Linear algebraic groups

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Chapter 1

First definitions and properties

1.1 Algebraic groups

1.1.1 Definitions

In this lectures, we will use basic notions of algebraic geometry. Our main reference for algebraic geometry will be the book [Har77] by R. Hartshorne. We will work over an algebraically closed field k of any characteristic. We will call variety a reduced separated scheme of finite type over k.

The basic definition is the following.

Definition 1.1.1 *An* algebraic group *is a variety G which is also a group and such that the maps defining the group structure $\mu : G \times G \to G$ with $\mu(x, y) = xy$, the multiplication, $i : G \to G$ with $i(x) = x^{-1}$ the inverse and $e_G : \mathrm{Spec}(k) \to G$ with image the identity element e_G of G are morphisms.*

There are several associated definitions.

Definition 1.1.2 *($\imath$) An algebraic group G is* linear *if G is an affine variety.*

($\imath\imath$) A connected algebraic group which is complete is called an abelian variety*.*

($\imath\imath\imath$) A morphism $G \to G'$ of varieties between two algebraic groups which is a group homomorphism is called a homomorphism of algebraic groups*.*

($\imath v$) A closed subgroup *H of an algebraic group G is a closed subvariety of G which is a subgroup.*

Fact 1.1.3 *Let G be an algebraic group and H a closed subgroup, then there is a unique algebraic group structure on H such that the inclusion map $H \to G$ is a morphism of algebraic groups.*

Proof. Exercise. □

Fact 1.1.4 *Let G and G' be two algebraic groups. The product $G \times G'$ with the direct product group structure is again an algebraic group. It is called the* direct product *of the algebraic groups G and G'.*

Proof. Exercise. □

1.1.2 Chevalley's Theorem

One usually splits the study of algebraic groups in two parts: the linear algebraic groups and the abelian varieties. This is because of the following result that we shall not try to prove.

Theorem 1.1.5 *Let G be an algebraic group, then there is a maximal linear algebraic subgroup G_{aff} of G. This subgroup is normal and the quotient $A(G) := G/G_{\mathrm{aff}}$ is an abelian variety. In symbols, we have an exact sequence of algebraic groups:*

$$1 \to G_{\mathrm{aff}} \to G \to A(G) \to 1.$$

Furthermore, the map $G \to A(G)$ is the Albanese map.

Let us now give the following result on abelian varieties.

Theorem 1.1.6 *An abelian variety is a commutative algebraic group.*

From now on we assume that all algebraic groups are affine.

1.1.3 Hopf algebras

Algebraic groups can be defined only by the existence of the morphisms $\mu : G \times G \to G$, $i : G \to G$ and $e_G : \mathrm{Spec}(k) \to G$ such that the following diagrams are commutative. We denote by $\pi : G \to \mathrm{Spec}(k)$ the structural map. In the last diagram, we identified G with $G \times \mathrm{Spec}(k)$ and $\mathrm{Spec}(k) \times G$. If we assume that the algebraic group G is linear, then $G = \mathrm{Spec}\,(A)$ for some finitely generated algebra A that we shall often denote by $k[G]$. The maps μ, i, e_G and π define the following algebra morphisms:$\Delta : A \to A \otimes A$ called the *comultiplication*, $\iota : A \to A$ called the *antipode*, $\epsilon : A \to k$ and $j : k \to A$. Let us furthermore denote by $m : A \otimes A \to A$ the multiplication in the algebra A and recall that the corresponding morphism is the diagonal embedding $\mathrm{Spec}\,(A) \to \mathrm{Spec}\,(A) \times \mathrm{Spec}\,(A)$. The above diagrams translate into the following commutative diagrams.

Definition 1.1.7 *A k-algebra A with morphisms Δ, ι, ϵ, j and m as above is called a* Hopf algebra.

Exercise 1.1.8 Give the meaning of a group morphism is terms of the map μ, i and e_G and its interpretation in terms of Hopf algebras. This will be called a *Hopf algebra morphism*.

1.1.4 Examples

The first basic two examples are $G = \mathbb{A}^1 = k$ and $G = \mathbb{A}^1 \setminus \{0\} = k^\times$.

Example 1.1.9 In the first case we have $k[G] = k[T]$ for some variable T. The comultiplication is $\Delta : k[T] \to k[T] \otimes k[T]$ defined by $\Delta(T) = T \otimes 1 + 1 \otimes T$, the antipode $\iota : k[T] \to k[T]$ is defined by $\iota(T) = -T$ and the map $\epsilon : k[T] \to k$ is defined by $\epsilon(T) = 0$. This group is called the *additive group* and is denoted by $\mathbb{G}_a$.

Example 1.1.10 In the second case we have $k[G] = k[T, T^{-1}]$ for some variable T. The comultiplication is $\Delta : k[T, T^{-1}] \to k[T, T^{-1}] \otimes k[T, T^{-1}]$ defined by $\Delta(T) = T \otimes T$, the antipode $\iota : k[T, T^{-1}] \to k[T, T^{-1}]$ is defined by $\iota(T) = T^{-1}$ and the map $\epsilon : k[T, T^{-1}] \to k$ is defined by $\epsilon(T) = 1$. This group is called the *additive group* and is denoted by $\mathbb{G}_m$ or GL_1.

Example 1.1.11 For n an integer, the $\mathbb{G}_m \to \mathbb{G}_m$ defined by $x \mapsto x^n$ is a group homomorphism. On the Hopf algebra level, it is given by $T \mapsto T^n$ if $k[\mathbb{G}_m] = k[T, T^{-1}]$.

Note that if $\mathrm{char}(k) = p$ and p divides n, then this morphism is bijective by is not an isomorphism.

Example 1.1.12 Consider the algebra $\mathfrak{gl}_n$ of $n \times n$ matrices and let D be the polynomial computing the determinant of a matrix. The vector space $\mathfrak{gl}_n$ can be seen as an affine variety with $k[\mathfrak{gl}_n] = k[(T_{i,j})_{i,j\in[1,n]}]$. The *general linear group* GL_n is the open set of $\mathfrak{gl}_n$ defined by the non vanishing of $\det = D(T_{i,j})$. We thus have $\mathrm{GL}_n = \mathrm{Spec}\ \big(k[(T_{i,j})_{i,j\in[1,n]}, \det^{-1}]\big)$

The comultiplication Δ is given by

$$\Delta(T_{i,j}) = \sum_{k=1}^{n} T_{i,k} \otimes T_{k,j}.$$

The value of $\iota(T_{i,j})$ is the (i,j)-entry in the inverse matrix $(T_{k,l})^{-1}$ or of the matrix $\det^{-1}{}^t\mathrm{Com}(T_{k,l})$ where $\mathrm{Com}(M)$ is the comatrix of M. The map ϵ is given by $\epsilon(T_{i,j}) = \delta_{i,j}$.

Since $\mathfrak{gl}_n$ is irreducible of dimension n^2, so is GL_n.

Exercise 1.1.13 Check that these maps indeed define the well known group structure on GL_n.

Example 1.1.14 Any subgroup of GL_n which is closed for the Zariski topology is again an algebraic group. For example:

- any finite subgroup;
- the group D_n of diagonal matrices;
- the group T_n of upper triangular matrices;
- the subgroup U_n of T_n of matrices with diagonal entries equal to 1;
- the *special linear group* SL_n of matrices with determinant equal to 1;
- the *orthogonal group* $\mathrm{O}_n = \{M \in \mathrm{GL}_n \ /\ {}^tXX = 1\}$;
- the *special orthogonal group* $\mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n$;
- the *symplectic group* $\mathrm{Sp}_{2n} = \{X \in \mathrm{GL}_{2n} \ /\ {}^tXJX = J\}$ with

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

For each simple Lie algebra, there exists at least one associated algebraic group. We shall see that conversely, any linear algebraic group is a closed subgroup of GL_n for some n.

Example 1.1.15 It is already more difficult to give the algebra of the group PGL_n which is the quotient of GL_n by its center $Z(\mathrm{GL}_n) = \mathbb{G}_m$. One can prove for example that PGL_n is the closed subgroup of $\mathrm{GL}(\mathfrak{gl}_n)$ of algebra automorphisms of $\mathfrak{gl}_n$.

Example 1.1.16 As last example, let us give a non linear algebraic group. If X is an elliptic curve then it has a group structure and is therefore the first example of an abelian variety. The group structure is defined via the isomorphism $X \to \mathrm{Pic}^0(X)$ defined by $P \mapsto \mathcal{O}_X(P - P_0)$ where P_0 is a fixed point.

1.2 First properties

1.2.1 Connected components

Proposition 1.2.1 *Let G be an algebraic group.*

($\imath$) There exists a unique irreducible component G^0 of G containing the identity element e_G. It is a closed normal subgroup of G of finite index.

($\imath\imath$) The subgroup G^0 is the unique connected component containing e_G. The connected components and the irreducible components of G coincide.

($\imath\imath\imath$) Any closed subgroup of G of finite index contains G^0.

Proof. *($\imath$)* Let X and Y be two irreducible components of G containing e_G. The product XY is the image of $X \times Y$ by μ and is therefore irreductible as well as its closure $\overline{XY}$. Furthermore X and Y are contained in $\overline{XY}$ (because e_G is in X and in Y). We thus have $X = \overline{XY} = Y$. This proves that there is a unique irreducible component $G^0 = X$ of G containing e_G and that it is stable under multiplication and closed. Therefore G^0 is a closed subgroup. Consider, for $g \in G$, the inner automorphism $\mathrm{Int}(g) : G \to G$ defined by $x \mapsto gxg^{-1}$. We have that $\mathrm{Int}(g)(G^0)$ is irreducible and contains e_G, therefore $\mathrm{Int}(g)(G^0) \subset G^0$ and G^0 is normal.

Note that G^0 being irreducible, it is connected. Let $g \in G$, using the isomorphism $G \to G$ defined by $x \mapsto gx$, we see that the irreducible components of G containing g are in one-to-one correspondence with the irreducible components of G containing e_G. There is a unique one which is gG^0. The irreducible components of G are therefore the G^0 orbits and are thus disjoint. They must coincide with the connected components. Because there are finitely many irreducible components, the group G^0 must have finite index. This proves also *($\imath\imath$)*.

($\imath\imath\imath$) Let H be a closed subgroup of finite index in G. Let H^0 be its intersection with G^0. The quotient G^0/H^0 is a subgroup of G/H therefore finite. Thus H^0 is open and closed in G^0 thus $H^0 = G^0$ and the result follows. □

Remark 1.2.2 Note that the former proposition implie that all the components of the group G have the same dimension.

1.2.2 Image of a group homomorphism

Lemma 1.2.3 *Let U and V be dense open subsets of G, then $UV = G$.*

Proof. Let $g \in G$, then U and gV^{-1} are dense open subset and must meet. Let u be in the intersection, then there exists $v \in V$ with $u = gv^{-1} \in U$ thus $g = uv$. □

Lemma 1.2.4 *Let H be a subgroup of G.*

($\imath$) The closure $\overline{H}$ of H is a subgroup of G.

($\imath\imath$) If H contains a non-empty open subset of $\overline{H}$, then H is closed.

Proof. *($\imath$)* Let $h \in H$, then $hH \subset H \subset \overline{H}$ thus, because $h\overline{H}$ is the closure of hH we have $h\overline{H} \subset \overline{H}$. This gives $H\overline{H} \subset \overline{H}$.

Now let $h \in \overline{H}$, by the last inclusion, we have $Hh \subset \overline{H}$ thus, because $\overline{H}h$ is the closure of Hh we have $\overline{H}h \subset \overline{H}$. This gives $\overline{H}\,\overline{H} \subset \overline{H}$.

Because i is an isomorphism, we have $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$ proving the first part.

($\imath\imath$) If H contains a non-empty open subset U of $\overline{H}$, then $H = \cup_{h \in H} hU$ is open in $\overline{H}$ and by the previous lemma, we have $\overline{H} = HH = H$. □

Proposition 1.2.5 *Let $\phi : G \to G'$ be a morphism of algebraic groups.*

($\imath$) The kernel $\ker\phi$ *is a closed normal subgroup.*

($\imath\imath$) The image $\phi(G)$ is a closed subgroup of G.

($\imath\imath\imath$) We have the equality $\phi(G^0) = \phi(G)^0$.

Proof. *($\imath$)* The kernel is normal and the inverse image of the closed subset $\{e_{G'}\}$ therefore closed.

($\imath\imath$) By Chevalley's Theorem (in algebraic geometry, see [Har77, Exercise II.3.19]), the image $\phi(G)$ contains an open subset of its closure. By the previous lemma, it has to be closed.

($\imath\imath\imath$) G^0 being irreducible, the same is true for $\phi(G^0)$ which is therefore connected and thus contained in $\phi(G)^0$. Furthermore, we have that $\phi(G)/\phi(G^0)$ is a quotient of G/G^0 therefore finite. Thus $\phi(G^0)$ is of finite index in $\phi(G)$ and $\phi(G)^0 \subset \phi(G^0)$. □

1.2.3 Subgroup generated by subvarieties

Proposition 1.2.6 *Let $(X_i)_{i\in I}$ be a family of irreducible varieties together with morphisms $\phi_i : X_i \to G$. Let H be the smallest closed subgroup containing the images $Y_i = \phi_i(X_i)$. Assume that $e_G \in Y_i$ for all $i \in I$.*

($\imath$) Then H is connected.

($\imath\imath$) There exist an integer n, a sequence $a = (a(1), \cdots, a(n)) \in I^n$ and $\epsilon(k) = \pm 1$ for $k \in [1, n]$ such that $H = Y_{a(1)}^{\epsilon(1)} \cdots Y_{a(n)}^{\epsilon(n)}$.

Proof. Let us prove *($\imath\imath$)*, this will imply *($\imath$)* since the Y_i are irreducible.

Enlarging the family, we may assume that $Y_i^{-1} = Y_j$ for some j and we get rid of the signs $\epsilon(k)$. For $a = (a(1), \cdots, a(n))$, let $Y_a = Y_{a(1)} \cdots Y_{a(n)}$. It is an irreducible variety as well as its closure $\overline{Y_a}$. Furthermore, we have by the same argument as is the former lemma the inclusion $\overline{Y_a} \cdot \overline{Y_b} \subset \overline{Y_{(a,b)}}$. Let a be such that $\overline{Y_a}$ is maximal for the inclusion *i.e.* for any b, we have $\overline{Y_a} \cdot \overline{Y_b} \subset \overline{Y_a}$. This is possible because the dimensions are finite. Now $\overline{Y_a}$ is irreducible, closed and closed under taking products. Note that for all b we have $\overline{Y_a} \cdot \overline{Y_b} \subset \overline{Y_a}$ therefore because e_G lies in all Y_i we have $\overline{Y_b} \subset \overline{Y_a}$. Furthermore $\overline{Y_a}^{-1} = \overline{Y_a^{-1}}$ and is the closure of the product $Y_{a(n)}^{-1} \cdots Y_{a(1)}^{-1}$ and thus contained in $\overline{Y_a}$. Therefore $\overline{Y_a}$ is a closed subgroup of G containing the Y_i thus $H \subset \overline{Y_a}$ but obviously $\overline{Y_a} \subset H$ so the result follows. □

Corollary 1.2.7 *($\imath$) If $(G_i)_{i\in I}$ is a family of closed connected subgroups of G, then the subgroup H generated by them is closed and connected. Furthermore, there is an integer n such that $H = G_{a(1)} \cdots G_{a(n)}$.*

Definition 1.2.8 *Let H and K be subgroups of a group G, we denote by (H, K) the subgroup generated by the elements $hkh^{-1}k^{-1}$ (called the commutators).*

Corollary 1.2.9 *If H and K are closed subgroups such that one of them is connected, then (H, K) is closed and connected.*

Proof. Assume that H is connected. This follows from the previous proposition using the family $\phi_k : H \to G$ with $\phi_k(h) = hkh^{-1}k^{-1}$ which is indexed by K. □

1.3 Action on a variety

1.3.1 Definition

Definition 1.3.1 *($\imath$) Let X be a variety with an action of an algebraic group G. Let $a_X : G \times X \to X$ with $a_X(g, x) = g \cdot x$ be the map given by the action. We say that X is a G-*variety *or a G-*space *if $a - X$ is a morphism.*

($\imath\imath$) A G-space with a transitive action of G is called a homogeneous space*.*

($\imath\imath\imath$) A morphism $\phi : X \to Y$ between G-spaces is said to be equivariant *if the following diagram commutes:*

$$\begin{tikzcd} G \times X \arrow[r, "a_X"] \arrow[d, "\mathrm{Id}\times\phi"'] & X \arrow[d, "\phi"] \\ G \times Y \arrow[r, "a_Y"] & Y \end{tikzcd}$$

($\imath v$) Let X be a G-space and $x \in X$. The orbit *of x is the image $G \cdot x = a_X(G \times \{x\})$. The* isotropy group *of x or* stabiliser *of x is the subgroup $G_x = \{g \in G \ / \ g \cdot x = x\}$.*

Exercise 1.3.2 Prove that the stabiliser G_x is the reduced scheme build on the fiber product $G_x = (G \times \{x\}) \times_X \{x\}$.

