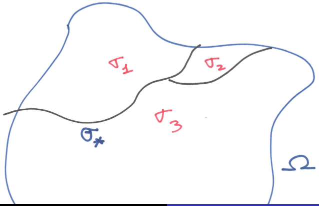
Theorem

There exist a subset $\mathfrak{S}_\star$ *in* Ω, *and a subsequence of* $(\varepsilon_n)_{n\in\mathbb{N}}$ *still denoted* $(\varepsilon_n)_{n\in\mathbb{N}}$ *such that the following properties hold:*

i) $\mathfrak{S}_\star$ *is a closed 1-dimensional rectifiable, with locally finite many connected components and such that* $\mathscr{H}^1(\mathfrak{S}) \leq \mathrm{C_H}\,\mathrm{M_0}$, *where* $\mathrm{C_H}$ *is a constant depending only on the potential* V.

ii) *Set* $\mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$, *and* $(\mathfrak{U}^i_\star)_{i\in I}$ *be the connected components of* $\mathfrak{U}_\star$. *For each* $i \in I$ *there exists an element* $\sigma_i \in \Sigma$ *such that*

$$u_\varepsilon \to \sigma_i \text{ uniformly on every compact subset of } \mathfrak{U}^i_\star.$$



Comments on the results

- At this stage, I have not been able to prove stationary, nor positivity of discrepancy.
- The set $\mathfrak{S}_\star$ in the above theorem represents the concentration set for the energy
- The argument for the proof of rectifiability of the singular set $\mathfrak{S}_\star$ is quite specific , namely compact set of hausdordd dimension 1 are rectifiable.
- Most of the statement relies on the two clearing-out properties which follow :

Clearing-out properties for the measure $\nu_\star$

The first one is a classical clearing-out result for the measure $\nu_\star$.

Theorem

Let $x_0 \in \Omega$ and $r > 0$ be given such that $\mathbb{D}^2(x_0, r) \subset \Omega$. There exists a constant $\eta_0 > 0$ such that, if we have

$$\frac{\nu_\star\left(\overline{\mathbb{D}^2(x_0,r)}\right)}{r} < \eta_0, \text{ then it holds } \nu_\star\left(\overline{\mathbb{D}^2(x_0,\frac{r}{2})}\right) = 0. \tag{7}$$

we set

$$\theta_\star(x_0) = \liminf_{r\to 0} \frac{\nu_\star\left(\overline{\mathbb{D}^2(x_0,r)}\right)}{r}$$

and define $\mathfrak{S}_\star$ as

$$\mathfrak{S}_\star = \{x \in \Omega, \theta_\star(x_0) \geq \eta_0.\}. \tag{8}$$

The fact that $\mathfrak{S}_\star$ is closed of finite one-dimensional Hausdorff measure is a direct consequence of the clearing-out property for the measure $\nu_\star$.

The connectedness properties of $\mathfrak{S}_\star$ require a different type of clearing-out result. Let $\mathscr{U} \subset \Omega$ be open. For $\delta > 0$, we consider the sets

$$\begin{cases} \mathscr{U}_\delta = \{x \in \Omega, \mathrm{dist}(x, \mathscr{U}) \leq \delta\} \text{ and} \\ \mathscr{V}_\delta = \mathscr{U}_\delta \setminus \mathscr{U} = \{x \in \Omega, 0 \leq \mathrm{dist}(x, \mathscr{U}) \leq \delta\}. \end{cases} \tag{9}$$

Theorem

Let $\mathscr{U} \subset \Omega$ be a open subset of Ω, let $\delta > 0$ be given. If we have

$$\nu_\star(\mathscr{V}_\delta) = 0, \text{ then it holds } \nu_\star\left(\overline{\mathscr{U}}\right) = 0. \tag{10}$$

In other terms, if the measure $\nu_\star$ vanishes in some neighborhood of the set $\mathscr{U}$, then it vanishes on $\overline{\mathscr{U}}$.

- allows us to establish the connectedness properties of $\mathfrak{S}_\star$.
- yields rectifiability invoking standard results on continua of bounded one-dimensional Hausdorff measure.

Elements in the proof : 1) scale invariance

Proofs are derived from corresponding PDE results at the ε level for u_ε.

For $r>0$ and $\varepsilon>0$, set $\epsilon=\frac{\varepsilon}{r}$. For $u_\varepsilon:\mathbb{D}^2(x_0,r)\to\mathbb{R}^k$, consider $v_\epsilon:\mathbb{D}^2\to\mathbb{R}^k$ defned by

$$v_\epsilon(x)=u_\varepsilon(rx+x_0)), \forall x\in\mathbb{D}^2.$$

If u_ε is a solution to the PDE, when v_ϵ is a solution to the PDE with parameter ϵ. The scaling for the energy are

$$\begin{cases} e_\epsilon(v_\epsilon)(x)=re_\varepsilon(u)(rx+x_0), \forall x\in\mathbb{D}^2 \\ \mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)=r\mathrm{E}_\epsilon\left(v_\epsilon,\mathbb{D}^2(1)\right) \text{ and } \mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)=r\mathbb{V}_\epsilon\left(v_\epsilon,\mathbb{D}^2(1)\right) \text{ with} \end{cases}$$

$$\mathrm{E}_\varepsilon(u,G)\equiv\int_G e_\varepsilon(u)\mathrm{d}x \text{ and } \mathbb{V}_\varepsilon(u,G)\equiv\int_G\frac{V(u)}{\varepsilon}\mathrm{d}x.$$

- The parameter ε as well as the energy E_ε behave as lengths
- $\varepsilon^{-1}E_\varepsilon$ is scale invariant, according to the previous scale changes.

