The role of defect and splitting in finite generation of extensions of associated graded rings along a valuation.

Abstract. Suppose that R is a 2 dimensional excellent local domain with quotient field K. Further suppose that K^* is a finite separable extension of K and S is a 2 dimensional local domain with quotient field K^* such that S dominates R.

Suppose that ν^* is a valuation of K^* such that ν^* dominates S. Let ν be the restriction of ν^* to K. The associated graded ring $\mathrm{gr}_\nu(R)$ was introduced by Bernard Teissier. We show that the extension $(K,\nu)\rightarrow(K^*,\nu^*)$ is without defect if and only if there exist regular local rings R_1 and S_1 such that R_1 is a local ring of a blow up of R, S_1 is a local ring of a blowup of S, ν^* dominates S_1, S_1 dominates R_1 and the associated graded ring $\mathrm{gr}_{\nu^*}(S_1)$ is a finitely generated $\mathrm{gr}_\nu(R_1)$-algebra.

We also investigate the role of splitting of the valuation ν in K^* in finite generation of the extensions of associated graded rings along the valuation. We will say that ν does not split in S if ν^* is the unique extension of ν to K^* which dominates S. We show that if R and S are regular local rings, ν^* has rational rank 1 and is not discrete and $\mathrm{gr}_{\nu^*}(S)$ is a finitely generated $\mathrm{gr}_\nu(R)$-algebra, then ν does not split in S. We give examples showing that such a strong statement is not true when ν does not satisfy these assumptions. As a consequence of our theorem, we deduce that if ν has rational rank 1 and is not discrete and if $R\rightarrow R'$ is a nontrivial sequence of quadratic transforms along ν, then $\mathrm{gr}_\nu(R')$ is not a finitely generated $\mathrm{gr}_\nu(R)$-algebra.