Example 1.3.3 The group G can be seen as a G-space in several ways. Let $a_G : G \times G \to G$ be defined by $a_G(g, h) = ghg^{-1}$. The orbits are the conjugacy classes while the isotropy subgroups are the centralisers of elements.

Definition 1.3.4 *If X is a homogneous space for the action of G and furthermore all the isotropy subgroups are trivial, then we say that X is a* pricipal homogeneous space *or* torsor*.*

Example 1.3.5 The group G can also act on itself by left (resp. right) translation *i.e.* $a_G : G \times G \to G$ defined by $a(g, h) = gh$ (resp. $a(g, h) = hg$). The action is then transitive and G is a principal homogeneous space for this action.

Example 1.3.6 Let V be a finite dimensional vector space then the map $a_V : \mathrm{GL}(V) \times V \to V$ defined by $a_V(f, v) = f(v)$ defines a $\mathrm{GL}(V)$-space structure on V.

Example 1.3.7 Let V be a finite dimensional vector space and a homomorphism of algebraic group $r : G \to \mathrm{GL}(V)$. Then the map $G \times V \to V$ given by the composition of $r \times \mathrm{Id}$ with the map a_V of the previous example defined a G-space structure on V. We also have a G-structure on $\mathbb{P}(V)$.

Definition 1.3.8 *A morphism of algebraic groups $G \to \mathrm{GL}(V)$ is called a* rational representation *of G in V.*

1.3.2 First properties

Lemma 1.3.9 *Let X be a G-space.*

($\imath$) Any orbit is open in its closure.

($\imath\imath$) There is at least one closed orbit in X.

Proof. ($\imath$) An orbit $G \cdot x$ is the image of G under the morphism $G \to X$ defined by $g \mapsto g \cdot x$. By Chevalley's theorem, we know that $G \cdot x$ contains an open subset U of its closure. But then $G \cdot x = \cup_{g \in G} g \cdot U$ is open in $\overline{G \cdot x}$.

(ıı) Let $G \cdot x$ be an orbit of minimal dimension. It is open in $\overline{G \cdot x}$ therefore $\overline{G \cdot x} \setminus G \cdot x$ is closed of smaller dimension. However it is an union of orbits, therefore it is empty by minimality. □

Let X be a G-space and assume that X is affine. Write $X = \operatorname{Spec} k[X]$. The action $a_X : G \times X \to X$ is given by a map $a_X^\sharp : k[X] \to k[G] \otimes k[X]$. We may define a representation of abstract groups

$$G \xrightarrow{r} \mathrm{GL}(k[X])$$

defined by $(r(g)f)(x) = f(g^{-1}x)$. On the level of algebras, this map is defined as follows. An element $g \in G$ defines a map $ev_g : k[G] \to k$ and we can form the composition

$$r(g) : k[X] \xrightarrow{a_X^\sharp} k[G] \otimes k[X] \xrightarrow{ev_{g^{-1}}} k \otimes k[X] = k[X].$$

Proposition 1.3.10 *Let V be a finite dimensional subspace of $k[X]$.*

(ı) There is a finite dimensional subspace W of $k[X]$ which contains V and is stable under the action of $r(g)$ for all $g \in G$.

(ıı) The subspace V is stable under $r(g)$ for all $g \in G$ if and only if we have $a_X^\sharp(V) \subset k[G] \otimes V$. In that case the map $r_V : G \times V \to V$ defined by $(g, f) \mapsto (ev_g \otimes \mathrm{Id}) \circ a_X^\sharp(f)$ is a rational representation.

Proof. *(ı)* It is enough to prove this statement for V of dimension one. So let us assume that V is spanned by an element $f \in k[X]$. Let us write

$$a_X^\sharp(f) = \sum_{i=1}^{n} v_i \otimes f_i$$

with $v_i \in k[G]$ and $f_i \in k[X]$. For any $g \in G$, we have

$$r(g)f = \sum_{i=1}^{n} v_i(g) f_i$$

therefore for all $g \in G$, the element $r(g)f$ is contained in the finite dimensional vector subspace of $k[X]$ spanned by the elements $(f_i)_{i \in [1,n]}$. Therefore the span W of the elements $r(g)f$ for all $g \in G$ is finite dimensional. This span is obviously spable under the action of $r(g)$ for all $g \in G$ since $r(g)r(g')f = r(gg')f$.

(ıı) Assume that V is stable by $r(g)$ for all $g \in G$. Let us fix a base $(f_i)_{i \in [1,n]}$ of V and complete it with the elements $(g_j)_j$ to get a base of $k[X]$. Let $f \in V$ and write

$$a_X^\sharp(f) = \sum_{i=1}^{n} v_i \otimes f_i + \sum_j u_j \otimes g_j$$

with $v_i, u_j \in k[G]$. If for all $g \in G$ we have $r(g)f \in V$, then for all $g \in G$, we have $u_j(g^{-1}) = 0$ thus $u_j = 0$ thus $a_X^\sharp(V) \subset k[G] \otimes V$.

Conversely, if $a_X^\sharp(V) \subset k[G] \otimes V$, then we may write

$$a_X^\sharp(f) = \sum_{i=1}^{n} v_i \otimes f_i$$

with $v_i \in k[G]$ and $f_i \in V$. For any $g \in G$, we have

$$r(g)f = \sum_{i=1}^{n} v_i(g) f_i \in V$$

and the result follows. □

1.3.3 Affine algebraic groups are linear

In this section we consider the action of G on itself by left and right multiplication. Let us fix some notation. We denote by λ and ρ the representations of G in $\mathrm{GL}(k[G])$ induced by left and right action. That is to say, for $g \in G$, we define $\lambda(g) : k[G] \to k[G]$ and $\rho(g) : k[G] \to k[G]$. Explicitly, for $h \in G$ and for $f \in k[G]$, we have

$$(\lambda(g)f)(x) = f(g^{-1}x) \text{ and } (\rho(g)f)(x) = f(xg).$$

Exercise 1.3.11 If $\iota : k[G] \to k[G]$ is the antipode isomorphism, then, for all $g \in G$, we have the equality $\rho(g) = \iota \circ \lambda(g) \circ \iota^{-1}$.

Lemma 1.3.12 *The representations λ and ρ are faithful.*

Proof. We only deal with λ, the proof with ρ is similar or we can use the former exercise. Let us assume that $\lambda(g) = e_{\mathrm{GL}(k[G])}$. Then $\lambda(g)f = f$ for all $f \in k[G]$. Therefore, for all $f \in k[G]$ we have $f(g^{-1}e_G) = f(e_G)$. This implies $g^{-1} = e_G$. □

Theorem 1.3.13 *Any linear algebraic group is a closed subgroup of* GL_n *for some* n.

Proof. Let V be a finite dimensional subspace of $k[G]$ which spans $k[G]$ as an algebra. By Proposition 1.3.10, there exists a finite dimensional subspace W containing V and stable under the action of $\lambda(g)$ for all $g \in G$. Let us choose a basis $(f)i)_{i\in[1,n]}$ of W. Because W is stable, again by Proposition 1.3.10, we may write

$$a^\sharp_W(f_i) = \sum_{j=1}^n m_{i,j} \otimes f_j$$

with $a^\sharp_W : W \to k[G] \otimes W$ associated to the action λ_W and $m_{i,j} \in k[G]$. We may define the following morphism

$$\phi^\sharp : k[\mathrm{GL}_n] = k[(T_{i,j})_{i,j\in[1,n]}, \det{}^{-1}] \to k[G]$$

by $T_{i,j} \mapsto m^{j,i}$ and $\det^{-1} \mapsto \det(m_{j,i})$ where here $m^{i,j}$ are the coefficients of the inverse of $(m_{i,j})$. On the level of points, this defines a morphism $\phi : G \to \mathrm{GL}_n$ given by $g \mapsto (m_{j,i}(g^{-1}))_{i,j\in[1,n]}$. Note that because $\lambda(gg')f = \lambda(g)\lambda(g')f$ we easily get that this map is a group morphism. We thus have a morphism of algebraic groups $\phi : G \to \mathrm{GL}_n$. Furthermore the image of $\phi^\sharp$ contains the elements f_i which generate $k[G]$ therefore $\phi^\sharp$ is surjective and ϕ is an embedding. □

Lemma 1.3.14 *Let H be a closed subgroup of G and let I_H be its ideal in $k[G]$. Then we have the equalities:*

$$H = \{g \in G \ / \ \lambda(g)I_H = I_H\} = \{g \in G \ / \ \rho(g)I_H = I_H\}.$$

Proof. It is enough to prove it for λ. Let $g \in G$ with $\lambda(g)I_H = I_H$, then for all $f \in I_H$, we have $f(g^{-1}) = \lambda(g)f(e - G) = 0$ since $\lambda(g)f \in I_H$ and $e - G \in H$. Therefore $g^{-1} \in H$.

Conversely if $g \in H$, let $f \in I_H$ and $h \in H$. We have $\lambda(g)f(h) = f(g^{-1}h) = 0$ since $g^{-1}h \in H$. Therefore $\lambda(g)f \in I_H$. □

Chapter 2

Tangent spaces and Lie algebras

In this chapter we define tangent spaces for algebraic varieties and apply the definition to linear algebraic groups. This enables one to define the Lie algebra of an algebraic group.

2.1 Derivations and tangent spaces

2.1.1 Derivations

Definition 2.1.1 *Let R be a commutative ring, A be an R algebra and M be an A-module. An R-*derivation *of A in M is a linear map $D : A \to M$ such that for all $a, b \in A$ we have:*

$$D(ab) = aD(b) + D(a)b.$$

The set of all such derivations is denoted by $\mathrm{Der}_R(A, M)$.

Remark 2.1.2 (i) We have the equality $D(1) = 0$ thus for all $r \in R$ we have $D(r) = 0$.

(ii) The set $\mathrm{Der}_R(A, M)$ is a A-module: if D and D' are derivations, then so is $D + D'$ and if $a \in A$, then aD is again a derivation.

Exercise 2.1.3 Prove the assertion of the former remark.

Let $\phi : A \to B$ be a morphism of R-algebras and let $\psi : M \to N$ be a morphism of B-modules. This is also a morphism of A-modules.

Proposition 2.1.4 *(i) The map $\mathrm{Der}_R(B, M) \to \mathrm{Der}_R(A, M)$ defined by $D \mapsto D \circ \phi$ is well defined, it is a morphism a A-modules and its kernel is $\mathrm{Der}_A(B, M)$.*

(ii) The map $\mathrm{Der}_R(A, M) \to \mathrm{Der}_R(A, N)$ defined by $D \mapsto \psi \circ D$ is well defined, it is a morphism a A-modules.

(iii) Let S be a multiplicative subset of A and M an $S^{-1}A$-module, then we have a natural isomorphism $\mathrm{Der}_R(S^{-1}A, N) \to \mathrm{Der}_R(A, N)$.

(iv) Let A_1 and A_2 be two R-algebras, let $A = A_1 \otimes_R A_2$ and let M ne an A-module, then $\mathrm{Der}_R(A, M) \simeq \mathrm{Der}_R(A_1, M) \oplus \mathrm{Der}_R(A_2, M)$.

Proof. Exercice. The map in *(iv)* is given by $(D_1, D_2) \mapsto D$ with $D(a \otimes a') = D_1(a)a' + aD_2(a')$ $\square$

2.1.2 Tangent spaces

Definition 2.1.5 *Let X be an algebraic variety and let $x \in X$. The* tangent space *of X at x is the vector space* $\mathrm{Der}_k(\mathcal{O}_{X,x}, k(x))$ *(where $k(x) = \mathcal{O}_{X,x}/\mathfrak{M}_{X,x}$). We denote it by T_xX.*

Fact 2.1.6 *Let X be an affine variety , then $T_xX = \mathrm{Der}_k(k[X], k(x))$.*

Proof. Indeed this is an application of Proposition 2.1.4 *($\imath\imath\imath$)*. □

Fact 2.1.7 *Let $x \in X$ and U an open subvariety of X containing x, then $T_xU = T_xX$.*

Proof. This is simply because $\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$. □

Lemma 2.1.8 *($\imath$) Let $\phi : X \to Y$ be a morphism of algebraic varieties, then there exists a linear map $d_x\phi : T_xX \to T_{f(x)}Y$. This map is called the* differential of ϕ *at x.*

($\imath\imath$) Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms, then we have the equality $d_x(\psi\circ\phi) = d_{f(x)}\psi\circ d_x\phi$.

($\imath\imath\imath$) If $\phi : X \to Y$ is an isomorphism or the identity, then so is $d_x\phi$.

($\imath v$) If $\phi : X \to Y$ is a constant map, then $d_x\phi = 0$ for any $x \in X$.

Proof. *($\imath$)* It suffices to define $d_x\phi : \mathrm{Der}_k(\mathcal{O}_{X,x}, k) \to \mathrm{Der}_k(\mathcal{O}_{Y,f(x)}, k)$ by $D \mapsto D \circ \phi^\sharp$ and to apply Proposition 2.1.4.

($\imath\imath$) We have $d_x(\psi \circ \phi)(D) = D \circ (\psi \circ \phi)^\sharp = D \circ \phi^\sharp \circ \psi^\sharp = d_{f(x)}\psi(d_x\phi(D))$.

($\imath\imath\imath$) The inverse is $d_{\phi(x)}\phi^{-1}$.

($\imath v$) The map factors through $\mathrm{Spec}\, k$ whose tangent space is the zero space. Therefore the differential factors through the zero space. □

Lemma 2.1.9 *We have an isomorphism $T_xX \simeq (\mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x})^\vee$.*

Proof. Let us define a map $\pi : T_xX \to (\mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x})^\vee$ by $\pi(D)(m) = D(m)$ where $D \in \mathrm{Der}_k(\mathcal{O}_{X,x}, k)$ and $m \in \mathfrak{M}_{X,x}$. To check that this is well defined we need to prove that $D(\mathfrak{M}^2_{X,x}) = 0$. But for $m, m' \in \mathfrak{M}_{X,x}$, we have $D(mm') = \bar{m}D(m') + D(m)\bar{m}'$ with $\bar{a}$ the class $a \in \mathcal{O}_{X,x}$ in k. Thus $\bar{m} = \bar{m}' = 0$ and $D(\mathfrak{M}^2_{X,x}) = 0$.