Clearing-out for the PDE

Choose $\mu_0 > 0$ so that $B^k(\sigma_i, 2\mu_0) \cap \mathbb{B}^k(\sigma_j, 2\mu_0) = \emptyset$ for all $i \neq j$ in $\{1, \cdots, q\}$ and such that and

$$\frac{1}{2}\lambda_i^- \mathrm{Id} \leq \nabla^2 V(y) \leq 2\lambda_i^+ \mathrm{Id} \quad \text{for all } i \in \{1, \cdots, q\} \text{ and } y \in B(\sigma_i, 2\mu_0). \tag{11}$$

Theorem

Let $0 < \varepsilon \leq 1$ and u_ε be solution of the $(PDE)_\varepsilon$ on $\mathbb{D}^2$. There exists $\eta_0 > 0$ s.t. if

$$\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq \eta_0,$$

then there exists some $\sigma \in \Sigma$ such that

$$|u_\varepsilon(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for every } x \in \mathbb{D}^2(\frac{3}{4}),$$

We have the energy estimate, with $\mathrm{C_{nrg}} > 0$ depending only on V

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(\frac{5}{8}\right)\right) \leq \mathrm{C_{nrg}}\, \varepsilon E_\varepsilon(u_\varepsilon, \mathbb{D}^2).$$

The previous result relies on:

Proposition

Let $0 < \varepsilon \leq 1$ *and* u_ε *be a solution of* $(PDE)_\varepsilon$ *on* $\mathbb{D}^2$*. There exists a constant* $\mathrm{C}_{\mathrm{dec}} > 0$ *such that*

$$\int_{\mathbb{D}^2(\frac{9}{16})} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \mathrm{C}_{\mathrm{dec}}\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right]. \tag{12}$$

This proposition is perhaps the main new ingredient: When both $\mathrm{E}_\varepsilon(u_\varepsilon)$ and ε are small, it provides a fast decay of the energy on smaller balls. Iterating this decay estimate, we are led to the proof of the Clearing-out Theorem.

One-dimensional (Modica-Modica type) estimates

In dimension 1 energy bounds directly lead to uniform bound. Set $\mathbb{S}^1(r) = \{x \in \mathbb{R}^2, |x| = r\}$ and consider $u : \mathbb{S}^1(r) \to \mathbb{R}^k$.

Lemma

Let $0 < \varepsilon \leq 1$ *and* $0 < r < 1$ *be given. There exists a constant* $\mathrm{C_{unf}} > 0$ *such that, for any given* $u : \mathbb{S}^1(r) \to \mathbb{R}^k$*, there exists an element* $\sigma \in \Sigma$ *such that*

$$|u(\ell) - \sigma| \leq \mathrm{C_{unf}} \sqrt{\int_{\mathbb{S}^1(r)} \mathrm{e}_\varepsilon(u) \mathrm{d}\ell}, \quad \text{for all } \ell \in \mathbb{S}^1.$$

Comment On the disk $\mathbb{D}^2$, the result shows that if a map has small E_ε energy, then oscillations around an element of Σ are small for many circles $\mathbb{S}^1(r)$.

Elements in the proof: Standard elliptic estimates:

- **uniform bounds** $|\nabla u_\varepsilon| \leq \frac{K_{\rm dr}}{\varepsilon}$ and $|u_\varepsilon| \leq M$. In particular

$$e_\varepsilon(u_\varepsilon) \leq C_{\rm T} \frac{V(u_\varepsilon)}{\varepsilon} \quad \text{on } \Theta_\varepsilon = {u_\varepsilon}^{-1}\left(\bigcup_{i=1}^{q} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4})\right)$$

- **Pohozaev type bounds**: (specific to dimension 2) for $\delta > 0$ small, $\mathscr{U}$ open subset

$$\frac{1}{\varepsilon} \int_{\mathscr{U}_{\frac{\delta}{2}}} V(u_\varepsilon) {\rm d}x \leq C(\mathscr{U}, \delta) \int_{\mathscr{V}_\delta} e_\varepsilon(u_\varepsilon) {\rm d}x, \text{ where}$$

$$\begin{cases} \mathscr{U}_\delta = \{x \in \Omega, {\rm dist}(x, \mathscr{U}) \leq \delta\} \text{ and} \\ \mathscr{V}_\delta = \mathscr{U}_\delta \setminus \mathscr{U} = \{x \in \Omega, 0 \leq {\rm dist}(x, \mathscr{U}) \leq \delta\}. \end{cases}$$