Conversely, if $f \in (\mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x})^\vee$, let us define $D_f \in \mathrm{Der}(\mathcal{O}_{X,x}, k)$ by $D_f(a) = f(a - \bar{a})$. This is obviously k-linear and for $a, b \in \mathcal{O}_{X,x}$, we have $D_f(ab) = f(ab-\bar{a}\bar{b}) = f((a-\bar{a})(b-\bar{b})+\bar{a}(b-\bar{b})+\bar{b}(a-\bar{a}))$. But $(a-\bar{a})(b-\bar{b}) \in \mathfrak{M}^2_{X,x}$ thus $D_f(ab) = \bar{a}f(b-\bar{b})+\bar{b}f(a-\bar{a}) = aD(b)+D(a)b$ *i.e.* D_f is a derivation.

Finally we check $\pi(D_f)(m) = D_f(m) = f(m - \bar{m}) = f(m)$, thus $\pi(D_f) = f$. And we check $D_{\pi(D)}(a) = \pi(D)(a - \bar{a}) = D(a - \bar{a}) = D(a)$ because $D|_k = 0$. □

Fact 2.1.10 *Let $\phi : X \to Y$. Under the above identification, the differential $d_x\phi : T_xX \to T_{f(x)}Y$ is given by the transpose of the map $\phi^\sharp : \mathfrak{M}_{Y,f(x)}/\mathfrak{M}^2_{Y,f(x)} \to \mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x}$*

Proof. Exercise. □

Definition 2.1.11 *The* cotangent space *of X at x is $\mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x}$. It is isomorphic to $(T_xX)^\vee$.*

Lemma 2.1.12 *Let $\phi : X \to Y$ be a closed immersion, then $d_x\phi$ is injective for any $x \in X$. Therefore we may identify the tangent space T_xX with a subspace of $T_{\phi(x)}Y$.*

Proof. We may assume that X and Y are affine and $k[X] = k[Y]/I$. We then have the equality $\mathfrak{M}_{X,x} = \mathfrak{M}_{\phi(x),Y}/I$ and a surjection

$$T_{\phi(x)}T^\vee = \mathfrak{M}_{\phi(x),Y}/\mathfrak{M}^2_{\phi(x),Y} \to \mathfrak{M}_{\phi(x),Y}/(\mathfrak{M}^2_{\phi(x),Y} + I) \simeq \mathfrak{M}_{x,X}/\mathfrak{M}^2_{x,X} = T_xX^\vee$$

giving the result by duality. □

Proposition 2.1.13 *Let X be a closed subvariety of k^n and let I be the defining ideal of X. Assume that I is generated by the elements $f_1, \cdots, f_r$. Then for all $x \in X$, we have the equality*

$$T_xX = \bigcap_{k=1}^r \ker d_xf_k = \left\{ v \in k^n \ / \ \sum_{i=1}^n v_i \frac{\partial f_k}{\partial x_i}(x) = 0 \textit{ for all } k \in [1,r] \right\}.$$

Proof. Let $\mathfrak{M}_{k^n,x} = (X_i - x_i)_{i\in[1,n]}$ be the ideal of x in $k[k^n] = k[(X_i)_{i\in[1,n]}]$ and let $\mathfrak{M}_{X,x}$ be its image in $k[X]$. We have the equality

$$T_xX^\vee = \mathfrak{M}_{X,x}/\mathfrak{M}^2_{X,x} = \mathfrak{M}_{k^n,x}/(\mathfrak{M}^2_{k^n,x} + I).$$

But for any polynomial $P \in k[(X_i)_{i\in[1,n]}]$, we have the equality

$$P = P(x) + \sum_{i=1}^n \frac{\partial P}{\partial x_i}(x)(X_i - x_i) \mod \mathfrak{M}^2_{k^n,x}.$$

Let us define $\delta_x P = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(x)(X_i - x_i)$, we have the equality

$$T_xX^\vee = \mathfrak{M}_{k^n,x}/(\mathfrak{M}^2_{k^n,x}, (\delta_x f_i)_{i\in[1,n]}).$$

By duality this gives the result. □

Proposition 2.1.14 *Let $\phi : X \times Y \to Z$ be a morphism and let $x \in X$ and $y \in Y$. Then we have an isomorphism $T_{(x,y)}X \times Y \simeq T_xX \oplus T_yY$. Furthermore, modulo this isdentification we have an equality*

$$d_{(x,y)}\phi = d_x\phi_y + d_y\phi_x$$

with $\phi_x : Y \to Z$ defined by $\phi_x(y) = \phi(x,y)$ and $\phi_y : X \to Z$ defined by $\phi_y(x) = \phi(x,y)$.

Proof. Exercice. □

Corollary 2.1.15 *For G an algebraic group, we have the formulas: $d_{(e_G,e_G)}\mu(X,Y) = X + Y$ and $d_{e_G}i(X) = -X$.*

Proof. With notation as in the former proposition, we have $\mu_x(y) = xy = \mu_x(y)$. If the point $(x,y) = (e_G, e_G)$, then $\mu_x = \mu_y = \mathrm{Id}_G$. This gives the first formula.

The map $\mu \circ (\mathrm{Id}, i)$ is the constant map $G \to \mathrm{Spec}(k)$ defined by $g \mapsto e_G$. Its differential at e_G must vanish but also equals $\mathrm{Id} + d_{e_G}i$ giving the result. □

2.1.3 Distributions

Let X be a variety and $x \in X$. We have a direct sum $\mathcal{O}_{X,x} = k1 \oplus \mathfrak{M}_{X,x}$ thus we may identify $\mathfrak{M}_{X,x}^\vee$ as the subspace of $\mathcal{O}_{X,x}^\vee$ of linear forms ϕ with $\phi(1) = 0$. In symbols

$$\mathfrak{M}_{X,x}^\vee \simeq \{\phi \in \mathcal{O}_{X,x}^\vee \ / \ \phi(1) = 0\}.$$

Definition 2.1.16 *($\imath$) For n a non negative integer, we define the following vector spaces:*

$$\mathrm{Dist}_n(X,x) = \{\phi \in \mathcal{O}_{X,x}^\vee \ / \ \phi(\mathfrak{M}_{X,x}^{n+1}) = 0\} \simeq (\mathcal{O}_{X,x}/\mathfrak{M}_{X,x}^{n+1})^\vee.$$

$$\mathrm{Dist}_n^+(X,x) = \{\phi \in \mathrm{Dist}_n(X,x) \ / \ \phi(1) = 0\} \simeq (\mathfrak{M}_{X,x}/\mathfrak{M}_{X,x}^{n+1})^\vee.$$

($\imath\imath$) We set

$$\mathrm{Dist}(X,x) = \bigcup_n \mathrm{Dist}_n(X,x) \quad \text{and} \quad \mathrm{Dist}^+(X,x) = \bigcup_n \mathrm{Dist}_n^+(X,x).$$

The elements of $\mathrm{Dist}(X,x)$ *are called the distributions of X with support in x.*

Remark 2.1.17 We have the identification $\mathrm{Dist}_1^+(X,x) = T_xX$. The distributions are an algebraic version of the higher order differential operators on a differential manifold.

Lemma 2.1.18 *Let $f : X \to Y$ be a morphism and $x \in X$. Then ${}^tf^\sharp$ maps* $\mathrm{Dist}(X,x)$, $\mathrm{Dist}^+(X,x)$, $\mathrm{Dist}_n(X,x)$ *and* $\mathrm{Dist}_n^+(X,x)$ *to* $\mathrm{Dist}(Y,f(x))$, $\mathrm{Dist}^+(Y,f(x))$, $\mathrm{Dist}_n(Y,f(x))$ *and* $\mathrm{Dist}_n^+(Y,f(x))$ *respectively.*

In particular ${}^tf^\sharp$ is a generalisation of the differential map and we shall denote it also by d_xf.

Proof. Recall that ${f^\sharp}^{-1}(\mathfrak{M}_{x,X}) = \mathfrak{M}_{f(x),Y}$. In particular $f^\sharp(\mathfrak{M}_{f(x),Y}^n) \subset \mathfrak{M}_{x,X}^n$ giving the result. □

Fact 2.1.19 *Let $x \in X$ and U an open subvariety of X containing x, then* $\mathrm{Dist}(U,x) = \mathrm{Dist}(X,x)$.

Proof. Exercise. □

Fact 2.1.20 *Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms, then we have the equality $d_x(\psi \circ \phi) = d_{f(x)}\psi \circ d_x\phi$ on the level of distributions.*

Proof. Exercise. □

2.2 Lie algebra of an algebraic group

2.2.1 Lie algebra

Recall the definition of a Lie algebra. We shall assume the reader familiar with this notion and we refer to the classical text books like [Bou60] or [Hum72] for further information.

Definition 2.2.1 *A* Lie algebra $\mathfrak{g}$ *is a vector space together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:*

- $[x,x] = 0$ *and*

- $[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$ *for all* x,y,z *in* $\mathfrak{g}$.

Remark 2.2.2 The last condition is called the Jacobi identity. It is equivalent to saying that the map $\mathrm{ad}\,(x):\mathfrak{g}\to\mathfrak{g}$ defined by $\mathrm{ad}\,(x)(y)=[x,y]$ is a derivation of the algebra $\mathfrak{g}$ *i.e.* to the equality $\mathrm{ad}\,(x)([y,z])=[\mathrm{ad}\,(x)(y),z]+[y,\mathrm{ad}\,(x)(z)]$ for all x,y,z in $\mathfrak{g}$.

Example 2.2.3 The basic example is obtained from an associative algebra A by setting $[a,b]=ab=ba$.

Example 2.2.4 If A is an associative algebra and $\mathrm{Der}_k(A)=\{D\in\mathrm{End}_k(A)\ /\ D(ab)=aD(b)+D(a)b\}$. Then $\mathrm{Der}_k(A)$ with the bracket $[D,D']=D\circ D'-D'\circ D$ is a Lie algebra.

Definition 2.2.5 *A* morphism of Lie algebra *is a linear map* $\phi:\mathfrak{g}\to\mathfrak{g}'$ *such that* $\phi([x,y])=[\phi(x),\phi(y)]$ *for all* x,y *in* $\mathfrak{g}$.

Definition 2.2.6 *A representation of a Lie algebra* $\mathfrak{g}$ *in a vector space* V *is a morphism of Lie algbra* $\mathfrak{g}\to\mathfrak{gl}(V)=\mathrm{End}_k(V)$ *where* $\mathfrak{gl}(V)$ *has the Lie structure associated to the commutators.*

2.2.2 Invariant derivations

Recall that we defined left and right actions λ and ρ of a linear algebraic group G on it algebra of functions $k[G]$. Note that we have the formulas:

$$\lambda(g)(ff')=(\lambda(g)f)\lambda(g)f')\text{ and }\rho(g)(ff')=(\rho(g)f)\rho(g)f').$$

These actions induce actions of G on $\mathfrak{gl}(k[G])=\mathrm{End}_k(k[G])$ by conjugation: for $F\in\mathfrak{gl}(k[G])$ and $g\in G$, we set $\lambda(g)\cdot F=\lambda(g)F\lambda(g)^{-1}$ and $\rho(g)\cdot F=\rho(g)F\rho(g)^{-1}$

Fact 2.2.7 *The left and right actions of* G *on* $\mathfrak{gl}(k[G])$ *preserve the subspace of derivations.*

Proof. Exercise. □

Fact 2.2.8 *The subspace* $\mathrm{Der}_k(k[G])^{\lambda(G)}$ *of invariant derivations for the left action is a Lie subalgebra of* $\mathrm{Der}_k(k[G])$.

Proof. Exercise. □

Definition 2.2.9 *The* Lie algebra $L(G)$ of the group G *is* $\mathrm{Der}_k(k[G])^{\lambda(G)}$.

Recall that we denote by ϵ the map $e_G^\sharp:k[G]\to k$.

Proposition 2.2.10 *The map* $L(G)\to T_{e_G}G$ *defined by* $D\mapsto\epsilon\circ D$ *is an isomorphism.*

Proof. Let us first remark that we have the following equalities $L(G)=\mathrm{Der}_k(k[G],k[G])^{\lambda(G)}$ and $T_{e_G}G=\mathrm{Der}_k(k[G],k(e_G))$. Let us define the inverse map as follows. For $\delta\in\mathrm{Der}_k(k[G],k(e_G))$, define $D_\delta\in L(G)$ by $D_\delta(f)(x)=\delta(\lambda(x^{-1})f)$. Note that we could also define D_δ by the composition $(\mathrm{Id}\otimes\delta)\circ\Delta:k[G]\to k[G]$.

We first check that D_δ is a derivation: $D_\delta(fg)(x)=\delta(\lambda(x^{-1})fg)=\delta((\lambda(x^{-1})f)(\lambda^{-1}g))=f(x)\delta(\lambda(x^{-1}g)+\delta(\lambda(x^{-1}f)g(x)=f(x)D_\delta(g)(x)+D_\delta(f)(x)g(x)$.

We then check that D_δ is invariant *i.e.* $\lambda(g)D_\delta(f)=D_\delta\lambda(g)(f)$ for all $f\in k[G]$. But we have $\lambda(g)D_\delta(f)(x)=\delta(\lambda((g^{-1}x)^{-1})f)=\delta(\lambda(x^{-1})\lambda(g)f)$ while $D_\delta\lambda(g)(f)(x)=\delta(\lambda(x^{-1})\lambda(g)f)$.

Now we check that these maps are inverse to each other. On the one hand, we have $\epsilon\circ D_\delta(f)=D_\delta(f)(e_G)=\delta(\lambda(e_G^{-1})f)=\delta(f)$. On the other hand, for D invariant, we have $D_{\epsilon\circ D}(f)(x)=\epsilon\circ D(\lambda(x^{-1})f)=\epsilon\circ\lambda(x^{-1})(D(f))=D(f)(x)$. □

Remark 2.2.11 The tangent space $T_{e_G}G$ is thus endowed with a Lie algebra structure comming from the Lie algebra structure on $L(G)$.

2.2.3 The distribution algebra

Let us denote by $\mathrm{Dist}(G)$ the algebra of distributions at the origin *i.e.* $\mathrm{Dist}(G) = \mathrm{Dist}(G, e_G)$. We will realise the Lie algebra $\mathfrak{g} = L(G)$ of G as a subalgebra of a natural algebra structure on $\mathrm{Dist}(G)$. Let us first define such an algebra structure.

Theorem 2.2.12 *The space* $\mathrm{Dist}(G) = \bigcup_{n\geq 0} \mathrm{Dist}_n(G)$ *has a structure of filtered associative algebra* i.e. *we have* $\mathrm{Dist}_r(G)\mathrm{Dist}_s(G) \subset \mathrm{Dist}_{r+s}(G)$.

Proof. The Hopf algebra structure on $k[G]$ will give us the algebra structure on $\mathrm{Dist}(G)$. Indeed, let us write $k[G] = k1 \oplus \mathfrak{M}_e$ where $\mathfrak{M}_e$ is the maximal ideal corresponding to e_G. We then have

$$k[G] \otimes k[G] = k \cdot 1 \otimes 1 \oplus (\mathfrak{M}_e \otimes k[G] + k[G] \otimes \mathfrak{M}_e).$$

But we have $(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta$, thus we have $(\epsilon \otimes \epsilon)\Delta(\phi) = \phi(e)$ for any $\phi \in k[G]$. In particular, we get

$$\Delta(\mathfrak{M}_e) \subset \mathfrak{M}_e \otimes k[G] + k[G] \otimes \mathfrak{M}_e.$$

Because Δ is an algebra morphism, we deduce:

$$\Delta(\mathfrak{M}_e^n) \subset \sum_{i+j=n} \mathfrak{M}_e^i \otimes \mathfrak{M}_e^j.$$

In particular, for all r and s, the map Δ induces an algebra morphism

$$\Delta_{r,s} : k[G]/\mathfrak{M}_e^{r+s+1} \to k[G]/\mathfrak{M}_e^{r+1} \otimes k[G]/\mathfrak{M}_e^{s+1}.$$

Let $\eta \in \mathrm{Dist}_r(G)$ and $\xi \in \mathrm{Dist}_s(G)$. These are maps $\eta : k[G]/\mathfrak{M}_e^{r+1} \to k$ and $\xi : k[G]/\mathfrak{M}_e^{s+1} \to k$. We can therefore define a product

$$\eta\xi = (\eta \otimes \xi)\Delta_{r,s}.$$

This does not depend on r and s because if $t \geq r$ and $u \geq s$, then we have the commutative diagram:

$$\begin{tikzcd} k[G]/\mathfrak{M}_e^{t+u+1} \arrow[r, "\Delta_{t,u}"] \arrow[d] & k[G]/\mathfrak{M}_e^{t+1} \otimes k[G]/\mathfrak{M}_e^{u+1} \arrow[d] \\ k[G]/\mathfrak{M}_e^{r+s+1} \arrow[r, "\Delta_{r,s}"] & k[G]/\mathfrak{M}_e^{r+1} \otimes k[G]/\mathfrak{M}_e^{s+1}. \end{tikzcd}$$

The coassociativity of Δ implies that this product is associative. Furthermore, the equalities $(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta$ imply that $\epsilon \in \mathrm{Dist}_0(G) = k\epsilon$ is a unit. ☐

There is a natural Lie algebra struture on $\mathrm{Dist}(G)$, the Lie algebra structure associated to the algebra structure: $[\eta, \xi] = \eta\xi - \xi\eta$.

Theorem 2.2.13 *The subspace* $\mathrm{Dist}_1^+(G) = \{\eta \in k[G]^\vee \ / \ \eta(\mathfrak{M}_e^2) = 0$ *and* $\eta(1) = 0\}$ *is stable under the Lie bracket and therefore a Lie subalgebra.*

Proof. Let η and ξ be in $\mathrm{Dist}_1^+(G)$. By the previous statement, we already know that $\eta\xi$ and $\xi\eta$ lie in $\mathrm{Dist}_2(G)$. We have to prove that their difference vanishes on $\mathfrak{M}_e^2$. Let us first make the following computation:

$$\eta\xi(\phi\psi) = (\eta \otimes \xi)\Delta(\phi\psi) = (\eta \otimes \xi)(\Delta(\phi)\Delta(\psi)).$$

Because of the equalities $(\mathrm{Id} \otimes \epsilon) \circ \Delta = \mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta$, we get

$$\Delta(\phi) - 1 \otimes \phi - \phi \otimes 1 \in \mathfrak{M}_e \otimes \mathfrak{M}_e$$

and the same for ψ (check the vanishing on elements of the form (e_G, g) and (g, e_G)). We thus have the equality

$$\Delta(\phi)\Delta(\psi) = 1 \otimes \phi\psi + \phi\psi \otimes 1 + \phi \otimes \psi + \psi \otimes \phi \ (\mathrm{mod}\ (\mathfrak{M}_e \otimes \mathfrak{M}_e)^2).$$

Because $\eta(1) = \xi(1) = 0$, because $(\mathfrak{M}_e \otimes \mathfrak{M}_e)^2 \subset \mathfrak{M}_e^2 \otimes \mathfrak{M}_e^2$ and $\eta(\mathfrak{M}_e^2) = \xi(\mathfrak{M}_e^2) = 0$, we get the equality:

$$\eta\xi(\phi\psi) = \eta(\phi)\xi(\psi) + \eta(\psi)\xi(\phi).$$

This is symetric thus $[\eta, \xi](\phi\psi) = 0$ and the result follows. □

Recall that we proved that $\mathrm{Dist}_1^+(G)$ is isomorphic to the tangent space $T_{e_G}(G)$. We thus defined two Lie algebra structures on this space. They agree.

Proposition 2.2.14 *The map* $\mathrm{Dist}_1^+(G) \to L(G)$ *defined by* $\delta \mapsto D_\delta$ *is a Lie algebra isomorphism.*

Proof. Recall the definition of D_δ. We have $D_\delta(f)(x) = \delta(\lambda(x^{-1})f)$. We already checked that this is well defined and bijective and that its inverse is given by $D \mapsto \epsilon \circ D$. We only need to check that $D_{[\eta,\xi]} = [D_\eta, D_\xi]$ or for the inverses $[\eta, \xi] = \epsilon \circ [D_\eta, D_\xi]$. But we have seen (check again) the equality $D_\delta = (\mathrm{Id} \otimes \delta) \circ \Delta$. Let us write $\Delta(f) = \sum_i u_i \otimes v_i$. We get

$$\epsilon \circ D_\eta \circ D_\xi(f) = \eta \circ (\mathrm{Id} \times \xi) \circ \Delta(f) = \eta(\sum_i u_i\xi(v_i) = \sum_i \eta(u_i)\xi(v_i).$$

On the other hand, we have

$$\eta\xi(f) = (\eta \otimes \xi) \circ \Delta(f) = \sum_i \eta(u_i)\xi(v_i).$$

The result follows. □

Proposition 2.2.15 *Let* $\phi : G \to H$ *be a morphism of algebraic groups, then* $d\phi : \mathrm{Dist}(G) \to \mathrm{Dist}(H)$ *is a Lie algebra morphism. In particular, the map* $L(G) \to L(H)$ *is a Lie algebra morphism.*

Proof. The map $d\phi$ is given by $\delta \mapsto \delta \circ \phi^\sharp$. But ϕ being a morphism of algebraic groups, we have $\phi^\sharp \circ \Delta_H = \Delta_G \phi^\sharp$. Thus we have $d\phi(\eta)d\phi(\xi) = (\eta \circ \phi^\sharp \otimes \xi \circ \phi^\sharp) \circ \Delta_H = (\eta \otimes \xi) \circ \Delta_G \circ \phi^\sharp = \eta\xi \circ \phi^\sharp = d\phi(\eta\xi)$. This prove the result. □

Corollary 2.2.16 *If* H *is a closed algebraic subgroup of* G*, then* $L(H)$ *is a Lie aubalgebra of* $L(G)$.

Corollary 2.2.17 *We have the equalities* $\mathrm{Dist}(G) = \mathrm{Dist}(G^0)$ *and* $L(G) = L(G^0)$.

2.2.4 Envelopping algebra

We recall here the very definition and first properties of the envelopping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. For simplicity we shall assume that $\mathfrak{g}$ is finite dimensional.

Definition 2.2.18 *Let $\mathfrak{g}$ be a Lie algebra, its envelopping algebra is the quotient $U(\mathfrak{g}) = T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]; x, y \in \mathfrak{g})$.*

The universal envelopping algebra is the solution of the following universal problem. Let $\tau : \mathfrak{g} \to U(\mathfrak{g})$ be the natural map.

Proposition 2.2.19 *($\imath$) Let A be an associative algebra an let $\phi : \mathfrak{g} \to A$ be a Lie algebra morphism (where the Lie bracket on A is $[a, b] = ab - ba$), then there exists a unique algebra morphism $\Phi : U(\mathfrak{g}) \to A$ such that $\phi = \Phi \circ \tau$.*

($\imath\imath$) As a consequence, we have an equivalence of categories between $\mathrm{Rep}(\mathfrak{g})$ the categorie of Lie algebra representations of $\mathfrak{g}$ and $\mathrm{Mod}(U(\mathfrak{g}))$ the category of $U(\mathfrak{g})$-modules.

We also have the following result.

Theorem 2.2.20 (Poincaré-Birkhoff-Witt) *Let $(x_i)_{i \in [1,n]}$ be a base of $\mathfrak{g}$, then the ordered monomials $x_1^{\nu_1} \cdots x_n^{\nu_n}$ form a basis of $U(\mathfrak{g})$. In particular, the map $\tau : \mathfrak{g} \to U(\mathfrak{g})$ is injective.*

Corollary 2.2.21 *The isomorphism $L(G) \simeq \mathrm{Dist}_1^+(G)$ induces by the universal property a morphism of algebras $U(L(G)) \to \mathrm{Dist}(G)$ whose image is the Lie subalgebra generated by $\mathrm{Dist}_1^+(G)$.*

Example 2.2.22 For $G = \mathbb{G}_a$, we have $k[G] = k[T]$ and let $\eta_i \in \mathrm{Dist}_i(G)$ be defined by $\eta_i(T^j) = \delta_{i,j}$. The $(\eta_i)_i$ form a base for $\mathrm{Dist}(G)$ and we have

$$\eta_i \eta_j = \binom{i+j}{i} \eta_{i+j}.$$

We have $U(L(G)) = k[\eta_1]$ and the map $U(L(G)) \to \mathrm{Dist}(G)$ sends η_1^n to $n!\eta_n$. This is an isomorphism for $\mathrm{char} k = 0$ but its image is spanned by the η_i for $i \in [0, p-1]$ for $\mathrm{char} k = p$.

2.2.5 Examples

Let us first compute the Lie algebra of GL_n.

Proposition 2.2.23 *The Lie algebra of GL_n is $\mathfrak{gl}_n$* i.e. *the vector space of $n \times n$ matrices with the natural Lie algebra structure given by associative algebra structure of matrix multiplication.*

Proof. Let $T_{i,j}$ be generators of $k[\mathrm{GL}_n]$. A base of the space of derivations $\mathrm{Der}_k(k[G], k)$ is given by $e_{i,j}(T_{k,l}) = \delta_{i,k}\delta_{j,l}$. Let us check that the map $e_{i,j} \mapsto E_{i,j}$ is a Lie algebra isomorphism (here $E_{i,j}$ is the standard base for matrices). It is abviously an isomorphism of vector spaces. We have

$$\begin{aligned} e_{a,b}e_{c,d}(T_{i,j}) &= (e_{a,b} \otimes e_{c,d}) \circ \Delta(T_{i,j}) \\ &= (e_{a,b} \otimes e_{c,d})(\textstyle\sum_k T_{i,k} \otimes T_{k,j}) \\ &= \textstyle\sum_k \delta_{a,i}\delta_{b,k}\delta_{c,k}\delta_{d,j} \\ &= \delta_{b,c}\delta_{a,i}\delta_{d,j} \\ &= \delta_{b,c}e_{a,d}(T_{i,j}). \end{aligned}$$

We thus have $e_{a,b}e_{c,d} = \delta_{b,c}e_{a,d}$ which is the same multiplication rule as for matricies. □

Corollary 2.2.24 *The Lie algebra of* SL_n *is* $\mathfrak{sl}_n$.

Proof. We only need to check the equality

$$\sum_{i,j} \frac{\partial \det}{\partial T_{i,j}}(\mathrm{Id})T_{i,j} = \sum_i T_{i,i}.$$

∎

2.3 Derived action on a representation

2.3.1 Derived action

Let X be a right affine G-space and let $a_X^\sharp : k[X] \to k[X] \otimes k[G]$ be the comorphism of $a_X : X \times G \to X$. Let V be a stable vector subspace of $k[X]$ *i.e.* $a_X^\sharp(V) \subset V \otimes k[G]$. If V is finite dimensional, this is equivalent to a rational representation of G *i.e.* a morphism of algebraic groups $\phi : G \to \mathrm{GL}(V)$.

Proposition 2.3.1 *(ı) There is a* $\mathrm{Dist}(G)$*-module structure on* V *defined by* $\eta \cdot v = (\mathrm{Id} \otimes \eta) \circ a_V^\sharp(v)$. *In particular,* V *is a* $U(L(G))$*-module and therefore a* $L(G)$*-representation.*

(ıı) If V *is finite dimensional, then the map* $\mathrm{Dist}_1^+(G) \to \mathfrak{gl}(V)$ *obtained from the above representation is the differential* $d_{e_G}\phi$.

Proof. (ı) We first compute $\epsilon \cdot v = (\mathrm{Id} \otimes \epsilon) \circ a_X^\sharp(v)$ but the fact that the identity elements acts trivially gives $(\mathrm{Id} \otimes \epsilon) \circ a_X^\sharp = \mathrm{Id}$ then $\epsilon \cdot v = v$. We also compute (recall the formula $(a_X^\sharp \otimes \mathrm{Id}) \circ a_X^\sharp = (\mathrm{Id} \otimes \Delta) \circ a_X^\sharp$):

$$\begin{aligned}
\eta \cdot (\xi \cdot v) &= (\mathrm{Id} \otimes \eta \otimes \mathrm{Id}_k) \circ (a_X^\sharp \otimes \mathrm{Id}_k) \circ (\mathrm{Id} \otimes \xi) \circ a_X^\sharp(v) \\
&= (\mathrm{Id} \otimes \eta \otimes \mathrm{Id}_k) \circ (\mathrm{Id} \otimes \mathrm{Id} \otimes \xi) \circ (a_X^\sharp \otimes \mathrm{Id}) \circ a_X^\sharp(v) \\
&= (\mathrm{Id} \otimes \eta \otimes \xi) \circ (\mathrm{Id} \otimes \Delta) \circ a_X^\sharp(v) \\
&= (\mathrm{Id} \otimes \eta\xi) \circ a_X^\sharp(v) \\
&= (\eta\xi) \cdot v.
\end{aligned}$$

This proves the first point.

(ıı) Recall how the map ϕ is constructed (we did it for a left action but the same works for the right action). We fix a base $(f_i)_{i \in [1,n]}$ for V and look at the comorphism

$$a_X^\sharp(f_i) = \sum_j f_j \otimes m_{j,i}$$

for $m_{i,j} \in k[G]$. The morphism $G \to \mathrm{GL}(V)$ is defined by the comorphism $\phi^\sharp : k[T_{i,j}, \det^{-1}] \to k[G]$ defined by $\phi^\sharp(T_{i,j}) = m_{i,j}$. For $\eta \in \mathrm{Dist}(G)$ we then have $d_{e_G}\phi(\eta)(T_{i,j}) = \eta \circ \phi^\sharp(T_{i,j}) = \eta(m_{i,j})$. In terms of the base $e_{i,j}$ such that $e_{i,j}(T_{k,l}) = \delta_{i,k}\delta_{j,l}$ we thus get

$$d_{e_G}\phi(\eta) = \sum_{i,j} \eta(m_{i,j}) e_{i,j}.$$

By the identification of $T_{e_{\mathrm{GL}(V)}}\mathrm{GL}(V)$ with $\mathfrak{gl}(V)$ we get

$$d_{e_G}\phi(\eta)(f_k) = \sum_{i,j} \eta(m_{i,j}) E_{i,j}(f_k) = \sum_i \eta(m_{i,k}) f_i.$$

On the other hand, we have

$$\eta \cdot f_k = (\mathrm{Id} \otimes \eta) \circ a_X^\sharp(f_k) = \sum_j \eta(m_{j,k}) f_j$$

therefore $d_{e_G}\phi(\eta)(v) = \eta \cdot v$ and the result follows. ☐

Proposition 2.3.2 *(ı) The Lie algebra* $\mathrm{Dist}_1^+(G)$ *acts on* $k[X]$ *via derivations.*
(ıı) Assume that $X = G$ *with* G *acting on itself by right translation, then* $\eta \cdot f = D_\eta(f)$*.*

Proof. (ı) Let $\eta \in L(G)$ and let $f, f' \in k[X]$. Let us write

$$a_X^\sharp(f) = \sum_i u_i \otimes a_i \text{ and } a_X^\sharp(f) = \sum_j v_j \otimes b_j.$$

We have $\eta \cdot f = \sum_i \eta(a_i) u_i$ and $\eta \cdot f' = \sum_j \eta(b_j) v_j$. We compute:

$$\begin{aligned}
\eta \cdot ff' &= (\mathrm{Id} \otimes \eta) \circ a_X^\sharp(ff') \\
&= (\mathrm{Id} \otimes \eta) \circ (a_X^\sharp(f) a_X^\sharp(f')) \\
&= \textstyle\sum_i \sum_j \eta(a_i b_j) u_i v_j \\
&= \textstyle\sum_i \sum_j (a_i(e_G)\eta(b_j) + b_j(e_G)\eta(a_i)) u_i v_j.
\end{aligned}$$

But recall that $(\mathrm{Id} \otimes \epsilon) \circ a_X^\sharp = \mathrm{Id}$ thus $f = \sum_i a_i(e_G) u_i$ and $f' = \sum_j b_j(e_G) v_j$ thus we have

$$\eta \cdot ff' = f\eta(f') + f'\eta(f).$$

(ıı) Recall the definition $D_\eta(f)(x) = \eta(\lambda(x^{-1})f)$ or $D_\eta = (\mathrm{Id} \otimes \eta) \circ \Delta$. But this is exactely the action of η since $a_G^\sharp = \Delta$. ☐

Remark 2.3.3 In general, even if a representation of algebraic groups is faithful, the derived action need not be faithful. The problem comes from the fact that a bijective morphism of algebraic groups is not an isomorphism. For example, the map $\phi : \mathbb{G}_m \to \mathbb{G}_m$ defined by $T \mapsto T^p$ has a trivial differential in characteristic p. Therefore any representation factorising through this map with have a trivial derived action. Indeed we have $d_{e_G}\phi(\delta)(T) = \delta(T^p) = p\delta(T^{p-1}) = 0$.