Remark

A related relation is: For any radius $0 < r \leq 1$

$$\frac{1}{\varepsilon^2}\int_{\mathbb{D}^2(r)} V(u_\varepsilon) = \frac{r}{4}\int_{\partial D^2(r)} \left(\left|\frac{\partial u}{\partial \tau}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial r}\right|^2 + \frac{2}{\varepsilon^2} V(u) \right) \mathrm{d}\tau.$$

This identity leads to the monotonicity formula

$$\frac{d}{dr}\left(\frac{\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right)}{r} \right) = \frac{1}{r^2}\int_{\mathbb{D}^2(r)} \xi_\varepsilon(u_\varepsilon)\mathrm{d}x + \frac{1}{r}\int_{S^1(r)} |\frac{\partial u_\varepsilon}{\partial r}|^2 \mathrm{d}\ell.$$

Energy on level sets

let $u_\varepsilon : \mathbb{D}^2 \to \mathbb{R}^k$ be solution to the $(PDE)_\varepsilon$. Assume we are given $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$, $0 < \kappa < \frac{\mu_0}{2}$, $\sigma_{\rm main} \in \Sigma$ such that

$$|u_\varepsilon - \sigma_{\rm main}| < \kappa \text{ on } \partial\mathbb{D}^2(\varrho_\varepsilon). \tag{13}$$

Consider $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ defined by

$$\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa) = \left\{x \in \mathbb{D}^2(\varrho_\varepsilon) \text{ such that } |u_\varepsilon(x) - \sigma_i| < \kappa, \quad \text{for some } i = 1 \dots q\right\}$$

The set $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ is a truncation of the domain with points with values far from Σ removed. The solution u_ε on $\Upsilon_\varepsilon(\varrho, \kappa)$ is close, at least when the energy is small, to one of the points σ_i: Near this point the potential is close to a quadratic potential. We have

Proposition

We have, for $C_\Upsilon > 0$, *under above assumptions*

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq C_\Upsilon \left[\kappa \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right].$$

The next step is to specify the result of the previous proposition for special choices of κ and ϱ_ε. More precisely, we choose

$$\varrho_\varepsilon = \mathfrak{r}_\varepsilon \text{ and } \kappa_\varepsilon = 2C_{\rm bd}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)},\ C_{\rm bd} > 0 \text{ a constant,}$$

where $\frac{3}{4} \leq \mathfrak{r}_\varepsilon \leq 1$ is obtained by the following mean value argument:

Lemma

Let $0 \leq r_0 < r_1 \leq 1$ and $u : \mathbb{D}^2 \to \mathbb{R}^k$ be given. There exists a radius $\mathfrak{r}_\varepsilon \in [r_0, r_1]$ s.t.

$$\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} e_\varepsilon(u)\mathrm{d}\ell \leq \frac{1}{r_1 - r_0}\mathrm{E}_\varepsilon(u, \mathbb{D}^2(r_1)).$$

This specification yields

$$\int_{\Upsilon_\varepsilon(\mathfrak{r}_\varepsilon,\kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq 2C_\Upsilon\left[C_{\rm bd}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)}\int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon\int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right].$$

Proposition

There exists a constant $C_{\rm V} > 0$ *such that*

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(\frac{5}{8})} V(u_\varepsilon)\mathrm{d}x \leq C_{\rm V}\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right].$$

We may assume the energy is small and consider the restriction of u_ε to the set $\Omega = \mathbb{D}^2(\mathfrak{r}_\varepsilon)$. The coarea formula and a mean-value argument yield some $s_\varepsilon \in [C_{\rm bd}\sqrt{\mathrm{E}_\varepsilon(u)}, 2C_{\rm bd}\sqrt{\mathrm{E}_\varepsilon(u)}]$ such that the curve $\mathscr{C}_\varepsilon \equiv w^{-1}(s_\varepsilon) \cap \mathbb{D}^2(\mathfrak{r}_\varepsilon)$, where $w = |u_\varepsilon - \sigma|$, verifies

$$\mathscr{L}(\mathscr{C}_\varepsilon) \leq \mathscr{L}\left(w^{-1}(s_\varepsilon)\right) \leq C_L\sqrt{\mathrm{E}_\varepsilon(u)}. \qquad (14)$$

By a mean value argument, we may then choose a new radius $\varrho_\varepsilon \in [\frac{5}{8}, \mathfrak{r}_\varepsilon]$ such that

$$\begin{cases} |u-\sigma| \leq C_{\mathrm{bd}}\sqrt{\mathrm{E}_\varepsilon(u)} \text{ on } \mathbb{S}^1(\varrho_\varepsilon) \\ \displaystyle\int_{\mathbb{S}^1(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \leq \frac{1}{16}\int_{G_\varepsilon} e_\varepsilon(u_\varepsilon)\mathrm{d}x. \end{cases}$$

Invoking the potential estimate we are led to

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(\varrho_\varepsilon)} V(u_\varepsilon) \leq \frac{\varrho_\varepsilon}{8}\int_{\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \frac{1}{2}\int_{\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x. \tag{15}$$

which combined with previous estimates yield the conclusion.

Thank you for your attention!