Corollary 2.3.4 *The derived action of the right translation* ρ *of* G *on itself is a faithful representation* $d_{e_G}\rho : \mathfrak{g} \to \mathfrak{gl}(k[G])$*.*

Proof. We already know that this representation exists. It maps δ to D_δ and this map is injective. ☐

2.3.2 Stabilisor of the ideal of a closed subgroup

Let H be a closed subgroup of an algebraic group G and let I be the ideal of H in G. Let us consider the action ρ of G on $k[G]$ defined by $\rho(g)f(x) = f(xg)$.

Lemma 2.3.5 *(ı) We have the equality* $H = \{g \in G \ / \ \rho(g)I_H = I_H\}$*.*
(ıı) We have the equality $\mathrm{Dist}_1^+(H) = \{\delta \in \mathrm{Dist}_1^+(G) \ / \ \delta(I) = 0\}$*.*
(ııı) We have the equality $L(H) = \{D \in L(G) \ / \ D(I) \subset I\}$*.*

Proof. We already proved (i). For (ii), if $\delta \in \mathrm{Dist}_1^+(G)$ satisfies $\delta(I) = 0$, then δ induces a linear map $\delta : k[H] = k[G]/I \to k$ and lies therefore in $\mathrm{Dist}_1^+(H)$. The converse is also obvious.

(iii) If $D(I) \subset I$, then $\epsilon \circ D(I) = 0$ thus by (ii) $\epsilon \circ D \in \mathrm{Dist}_1^+(H)$ thus $D \in L(H)$. Conversely, if $D \in L(H)$ and for $f \in I$, $h \in H$, we have $\lambda(h^{-1})f \in I$ thus

$$D(f)(h) = \epsilon(\lambda(h^{-1})D(f)) = \epsilon \circ D(\lambda(h^{-1})f) = 0.$$

Thus $D(I) \subset I$ and the lemma follows. □

2.3.3 Adjoint actions

Let G be an algebraic group and let $\mathfrak{g} = \mathrm{Dist}_1^+(G)$ be its Lie algebra. For $g \in G$, let us denote by $\mathrm{Int}(g) : G \to G$ the morphism defined by $x \mapsto gxg^{-1}$. One can easily check that this is a isomorphism of algebraic groups. Let $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ be its differential at e_G *i.e.* $d_{e_G}\mathrm{Int}(g) = \mathrm{Ad}(g)$.

Fact 2.3.6 *The differential* $\mathrm{Ad}(g)$ *is an isomorphism of Lie algebras.*

Proof. Indeed, the inverse of $\mathrm{Int}(g)$ is $\mathrm{Int}(g^{-1})$ therefore $\mathrm{Ad}(g)$ is bijective with inverse $\mathrm{Ad}(g^{-1})$. Because $\mathrm{Int}(g)$ is a morphism of algebraic groups, we have that $\mathrm{Ad}(g)$ is a Lie algebra morphism. □

Fact 2.3.7 *The map* $\mathrm{Ad} : G \to \mathrm{Gl}(\mathfrak{g})$ *defined by* $g \mapsto \mathrm{Ad}(g)$ *is a homomorphism of abstract groups.*

Proof. Indeed, we have the equality $\mathrm{Int}(gg') = \mathrm{Int}(g) \circ \mathrm{Int}(g')$, we get $\mathrm{Ad}(gg') = \mathrm{Ad}(g) \circ \mathrm{Ad}(g')$. □

Theorem 2.3.8 *The map* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ *is a morphism of algebraic groups. Its differential at* e_G *is* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *defined by* $\mathrm{ad}(\eta)(\xi) = [\eta, \xi]$ *for all* η, ξ *in* $\mathfrak{g}$.

Proof. We first prove that it is enough to prove this result for $\mathrm{GL}(V)$. Indeed, embbed G in some $\mathrm{GL}(V)$. We have the commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\mathrm{Ad}_G"] \arrow[d, "\phi"'] & \mathrm{GL}(\mathfrak{g}) \arrow[d, "\psi"] \\ \mathrm{GL}(V) \arrow[r, "\mathrm{Ad}_{\mathrm{GL}(V)}"] & \mathrm{GL}(\mathfrak{gl}(V)), \end{tikzcd}$$

and $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Let us write $k[\mathrm{GL}(\mathfrak{g})] = k[T_{i,j}, 1 \leq i,j \leq n, \det^{-1}]$ which is a quotient of $k[\mathrm{GL}(\mathfrak{gl}(V))] = k[T_{i,j}, 1 \leq i,j \leq n+m, \det^{-1}]$. If $\mathrm{Ad}_{\mathrm{GL}(V)}$ is a morphism of algebraic groups, then the composition of the linear form $T_{i,j}$ on $\mathrm{GL}(\mathfrak{gl}(V))$ and of $\mathrm{Ad}_{\mathrm{GL}(V)} \circ \phi$ is a regular function on G *i.e.* an element in $k[G]$. This is true for all $1 \leq i,j \leq n+m$ and *a fortiori* for all $1 \leq i,j \leq n$. Thus the map Ad_G is a morphism. Being an abstract group morphism, it is a morphism of algebraic groups.

Now we may differentiate this diagram to get a diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}_G"] \arrow[d, "d_{e_G}\phi"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "d_{e_{\mathrm{GL}(\mathfrak{g})}}\psi"] \\ \mathfrak{gl}(V) \arrow[r, "\mathrm{ad}_{\mathrm{GL}(V)}"] & \mathfrak{gl}(\mathfrak{gl}(V)), \end{tikzcd}$$

all the morphisms being Lie algebra morphisms. To prove the result, because $d_{e_G}\psi$ is injective, we have to check it for the composition $d_{e_{\mathrm{GL}(\mathfrak{g})}}\psi \circ \mathrm{ad}_G = \mathrm{ad}_{\mathrm{GL}(V)} \circ d_{e_G}\phi$. Assuming the result true for $\mathrm{GL}(V)$, we get for $\eta \in \mathfrak{g}$ and $X \in \mathfrak{gl}(V)$,

$$\begin{aligned} d_{e_{\mathrm{GL}(\mathfrak{g})}}\psi \circ \mathrm{ad}_G(\eta)(X) &= \mathrm{ad}_{\mathrm{GL}(V)}(d_{e_G}\phi(\eta))(X) \\ &= [d_{e_G}\phi(\eta), X]. \end{aligned}$$

If $X = d_{e_G}\phi(\xi)$ for $\xi \in \mathfrak{g}$, we get

$$\begin{aligned} d_{e_{\mathrm{GL}(\mathfrak{g})}}\psi \circ \mathrm{ad}_G(\eta)(d_{e_G}(\xi)) &= [d_{e_G}\phi(\eta), d_{e_G}\phi(\xi)] \\ &= d_{e_G}\phi[\eta, \xi], \end{aligned}$$

which means $\mathrm{ad}_G(\eta)(\xi) = [\eta, \xi]$.

We are thus left to prove the result for $G = \mathrm{GL}(V)$. This is done in the next proposition. □

Proposition 2.3.9 *The morphism* $\mathrm{Ad} : \mathrm{GL}(V) \to \mathrm{GL}(\mathfrak{gl}(V))$ *is a morphism of algebraic groups defined by* $\mathrm{Ad}\,(g)(X) = gXg^{-1}$*. Its differential* ad *satisfies* $\mathrm{ad}\,(X)(Y) = [X, Y]$ *for all* X, Y *in* $\mathfrak{gl}(V)$*.*

Proof. To prove that Ad is an algebraic group morphism, it is enough to prove the formula $\mathrm{Ad}\,(g)(X) = gXg^{-1}$. But $\mathrm{Int}(g) : \mathrm{GL}(V) \to \mathrm{GL}(V)$ can be extended to a morphism $\mathrm{INT}(g) : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ defined by $X \mapsto gXg^{-1}$. This morphism is linear and it is then easy to check that its differential $\mathrm{AD}(g)$ at $e_{\mathrm{GL}(V)}$ is again $\mathrm{INT}(g)$. Because $\mathrm{GL}(V)$ is an open neihbourhood of $e_{\mathrm{GL}(V)}$ in $\mathfrak{gl}(V)$ we get $\mathrm{Ad}\,(g) = \mathrm{INT}(g)$ and the first part.

Now we need to compute $d_{e_{\mathrm{GL}(V)}}\mathrm{Ad}$. For this we first prove two lemmas on differentials. Recall that we denote by i and μ the inverse map and the multiplication map. For $g \in \mathrm{GL}(V)$, we denote by μ_g, resp. ${}_g\mu : \mathrm{GL}(V) \to \mathrm{GL}(V)$ the map $\mu(\cdot, g)$ resp. $\mu(g, \cdot)$. Note thqt these two maps can be extented to $\mathfrak{gl}(V)$ and are linear therefore they are equal to their differential.

Lemma 2.3.10 *Let* $g \in \mathrm{GL}(V)$ *and* $X \in \mathfrak{gl}(V)$*, then* $d_g i(X) = -g^{-1}Xg^{-1}$*.*

Proof. Let us consider the two compositions ${}_{g^{-1}}\mu \circ i$ and $i \circ \mu_g$. These maps are equal and so are their differential. We thus get (denoting by e the unit element of $\mathrm{GL}(V)$) for $Y \in \mathfrak{gl}(V)$:

$$d_g i(Yg) = d_g i \circ d_e\mu_g(Y) = d_e({}_{g^{-1}}\mu) \circ d_e i(Y) = -g^{-1}Y.$$

Setting $X = Yg$ *i.e.* $Y = Xg^{-1}$ we get the result. □

Lemma 2.3.11 *Let* $g, h \in \mathrm{GL}(V)$ *and* $X, Y \in \mathfrak{gl}(V)$*, then* $d_{g,h}\mu(X, Y) = Xh + gY$*.*

Proof. Let us consider the two compositions ${}_g\mu \circ \mu_h \circ \mu$ and $\mu \circ ({}_g\mu \times \mu_h)$. These maps are equal and so are their differential. We thus get (denoting by e the unit element of $\mathrm{GL}(V)$) for $A, B \in \mathfrak{gl}(V)$:

$$d_{(g,h)}\mu(gA, Bh) = d_{(g,h)}\mu \circ ({}_g\mu \times \mu_h)(A, B) = d_h({}_g\mu) \circ d_e\mu_h \circ d_{(e,e)}\mu(A, B) = g(A + B)h.$$

Setting $X = gA$ and $Y = Bh$ we get the result. □

Let us finish the computation of ad. For $Y \in \mathfrak{gl}(V)$, let us denote by $\mathrm{ev}_Y : \mathfrak{gl}(\mathfrak{gl}(V)) \to \mathfrak{gl}(V)$ be the linear map defined by $u \mapsto u(Y)$. Let $\theta_Y : \mathrm{GL}(V) \to \mathfrak{gl}(V)$ be the map defined by $\theta_Y(g) = \mathrm{Ad}\,(g)(Y)$. We have $\theta_Y = \mathrm{ev}_Y \circ \mathrm{Ad}$. Let us compute its differential:

$$d_e\theta_Y = d_e\mathrm{ev}_Y \circ d_e\mathrm{Ad} = \mathrm{ev}_Y \circ \mathrm{ad}\,.$$

On the other hand, $\theta_Y(g) = gYg^{-1}$ therefore we have the equality $\theta_Y|_{\mathrm{GL}(V)} = \mu \circ (\mu_Y, i)$ Computing the differential we get

$$d_e\theta_Y(X) = d_{(Y,e)}\mu \circ (\mu_Y, d_e i)(X) = d_{(Y,e)}\mu(XY, -X) = XY - YX.$$

Combining this with the previous formula we get

$$\mathrm{ad}\,(X)(Y) = \mathrm{ev}_Y(\mathrm{ad}\,(X)) = d_e\theta_Y(X) = XY - YX = [X, Y]$$

hence the result. $\square$

Let us now give some simple consequences of the above Theorem.

Corollary 2.3.12 *Let H be a closed normal subgroup of G and let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H and G. Then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

Corollary 2.3.13 *Let H be a closed subgroup and $N = N_G(H)$ be its normaliser. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H and G.*
(ı) N is a closed subgroup of G. Let $\mathfrak{n}$ be its Lie algebra.
(ıı) We have the inclusion $\mathfrak{n} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \{\eta \in \mathfrak{g} \ / \ [\eta, \mathfrak{h}] \subset \mathfrak{h}\}$.

Fact 2.3.14 *Let $g \in G$ and $\eta \in \mathfrak{g}$, then if $\gamma_g : G \to G$ is defined by $\gamma_g(h) = hgh^{-1}g^{-1}$, we have*

$$d_e\gamma_g(\eta) = (\mathrm{Id} - \mathrm{Ad}\,(g))(\eta).$$

Corollary 2.3.15 *Let H and K be closed subgroups of G, then the Lie algebra of (H, K) contains all the elements $\eta - \mathrm{Ad}\,(h)(\eta)$, $\xi - \mathrm{Ad}\,(k)(\xi)$ and $[\eta, \xi]$ for $h \in H$, $k \in K$, $\eta \in \mathrm{Dist}_1^+(H)$ and $\xi \in \mathrm{Dist}_1^+(K)$.*

Corollary 2.3.16 *The Lie algebra of (G, G) contains $[\mathfrak{g}, \mathfrak{g}]$.*

Corollary 2.3.17 *Let $g \in G$ and $C_G(g)$ be its centraliser.*
(ı) $C_G(g)$ is a closed subgroup of G. Let $\mathfrak{c}$ be its Lie algebra.
(ıı) We have the inclusion $\mathfrak{c} \subset \mathfrak{c}_{\mathfrak{g}}(g) = \{\eta \in \mathfrak{g} \ / \ \mathrm{Ad}\,(g)(\eta) = \eta\}$ with equality for $G = \mathrm{GL}(V)$.

Fact 2.3.18 *We have the inclusion $Z(G) \subset \ker \mathrm{Ad}$.*

Example 2.3.19 Let $\mathrm{char}(k) = p > 0$ and let G be the subgroup of GL_3 consisting of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix},$$

with $a \neq 0$. Then in this group all the above inclusions may be strict.

Chapter 3

Semisimple and unipotent elements

3.1 Jordan decomposition

3.1.1 Jordan decomposition in $\mathrm{GL}(V)$

Let us first recall some fact on linear algebra. See for example [Bou58] for proofs. Let V be a vector space.

Definition 3.1.1 *(ı) We call semisimple any endomorphism of V which is diagonalisable. Equivalently if $\dim V$ is finite, the minimal polynomial is separable.*

(ıı) We call nilpotent (resp. unipotent) any endomorphism x such that $x^n = 0$ for some n (resp. $x - \mathrm{Id}$ is nilpotent).

(ııı) We call locally finite any endomorphism x such that for all $v \in V$, the span of $\{x^n(v) \ / \ n \in \mathbb{N}\}$ is of finite dimension.

(ııı) We call locally nilpotent (resp. locally unipotent) any endomorphism x such that for all $v \in V$, there exists an n such that $x^n(v) = 0$ (resp. $\mathrm{Id} - x$ is locally nilpotent).

Fact 3.1.2 *Let x and y in $\mathfrak{gl}(V)$ such that x and y commute.*

(ı) If x is semisimple, then it is locally finite.

(ıı) If x and y are semisimple, then so are $x + y$ and xy.

(ııı) If x and y are locally nilpotent, then so are $x + y$ and xy.

(ıv) If x and y are locally unipotent, then so is xy.

Theorem 3.1.3 (Additive Jordan decomposition) *Let $x \in \mathfrak{gl}(V)$ be locally finite.*

(ı) There exists a unique decomposition $x = x_s + x_n$ in $\mathfrak{gl}(V)$ such that x_s is semisimple, x_n is nilpotent and x_s and x_n commute.

(ıı) There exists polynomial P and Q in $k[T]$ such that $x_s = P(x)$ and $x_n = Q(x)$. In particular x_s and x_n commute with any endomorphism commuting with x.

(ııı) If $U \subset W \subset V$ are subspaces such that $x(W) \subset U$, then x_s and x_n also map W in U.

(ıv) If $x(W) \subset W$, then $(x|_W)_s = (x_s)|_W$ and $(x|_W)_n = (x_n)|_W$ and $(x|_{V/W})_s = (x_s)|_{V/W}$ and $(x|_{V/W})_n = (x_n)|_{V/W}$.

Definition 3.1.4 *The elements x_s (resp. x_n) is called the* semisimple part *of $x \in \mathrm{End}(V)$ (resp.* nilpotent part *The decomposition $x = x_s + x_n$ is called the* Jordan-Chevalley decomposition.

Corollary 3.1.5 (Multiplicative Jordan decomposition) *Let $x \in \mathfrak{gl}(V)$ be locally finite and invertible.*

($\imath$) There exists a unique decomposition $x = x_s x_u$ in $\mathrm{GL}(V)$ *such that x_s is semisimple, x_u is unipotent and x_s and x_u commute.*

($\imath\imath$) The elements x_s and x_u commute with any endomorphism commuting with x.

($\imath\imath\imath$) If $U \subset W \subset V$ are subspaces such that $x(W) \subset U$, then x_s and x_n also map W in U.

($\imath v$) If $x(W) \subset W$, then $(x|_W)_s = (x_s)|_W$ and $(x|_W)_u = (x_u)|_W$ and $(x|_{V/W})_s = (x_s)|_{V/W}$ and $(x|_{V/W})_u = (x_u)|_{V/W}$.

Proof. We simply have to write $x = x_s + x_n$. Because x is inversible, so is x_s thus we may set $x_u = \mathrm{Id} + x_s^{-1} x_n$ which is easily seen to be unipotent and satisfies the above properties. $\square$

3.1.2 Jordan decomposition in G

Theorem 3.1.6 *Let G be an algebraic group and let $\mathfrak{g}$ be its Lie algebra.*

($\imath$) For any $g \in G$, there exists a unique couple $(g_s, g_u) \in G^2$ such that $g = g_s g_u$ and $\rho(g_s) = \rho(g)_s$ and $\rho(g_u) = \rho(g)_u$.

($\imath\imath$) For any $\eta \in \mathfrak{g}$, there exists a unique couple $(\eta_s, \eta_n) \in \mathfrak{g}^2$ such that $\eta = \eta_s + \eta_n$ and $d_{e_G}\rho(\eta_s) = d_{e_G}\rho(\eta)_s$ and $d_{e_G}\rho(\eta_n) = d_{e_G}\rho(\eta)_n$.

($\imath\imath\imath$) If $\phi : G \to G'$ is a morphism of algebraic groups, then $\phi(g_s) = \phi(g)_s$, $\phi(g_u) = \phi(g)_u$, $d_{e_G}\phi(\eta_s) = d_{e_G}\phi(\eta)_s$ and $d_{e_G}\phi(\eta_n) = d_{e_G}\phi(\eta)_n$.

Proof. Let us first note that because ρ and $d_e\rho$ are faithful, the unicity for $g \in G$ and $\eta \in \mathfrak{g}$ follows from the unicity of the Jordan decomposition for $\rho(g)$ and $d_e\rho(\eta)$.

We first prove ($\imath$) and ($\imath\imath$) for $\mathrm{GL}(V)$.

Proposition 3.1.7 *Let $g \in \mathrm{GL}(V)$ and $X \in \mathfrak{gl}(V)$.*

($\imath$) If g is semisimple, then so is $\rho(g)$.

($\imath\imath$) If X is semisimple, then so is $d_e\rho(X)$.

Therefore, if $g = g_s g_u$ and $X = X_s + X_n$ are the Jordan decompositions in $\mathrm{GL}(V)$ and $\mathfrak{gl}(V)$, then $\rho(g) = \rho(g_s)\rho(g_u)$ and $d_e\rho(X) = d_e\rho(X_s) + d_e\rho(X_n)$ are the Jordan decompositions of $\rho(g)$ and $d_e\rho(X)$.

Proof. Assume that g or X is semisimple (resp. unipotent or nilpotent), then let (f_i) be a base of V such that these endomorphisms are diagonal (resp. upper triangular with 1 or 0 on the diagonal). Recall also that for $f \in k[G]$ with $\Delta(f) = \sum_i a_i \otimes u_i$ we have

$$\rho(g)f = \sum_i a_i u_i(g) \text{ and } d_e\rho(X)f = \sum_i a_i X(g).$$

Applying this to the elements $T_{i,j}$ we get

$$\rho(g)T_{i,j} = \sum_k T_{k,j}(g) T_{i,k} \text{ and } d_e\rho(X)T_{i,j} = \sum_k T_{k,j} X(T_{i,k}).$$

But if g and X are diagonal, then $T_{i,j}(g) = \delta_{i,j}\lambda_i$ and $X(T_{i,j}) = \delta_{i,j}\lambda_i$. We thus get

$$\rho(g)T_{i,j} = \lambda_j T_{i,j} \text{ and } d_e\rho(X)T_{i,j} = \lambda_j T_{i,j}.$$

Furthermore for det we have $\Delta(\det) = \det \otimes \det$ thus

$$\rho(g)\det = \det(g)\det \text{ and } d_e\rho(X)\det = X(\det)\det = \mathrm{Tr}(X)\det.$$

We thus have in this case a base of eigenvectors. If g and X are unipotent of nilpotent, then the same will be true because in the lexicographical order base of the monomials, we also have a triangular matrix whose diagonal coefficients are those of g or $\mathrm{Tr}(X) = 0$. $\square$

We are therefore left to prove (ııı) to conclude. We deal with to cases which are enough: $\phi : G \to G'$ is injective or surjective. Any morphism can be decomposed in such two morphisms by taking the factorisation through the image.

Assume that ϕ is a closed immersion. Then we have $k[G'] \to k[G] = k[G']/I$. Let $g \in G$ resp. $\eta \in \mathfrak{g}$ and let $g = g_s g_u$ resp. $\eta = \eta_s + \eta_n$ the Jordan decomposition of g resp. η in G' resp. $\mathfrak{g}'$. We need to prove that these decompositions are in G resp. in $\mathfrak{g}$. For this we check that $\rho(g_s)I = I$, $\rho(g_u)I = I$, $d_e\rho(\eta_s)I \subset I$ and $d_e\rho(\eta_n)I \subset I$. But I is a vector subspace of $k[G']$ which is stable under g resp. X thus it is also stable under all these maps.

This applied to the inclusion of any algebraic group G in some $\mathrm{GL}(V)$ implies the existence of the decomposition.

Assume now that ϕ is surjective. This in particular implies that $\phi^\sharp : k[G'] \to k[G]$ is injective. Let $g \in G$ resp. $\eta \in \mathfrak{g}$ and let $g = g_s g_u$ resp. $\eta = \eta_s + \eta_n$ the Jordan decomposition of g resp. η in G resp. $\mathfrak{g}$.

We may realise $k[G']$ as a $\rho(G)$-submodule of $k[G]$. For $f \in k[G']$, $g \in G$ and $g' \in G'$, we have:

$$\rho(g)f(g') = f(g'\phi(g)).$$

We thus have the formula $\rho(g)|_{k[G']} = \rho(\phi(g))$. Applying this to g, g_s and g_u, we have

$$\rho(\phi(g)) = \rho(\phi(g_s))\rho(\phi(g_u)) = \rho(g_s)|_{k[G']}\rho(g_u)|_{k[G']}$$

but as $\rho(g_s)$ and $\rho(g_u)$ are semisimple and nipotent, so are their restriction thus this is the Jordan decomposition of $\rho(\phi(g))$ and thus of $\phi(g)$.

The above submodule structure means that we have an action $a_{G'}$ of G on G' whose action is given by

$$a^\sharp_{G'} = (\mathrm{Id} \otimes \phi^\sharp) \circ \Delta_{G'}.$$

Note that $a_{G'} = \Delta_G|_{k[G']}$. Thus for $f \in k[G']$ we have

$$\begin{aligned} d_e\rho(d_e\phi(\eta))f &= (\mathrm{Id} \otimes d_e\phi(\eta)) \circ \Delta_{G'}(f) \\ &= (id \otimes \eta) \circ (\mathrm{Id} \otimes \phi^\sharp) \circ \Delta_{G'}(f) \\ &= (\mathrm{Id} \otimes \eta) a^\sharp_{G'}(f) \\ &= (\mathrm{Id} \otimes \eta)\Delta_G(f) \\ &= \eta \cdot f. \end{aligned}$$

Therefore we have $d_e\rho(d_e\phi(\eta)) = d_e\rho(\eta)|_{k[G']}$ and the same argument as above applies. $\square$

3.2 Semisimple, unipotent and nilpotent elements

Definition 3.2.1 *(ı) Let $g \in G$, then g is called semisimple, resp. unipotent if $g = g_s$ resp. $g = g_u$.*

(ıı) Let $\eta \in \mathfrak{g}$, then η is called semisimple, resp. nilpotent if $\eta = \eta_s$ resp. $\eta = \eta_n$.

(ııı) We denote by G_s resp. G_u the set of semisimple, resp. unipotent elements in G.

(ıv) We denote by $\mathfrak{g}_s$ resp. $\mathfrak{g}_u$ the set of semisimple, resp. nilpotent elements in $\mathfrak{g}$.

Fact 3.2.2 *If $g \in G$ resp. $\eta \in \mathfrak{g}$ is semisimple and unipotent (resp. semisimple and nilpotent), then $g = e$ (resp. $\eta = 0$).*

Remark 3.2.3 Note that in the case of general Lie algebras, the Jordan decomposition does not always exists. This proves that any Lie algebra is not the Lie algebra of an algebraic group.

In general, if $\mathrm{char}(k) = p > 0$, for an algebraic group G defined over the field k with Lie algebra $\mathfrak{g} = \mathrm{Der}_k(k[G], k[G])^{\lambda(G)}$, we have an additional structure called the p-operation and given by taking the p-th power of the derivation (p-th composition). This maps invariant derivations to invariant derivations.

Definition 3.2.4 *A p-Lie algebra is a Lie algebra $\mathfrak{g}$ with a linear map $x \mapsto x^{[p]}$ called the p-operation such that*

- $(\lambda x)^{[p]} = \lambda^p x^{[p]}$,
- $\mathrm{ad}\,(x^{[p]}) = \mathrm{ad}\,(x)^p$,
- $(x + x')^{[p]} = x^{[p]} + x'^{[p]} + \sum_{i=1}^{p-1} i^{-1} s_i(x, x')$

where $x, x' \in \mathfrak{g}$, $\lambda \in k$ and $s_i(x, x')$ is the coefficient of a^i in $\mathrm{ad}\,(ax + x')^{p-1}(x')$.

Proposition 3.2.5 *The subset G_u resp. $\mathfrak{g}_n$ is closed in G resp. $\mathfrak{g}$.*

Proof. Let us embed G and $\mathfrak{g}$ in $\mathrm{GL}(V)$ and $\mathfrak{gl}(V)$. Then G_u is the intersection of G with the closed subset of elements g such that $(g - \mathrm{Id})^n = 0$ while $\mathfrak{g}_n$ is the intersection of $\mathfrak{g}$ and the closed subset of elements X such that $X^n = 0$. □

3.3 Commutative groups

3.3.1 Diagonalisable groups

Definition 3.3.1 *Let G be an algebraic groups.*

(ı) The group G is called unipotent if $G = G_u$.

(ıı) The group G is called diagonalisable if there exists a faithful representation $G \to \mathrm{GL}(V)$ such that the image of G is contained in the subgroup of diagonal matrices.

Proposition 3.3.2 *The following conditions are equivalent:*

(ı) The group G is diagonalisable.

(ıı) The group G is a closed subgroup of $\mathbb{G}_m^n$.

(ııı) The group G is commutative and all its elements are semisimple.

Proof. The implications (ı) ⇒ (ıı) ⇒ (ııı) are obvious. The last implication, follows from the next lemma. □

Lemma 3.3.3 *Let V be a finite dimensional vector space and let $\mathcal{F}$ be a family of self-commuting endomorphisms. Then*

(ı) there axists a base of V such that all matrices of the elements in $\mathcal{F}$ are upper triangular matrices in this base.

(ıı) Furthermore, for any subfamily $\mathcal{F}'$ of semisimple elements, the base can be chosen such that all the endomorphisms of $\mathcal{F}'$ have a diagonal matrix in that base.

Proof. We proceed by induction on $\dim V$. If all the elements in $\mathcal{F}$ are homotheties, then we are done. If not, then there exists $u \in \mathcal{F}$ and $a \in k$ such that $W = \ker(u - a\mathrm{Id})$ is not trivial and distinct from V. Then W is stable under any element in $\mathcal{F}$. We conclude by induction. □

3.3.2 Structure of commutative groups

Theorem 3.3.4 *(Structure of commutative groups) ($\imath$) Let G be a commutative group and let $\mathfrak{g}$ be its Lie algebra. Then G_s and G_u are closed subgroup of G (connected if G is connected) and the map $G_s \times G_u \to G$ defined by $(x, y) \mapsto xy$ is an isomorphism. Its inverse is the Jordan decomposition.*

($\imath\imath$) We have $L(G_s) = \mathfrak{g}_s$, $L(G_u) = \mathfrak{g}_n$ and $\mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_u$.

Proof. We shall consider G as a subgroup of $\mathrm{GL}(V)$ and $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}(V)$.

The group G being commutative, the subsets G_s and G_u are subgroups. Furthermore from the computation of ad, we know that the Lie bracket in $\mathfrak{g}$ is trivial *i.e* $\mathfrak{g}$ is commutative. This implies that $\mathfrak{g}_s$ and $\mathfrak{g}_n$ are subspaces. We also have $G_s \cap G_u = \{e\}$ and $\mathfrak{g}_s \cap \mathfrak{g}_n = 0$.

From the previous Lemma, we may embed G as a subgroup of the group of uppertriangular matrices in $\mathfrak{gl}(V)$. Therefore G_s is the intersection of G with the set of diagonal martrices which is closed thus G_s is a closed subgroup. The above map is obviously a morphism of varieties and thus of algebraic groups and by the Jordan decomposition it is a bijection.

Let us check that the map $g \mapsto g_s$ is a morphism. This will imply that the map $g \mapsto g_u = g_s^{-1} g$ is also a morphism. But in the above description of g as matricies, g_s is the diagonal part of g, therefore the map $g \mapsto g_s$ is a morphism. This also implies that if G is connected, so are G_s and G_u.

On the Lie algebra level, the Lie algebra of G_s is contained in the set of diagonal matrices and the Lie algebra of G_u is contained in the subspace of strictly upper triangular matrices. These subspaces are also the subspaces of semisimples resp. nilpotent elements thus $L(G_s) \subset \mathfrak{g}_s$ and $L5G_u) \subset \mathfrak{g}_n$. But $\dim G_s + \dim G_u = \dim G$ thus we have equality by dimension argument. □

Let us now quote without proof the following classification of algebraic groups of dimension 1.

Theorem 3.3.5 *Let G be a connected algebraic group of dimension 1, then $G = \mathbb{G}_m$ or $G = \mathbb{G}_a$.*

Chapter 4

Diagonalisable groups and Tori

The diagonalisable groups (commutative groups whose elements are all semisimple) play a very important role in the theory of reductive algebraic groups.

Definition 4.0.6 *An algebraic group is called a torus if it is isomorphic to $\mathbb{G}_m^n$ for some n.*

Example 4.0.7 The group D_n is a torus isomorphic to $\mathbb{G}_m^n$.

4.1 Structure theorem for diagonalisable groups

4.1.1 Characters

Definition 4.1.1 *Let G be an algebraic group.*

(ı) A character of G is a morphism of algebraic groups $\chi : G \to \mathbb{G}_m$. We denote by $X^*(G)$ the set of all characters of G.

(ıı) A cocharacter of G (or a one parameter subgroup, or 1-pm) is a morphism of algebraic groups $\lambda : \mathbb{G}_m \to G$. We denote by $X_*(G)$ the set of all cocharacters of G.

Remark 4.1.2 (ı) Note that $X^*(G)$ has a structure of abelian group given by $\chi\chi'(g) = \chi(g)\chi'(g)$. This group structure will often be written additively.

(ıı) Note that in general, $X_*(G)$ has only a multiplication by integers defined by $n \cdot \lambda(a) = \lambda(a)^n$. If G is commutative, then $X_*(G)$ has a group structure defined by $\lambda\mu(a) = \lambda(a)\mu(a)$.

Definition 4.1.3 *Let V be a rational representation of G. For any $\chi \in X^*(G)$ we define*

$$V_\chi = \{v \in V \ / \ \forall g \in G, \ g \cdot v = \chi(g)v\}.$$

Lemma 4.1.4 *(Dedekin's Lemma) Let G be any group.*

(ı) $\mathbb{X}(G) = \mathrm{Hom}_{Groups}(G, \mathbb{G}_m)$ is a linearly independent subset of k^G the set of all functions on G.

(ıı) For any G-module V, we have a direct sum decompopsition

$$V = \bigoplus_{\chi \in \mathbb{X}(G)} V\chi.$$

Proof. (ı) If there is a relation between the elements in $\mathbb{X}(G)$, let us choose such a relation with minimal length *i.e.* n minimal such that there is a relation

$$\sum_{i=1}^n a_i \chi_i = 0$$

with $a_i \neq 0$ and $\chi_i \in \mathbb{X}(G)$ all distinct.

Let g and h in G, we have

$$\sum_{i=1}^n a_i\chi_i(g)\chi_i(h) = \sum_{i=1}^n a_i\chi_i(gh) = 0 = \chi_1(g)\sum_{i=1}^n a_i\chi_i(h).$$

Taking the difference, we get $\sum_{i=2}^n a_i(\chi_i(g) - \chi_1(g))\chi_i(h) = 0$ and thus the relation

$$\sum_{i=2}^n a_i(\chi_i(g) - \chi_1(g))\chi_i = 0.$$

Because $\chi_2 \neq \chi_1$, there exists $g \in G$ such that $\chi_2(g) - \chi_1(g) \neq 0$ and we thus have a smaller relation. A contradiction.

(ıı) Assume that we have a minimal relation

$$\sum_{i=1}^n v_{\chi_i} = 0$$

for $v_{\chi_i} \in V_{\chi_i} \setminus \{0\}$ and all the χ_i distinct in $\mathbb{X}(G)$. We thus have for all $g \in G$:

$$\sum_{i=1}^n \chi_i(g)v_{\chi_i} = g \cdot \sum_{i=1}^n v_{\chi_i} = 0 = \chi_1(g)\sum_{i=1}^n v_{\chi_i}.$$

The same argument shows that we may produce a smaller relation. A contradiction. □

Corollary 4.1.5 *The subset $X^*(G)$ is linearly independent in $k[G]$.*

Lemma 4.1.6 *Let G and G' be two algebraic groups.*

(ı) There is a group isomorphism $X^*(G \times G') \simeq X^*(G) \times X^*(G')$ via $\chi \mapsto (\chi|_G, \chi|_{G'})$.

(ıı) If G is connected, then $X^*(G)$ is torsion free.

Proof. (ı) It is easy to check that the map $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$ is an inverse map.

(ıı) Assume that $\chi^n = 1$. Let $H = \ker\chi$. This is a closed subgroup of G of finite index: indeed $\chi(G)$ is contained in the subgroup of the n-th root of 1. Therefore $\chi(G)/\chi(H)$ is finite and G/H is also finite. Thus we have $G^0 \subset H \subset G$ but G being connected we have equality and $\chi = 1$. □

Example 4.1.7 For $G = \mathrm{D}_n$, write $x \in G$ as $x = \mathrm{diag}(\chi_1(x), \cdots, \chi_n(x))$. Then the χ_i are characters of G. Furthermore, we have $k[G] = k[\chi_i^{\pm}, i \in [1, n]]$. Indeed, from Dedekin's Lemma, all the monomials in the $\chi_i^{\pm}$ form a linearly independent family of functions. We thus have $X^*(G) = \mathbb{Z}^n$. Furthermore, a morphism $\lambda : \mathbb{G}_m \to D$ is of the form $x \mapsto \mathrm{diag}(x^{a_1}, \cdots, x^{a_n})$ therefore $X_*(G) = \mathbb{Z}^n$ and we have a perfect pairing between $X^*(G)$ and $X_*(G)$.

4.1.2 Structure Theorem

Theorem 4.1.8 (Structure theorem of diagonalisable groups) *Let G be an algebraic group. The following properties are equivalent:*

(ı) The group G is commutative and $G = G_s$.

(ıı) The group G is diagonalisable.

(ııı) The group $X^*(G)$ is abelian of finite type and spans $k[G]$ (and therefore forms a base of $k[G]$).

(ıv) Any representation V of G is a direct sum of representations of dimension 1.

Proof. We have already seen the equivalence of (ı) and (ıı).

Let us prove the implication (ıı)⇒(ııı). If G is diagonalisable, then G is closed subgroup of rmD_n thus we have a surjective map $k[T_1^{\pm}, \cdot, T_n^{\pm}] = k[\mathrm{D}_n] \to k[G]$. Furthermore, the T_i are characters of D_n. By restriction, we have that $T_i|_G$ is also a character of G thus the characters of G span $k[G]$. Furthermore, we have a surjective map $X^*(\mathrm{D}_n) \to X^*(G)$ thus the former is of finite type.

Let us prove the implication (ııı)⇒(ıv). Let $\phi : G \to \mathrm{GL}(V)$ be a representation. This can be seen as a map to $\mathfrak{gl}(V) \simeq k^{n^2}$ therefore, we have $\phi(g)_{i,j} \in k[G]$ and we may write $\phi(g)_{i,j} = \sum_\chi a(i,j)_\chi \chi$. Thus we have

$$\phi = \sum_\chi \chi A_\chi$$

for some linear map $A_\chi \in \mathrm{GL}(V)$. Note that only finitely many A_χ are non zero. Now the equality $\phi(gg') = \phi(g)\phi(g')$ yields the equality

$$\sum_\chi \chi(g)\chi(g')A_\chi = \sum_{\chi'} \sum_{\chi''} \chi'(g)\chi''(g')A_{\chi'}A_{\chi''}$$

which by Dedekin's Lemma applied twice gives $A_{\chi'}A_{\chi''} = \delta_{\chi',\chi''}A_{\chi'}$. Note that by evaluating at e, we have $\sum_\chi A_\chi = \mathrm{Id}$. In particular, if we set $V_\chi = \mathrm{im} A_\chi$, we have $V = \oplus_\chi V_\chi$. Furthermore, any $g \in G$ acts on V_χ as $\chi(g) \cdot \mathrm{Id}$. This proves the implication.

The implication (ıv)⇒(ıı) is obvious because if we embed G in $\mathrm{GL}(V)$, then G will be contained in the diagonal matrices given by the base coming from (ıv). □

Corollary 4.1.9 *Let G be a diagonalisable group, then $X^*(G)$ is an abelian group of finite type without p-torsion if $p = char(k)$. The algebra $k[G]$ is isomorphic to the group algebra of $X^*(G)$.*

Proof. We have already seen that $X^*(G)$ is an abelian group of finite type. Furthermore, if $X^*(G)$ has p-torsion, then there exists a character $\chi : G \to \mathbb{G}_m$ such that $\chi^p = 1$. This gives the relation $(\chi - 1)^p = \chi^p - 1 = 0$ in $k[G]$ which would not be reduced. A contradiction.

Recall that the group algebra of $X^*(G)$ has a base $(e(\chi))_{\chi \in X^*(G)}$ and the multiplication rule $e(\chi)e(\chi') = e(\chi\chi')$. But by the previous Theorem $k[G]$ has a base indexed by $X^*(G)$ with the same multiplication rule. □

Conversely, let M be any abelian group of finite type without p-torsion if $p = \mathrm{char}(k)$. We define $k[M]$ to be its groups algebra.

Proposition 4.1.10 *(ı) There is a diagonalisable algebraic group $G(M)$ with $k[G(M)] = k[M]$ defined by $\Delta(e(m)) = e(m) \otimes e(m)$, $\iota(e(m)) = e(-m)$ and $e_G(e(m)) = 1$.*

(ıı) We have an isomorphism $X^*(G(M)) \simeq M$.

(ııı) For G diagonalisable, we have an isomorphism $G \simeq G(X^*(G))$.

(ıv) Under the correspondence $G \mapsto X^*(G)$ and $M \mapsto G(M)$, we have the following isomorphism: $G(M \otimes M') = G(M) \times G(M')$

Proof. We start with (ıv). Indeed if such a group structure exists then taking the tensor product on the algebra level corresponds to taking the product of algebraic groups. Obviously the two group struture agree.

(ı) The abelian group M is a direct sum of copies of $\mathbb{Z}$ and of finite cyclic groups $\mathbb{Z}/n\mathbb{Z}$ with n prime to p. Therefore we are left to deal with these two case. For $M = \mathbb{Z}$, we recover the torus $\mathbb{G}_m$ whose algebra $k[\mathbb{G}_m] = k[T, T^{-1}]$ is isomorphic to $k[M]$ by sending T to $e(1)$.

For $M = \mathbb{Z}/n\mathbb{Z}$, the algebra $k[M]$ a quotient of $k[\mathbb{Z}]$ given by $\phi^\sharp : k[\mathbb{Z}] \to k[M]$ with $\phi^\sharp(e_\mathbb{Z}(1)) = e_M(1)$. Therefore $G(M) = \operatorname{Spec} k[M]$ is a closed finite subset of $\mathbb{G}_m$. But the comutiplication Δ_M is compatible with $\Delta_{\mathbb{G}_m}$ thus $G(M)$ is a closed subgroup of $\mathbb{G}_m$.

(ıı) From the definition of the comultiplication, any element $e(m) \in k[M]$ for $m \in M$ is a character of $G(M)$. Conversely, if χ is a character, then $\chi = \sum_i a_i e(m_i)$ but by Dedekin's Lemma we get that $\chi = a_i e(m_i)$ for some i. Furthermore, because $\chi(e) = 1 = e(m_i)(e)$ we get $a_i = 1$ and the result.

(ııı) We have an isomorphism $k[G] = k[X^*(G)]$. □

Corollary 4.1.11 *Let G be a diagonalised algebraic group, then the following are equivalent.*

(ı) The group G is a torus.

(ıı) The group G is connected.

(ııı) The group $X^*(G)$ is a free abelian group.

Proof. Obviously (ı) implies (ıı) and we have seen that (ıı) implies (ııı). Furthermore if (ııı) holds, then $X^*(G) \simeq \mathbb{Z}^r$, thus $G \simeq G(X^*(G)) \simeq G(\mathbb{Z})^r \simeq \mathbb{G}_M^r$. □

Corollary 4.1.12 *A diagonalisable algebraic group is a product of a torus and a finite abelian group of ordre primes to $p = char(k)$.*

4.2 Rigidity of diagonalisable groups

Let us start with the following proposition.

Proposition 4.2.1 *Let G and H be diagonalisable algebraic groups and let V be an affine connected variety. Assume that $\phi : G \times V \to H$ is a morphism such that for all $v \in V$, the induced morphism $\phi_v : G \to H$ is a algebraic group morphism.*

*Then the morphism ϕ is constant on V (*i.e.* it factors through G).*

Proof. Let $\phi^\sharp : k[H] \to k[G] \otimes k[V]$ be the comorphism and let $\psi \in X^*(H)$. We may write

$$\phi^\sharp(\psi) = \sum_{\chi \in X^*(G)} \chi \otimes f_{\psi,\chi},$$

with $f_{\psi,\chi} \in k[V]$. Now $\psi \circ \phi_v$ is a character of G and we have

$$\psi \circ \phi_v = \phi_v^\sharp(\psi) = \sum_{\chi \in X^*(G)} \chi f_{\psi,\chi}(v).$$

By Dedekin's Lemma, we get $\psi \circ \phi_v = f_{\psi,\chi_v}(v)\chi_v$ for some χ_v, $f_{\psi,\chi_v}(v) \in \mathbb{Z}$ and $f_{\psi,\chi}(v) = 0$ for $\chi \neq \chi_v$. Thus f_{ψ,χ_v} maps V to $\mathbb{Z}$ and because V is connected, it is constant. Therefore, for any χ, we have $f_{\psi,\chi} \in \mathbb{Z}$ (a constant function with value in $\mathbb{Z}$). We may thus define $\Phi : G \to H$ by

$$\Phi^\sharp(\psi) = \sum_{\chi \in X^*(G)} \chi f_{\psi,\chi} \in k[G].$$

It is enough to define $\Phi^\sharp$ on $X^*(H)$ since H is diagonalisable. The map ϕ factors through Φ. □

Definition 4.2.2 *Let G be an algebraic group and H be a closed subgroup. We denote by $N_G(H)$ and $C_G(H)$ the normaliser of H and centraliser of H. These are closed subgroups.*

Exercise 4.2.3 Prove that $N_G(H)$ and $C_G(H)$ are closed subgroups of G.

Corollary 4.2.4 *Let G be any algebraic group and let H be a diagonalisable subgroup of G. Then we have the equality $N_G(H)^0 = C_G(H)^0$ and $W(G,H) = N_G(H)/C_G(H)$ is finite.*

Proof. Consider the morphism $H \times N_G(H)^0 \to H$ defined by $\phi(x,y) = yxy^{-1}$. By the previous proposition, this map does not depend on y therefore $\phi(x,y) = \phi(x,e_H) = x$ and $N_G(H)^0 \subset C_G(H)$ proving the first equality.

Now $C_G(H)$ is a closed subgroup of $N_G(H)$ and contains $N_G(H)^0$ therefore it is of finite index. $\square$

Definition 4.2.5 *The Weyl group of an algebraic group G with respect to a torus T of G is the finite group $W(G,T) = N_G(T)/C_G(T)$.*

4.3 Some properties of tori

4.3.1 Centraliser of Tori

Proposition 4.3.1 *Let T be a torus contained in G. Then there exists $t \in T$ such that*

$$C_G(t) = C_G(T) \text{ and } \mathfrak{g}^t = \{\eta \in \mathfrak{g} \ / \ \mathrm{Ad}\,(t)(\eta) = \eta\} = \mathfrak{g}^T.$$

Proof. Let us embed G in $\mathrm{GL}(V)$ and $\mathfrak{g}$ in $\mathfrak{gl}(V)$. We have the equalities $C_G(T) = G \cap C_{\mathrm{GL}(V)}(T)$, $C_G(t) = G \cap C_{\mathrm{GL}(V)}(t)$, $\mathfrak{g}^t = \mathfrak{g} \cap \mathfrak{gl}(V)^t$ and $\mathfrak{g}^T = \mathfrak{gl}(V)^T$. Thus we are reduced to prove this result for $\mathrm{GL}(V)$.

We can write $V = \oplus_{i=1}^r V_{\chi_i}$ for some characters $\chi_i \in X^*(T)$ such that the χ_i are pairwise distinct and $\ker(\chi_i\chi_j^{-1})$ is a closed proper subgroup of T for $i \neq j$. Taking t not in these proper subgroups, we get that all the $\chi_i(t)$ are different. Therefore, we have the equalities

$$C_{\mathrm{GL}(V)}(t) = \prod_{i=1}^r \mathrm{GL}(V_{\chi_i}) = C_{\mathrm{GL}(V)}(T)$$

$$\mathfrak{gl}(V)^t = \prod_{i=1}^r \mathfrak{gl}(V_{\chi_i}) = \mathfrak{gl}(V)^T.$$

The result follows. $\square$

4.3.2 Pairing

Note that $X^*(\mathbb{G}_m) \simeq \mathbb{Z}$. Let us identify $X^*(\mathbb{G}_m)$ with $\mathbb{Z}$. We may define a pairing

$$\langle , \rangle : X^*(G) \times X_*(G) \to \mathbb{Z}$$

by $\langle \chi, \lambda \rangle = \chi \circ \lambda \in X^*(\mathbb{G}_m) = \mathbb{Z}$. Explicitely, we have $\chi \circ \lambda(z) = z^{\langle \chi, \lambda \rangle}$ for all $z \in \mathbb{G}_m$.

Proposition 4.3.2 *Let T be a torus, then the above pairing is perfect. In particular $X_*(T)$ is a free abelian group.*

Proof. It suffices to check the case of D_n which we did explicitely. $\square$

Chapter 5

Unipotent and sovable groups

5.1 Definitions

5.1.1 Groups

Definition 5.1.1 *Let G be an algebraic group.*

(i) We denote by $D(G)$ the closed subgroup (G,G). We define by induction $D^{i+1}(G)$ as the group $(D^i(G),D^i(G))$ and $D^0(G)=G$ (and thus $D^1(G)=D(G)$).

(ii) We define by induction $C^{i+1}(G)$ as $(G,C^i(G))$ and $C^0(G)=G$ (and thus $C^1(G)=D(G)$).

(iii) The group G is called solvable (resp. nilpotent) if $D^i(G)=\{e_G\}$ (resp. $C^i(G)=\{e_G\}$) for some i.

Fact 5.1.2 *Because we have the inclusions, $D^i(G)\subset C^i(G)$, if the group G is nilpotent, then it is solvable.*

Lemma 5.1.3 *If H and K are normal closed subgroups, then (H,K) is a closed normal subgroup.*

In particular, the subgroups $D^i(G)$ and $C^i(G)$ are closed normal (and even characteristic) subgroups for all $i\geq 0$.

Proof. The fact that the subgroup is normal is a classical fact from group theory. We know that the two groups (H^0,K) and (H,K^0) are connected and closed. Their product C is again connected and closed. One can prove (this is a purely group theoretic fact, exercise!) that C has finite index in (H,K) therefore (H,K) is a finite union of translates of C and therefore is closed. □

Fact 5.1.4 *If $1\to H\to G\to K\to 1$ is an exact sequence, then G is solvable if and only if H and K are solvable. If furthermore G is nilpotent, then so are H and K.*

Proof. Exercise. □

Definition 5.1.5 *The sequences $(D^i(G))_{i\geq 0}$ and $(C^i(G))_{i\geq 0}$ are decreasing sequences of closed subsets of G. Therefore, they are constant for i large enough. We define*

$$D^\infty(G)=\bigcap_{i\geq 0}D^i(G)\ \textit{and}\ C^\infty(G)=\bigcap_{i\geq 0}C^i(G).$$

5.1.2 Lie algebras

We can give the corresponding definitions for Lie algebras. We then get.

Proposition 5.1.6 *Let G be an algebraic group and $\mathfrak{g}$ be its Lie algebra.*
If G is solvable (resp. nilpotent) then so is $\mathfrak{g}$.

5.1.3 Upper triangular matrices

We will denote by T_n or B_n the subgroup in $\mathrm{GL}(V)$ of upper triangular matrices and recall that U_n is the subgroup of matrices with 1 on the diagonal. One easily check the inclusions $D(\mathrm{B}_n) \subset \mathrm{U}_n$ therefore U_n is normal in B_n.

Proposition 5.1.7 *The groups B_n and U_n are connected and respectively solvable and nilpotent.*

Proof. These varieties are isomorphic to open subspaces of affine spaces thus connected. Note that it is enough to prove that U_n is nilpotent. But we easily check that $C^i(\mathrm{U}_n)$, for $i \geq 1$, is contained in the subgroup of U_n of matrices with $a_{k,l} = 0$ for $1 \leq l - k \leq i$. □

5.2 Lie-Kolchin Theorems

5.2.1 Burnside and Wederburn Theorem

Lemma 5.2.1 (Schur's Lemma) *Let A be a k-algebra and V be a simple finite dimensional A-module. Then any endomorphism $u \in \mathrm{End}_A(V)$ is of the form $\lambda \mathrm{Id}_V$.*

Proof. First we know that if u is non zero, then it is an isomorphism since $\ker u$ and $\mathrm{im} u$ are proper resp. non trivial subspaces. Therefore $\mathrm{End}_A(V)$ is a division algebra (event. non commutative field).

Now let $f : k[T] \to \mathrm{End}_A(V)$ be defined by $T \mapsto u$. Because $\mathrm{End}_A(V)$ is a subspace of $\mathrm{End}_k(V)$, it is of finite dimension and the image of f is a subalgebra of $\mathrm{End}_A(V)$ which is integral (since $\mathrm{End}_A(V)$ is a division algebra). Thus $\ker f = (P)$ is a prime ideal *i.e.* P is irreducible ($\ker f$ is not trivial by the dimension argument). But k being algebraically closed, P is of degree one and the result follows. □

Theorem 5.2.2 (Burnside-Wederburn) *Let A be a subalgebra of $\mathfrak{gl}(V)$ with V finite dimensional. If V is a simple A-module, then $A = \mathfrak{gl}(V)$.*

Proof. Let us start with a Lemma.

Lemma 5.2.3 *Let W be a proper subspace of V and $v \in V \setminus W$, then there exists $u \in A$ such that $u|_W = 0$ and $u(v) \neq 0$.*

Proof. We proceed by induction on $\dim W$. For $w = 0$ we take $u = 1 \in A$. Let us assume that this is true for W or dimension $r - 1$ and prove it for W of dimension$r$. We write $W = W' \oplus kw$ for some W' and w. Let $I = Ann_A(W')$ and $J = Ann_A(W)$. We want to prove that $Jv \neq 0$ for all $v \in V \setminus W$. Assume on the contrary that $Jv = 0$ for some $v \in V \setminus W$.

Let us consider $Iw \subset V$. This is an A-submodule of V because I is an ideal. But by induction hypothesis, it is non trivial and because V is simple $Iw = V$. Therefore for any $x \in V$, we have $x = i(w)$ for some $i \in I$ (not necessarily unique). Define $\phi \in \mathfrak{gl}(V)$ by

$$\phi(x) = i(v),$$

with v as above satisfying $Jv = 0$. This is well defined, indeed, if $i(w) = j(w)$ for $i, j \in I$, then $(i - j)(w) = 0$ thus $i - j \in J$ and $(i - j)(v) = 0$. Furthermore ϕ is A-linear:

$$\phi(ax) = \phi(ai(w)) = ai(v) = a \cdot i(v) = a\phi(x)$$

because $i \in I \Rightarrow ai \in I$ as I is a left ideal. By Schur's Lemma, we get $\phi = \lambda \mathrm{Id}_V$. Thus for any $i \in I$, we have

$$\lambda i(w) = \phi(i(w)) = i(v)$$

thus $i(v - \lambda w) = 0$ for all $i \in I$ and by induction hypothesis, this implies that $v - \lambda w \in W'$ and thus $v \in W$, a contradiction. □

Let us now prove the theorem. For this we prove, choosing a basis $(e_i)_{i\in[1,n]}$ of V, that all the elementary matrices $E_{i,j}$ are in A. Let $V^j = \mathrm{span}((e_i)_{i\neq j})$ and apply the previous lemma to V^j and e_i, we get that there exists an element $u \in A$ with $u|_{V^j} = 0$ and $u(e_j) \neq 0$. But V being simple, there exists $a \in A$ with $au(e_j) = e_i$ and the composition au does the job. □

5.2.2 Unipotent groups

Theorem 5.2.4 (Lie-Kolchin Theorem 1) *(ı) Let V be a vector space and G be a subgroup of $\mathrm{GL}(V)$ whose elements are unipotent. Then there exists a basis of V such that all the elements of G become upper-triangular matrices.*

(ıı) Let G be an unipotent algebraic group and let $\rho : G \to \mathrm{GL}_n$ be a representation of G, then $\rho(G)$ is conjugated to a subgroup of the group of upper-triangular matrices.

In particular G is nilpotent as well as $\mathfrak{g}$.

Proof. (ı) We proceed by induction on the dimension of V. It is enough to prove a non-zero vector in V which is fixed by G. We may assume that V is a simple G-module (it is enough to find such a vector in a simple component of V). Let A be the subalgebra of $\mathfrak{gl}(V)$ spanned by G. By Burnside-Wedderburn Theorem, we have $A = \mathfrak{gl}(V)$.

Let us now prove that V has dimension 1. For this remark that because all the elements in G are unipotent, we have the equalities for $g, h \in G$: $\mathrm{Tr}(g) = \dim V = \mathrm{Tr}(gh)$. Therefore, we have $\mathrm{Tr}((g-1)h) = 0$ for all $g, h \in G$ and by linearity $\mathrm{Tr}((g-1)u) = 0$ for all $u \in A = \mathfrak{gl}(V)$. In particular $0 = \mathrm{Tr}((g-1)E_{i,j}) = g_{j,i} - delta_{i,j}$ thus $g = 1$ and $G = \{e_G\}$. The space V being simple it is of dimension 1.

(ıı) The image is a subgroup composed by unipotent matrices. Taking the Lie algebras we get the inclusion of $d\rho(\mathfrak{g})$ in the upper-triangular matrices.

For the last assertions, use a faithful representation. □

5.2.3 Solvable groups

Theorem 5.2.5 (Lie-Kolchin Theorem 2) *Let G be a solvable connected algebraic group and let $\rho : G \to \mathrm{GL}_n$ be a representation. Then $\rho(G)$ is conjugated to a subgroup of B_n.*

Proof. We proceed by induction on $n + \dim G$. It is enough to produce a line stable under G. We may therefore assume that $V = k^n$ is simple. By induction hypoethesis, there is a stable line L for $D(G)$ because it is a proper closed subgroup of G (since G is solvable). Looking at the map $\chi : D(G) \to \mathrm{GL}(L) = \mathbb{G}_m$, we get a character $\chi_0 \in X^*(D(G))$ with $V_{\chi_0} \neq 0$.

Fact 5.2.6 *For any character $\chi \in X^*(D(G))$, there exists a finite dimensional vector subspace W of $k[D(G)]$ containing χ with an action of G.*

Proof. Consider the action of G on $D(G)$ defined by $a : G \times D(G) \to D(G)$ with $a(g,h) = ghg^{-1}$. We now that there is a finite dimensional vector space W of $k[D(G)]$ containing χ such that $a^\sharp(W) \subset k[G] \otimes W$. Thus we have a rational representation of G on W. Recall that g acts on χ by $g \cdot \chi(h) = \chi(g^{-1} \cdot h)$. □

In particular, the stabiliser G_χ of any character is a closed subgroup of G. Furthermore, we have that the subspaces $V_{g\cdot\chi_0}$ are indirect sum for all $g \in G$ therefore the orbit of χ_0 has to be finite. Because G is connected, we have $G_{\chi_0} = G$. Therefore, for $g \in G$, $h \in D(G)$ and $v \in V_{\chi_0}$ we have

$$hg \cdot v = gg^{-1}hg \cdot v = g \cdot \chi_0(g^{-1}hg)v = g \cdot (g \cdot \chi_0)(h)v = g \cdot \chi_0(h)v$$

thus $g \cdot v \in V_{\chi_0}$. Therefore V_{χ_0} is a G-submodule and because V is simple, we have $V_{\chi_0} = V$. We thus get that for any $h \in D(G)$, we have $\rho(h) = \chi_0(h)\mathrm{Id}_V$. But the elements in $D(G)$ being commutators, we have $\det(\rho(h)) = 1$ thus $\chi_0(h)^n = 1$ and $\rho(D(G))$ is contained in the finite group of the n-th roots of Id_V. The group G being connected, so is $D(G)$ and $\rho(D(G)) = \{\mathrm{Id}_V\}$. In particular, $\rho(G)$ is commutative and the result follows. □

Corollary 5.2.7 *Let G be a solvable connected group.*
(ı) Then $D(G)$ is connected and unipotent.
(ıı) Then G_u is a closed normal subgroup.

Proof. We embed G in GL_n then G is contained in B_n and therefore $D(G)$ is contained in $D(\mathrm{B}_n) \subset \mathrm{U}_n$ therefore all its elements are unipotent. Furthermore we easily have $G_u = G \cap \mathrm{U}_n$ therefore is it a closed normal subgroup. □

Remark 5.2.8 The connectedness hypothesis is important since the normaliser N_2 of the subgroup D_2 in GL_2 is solvable but not conjugated to a subgroup of B_2.

Theorem 5.2.9 *Assume char$(k) = 0$, let $\mathfrak{g}$ be a solvable Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then there exists a base such that $\rho(\mathfrak{g})$ is contained in the subspace of upper-triangular matrices.*

Proof. Cf. last year lecture. Note that we need the assumption $\mathrm{char}(k) = 0$. □

Remark 5.2.10 In the previous statement, if we assume that $\mathfrak{g}$ is the Lie algebra of an algebraic group G which is solvable, then we do not need the characteristic assumption.

5.3 Structure Theorem

5.3.1 Statement of the existence of quotients

We shall need some fact on homogeneous G-spaces.

Lemma 5.3.1 *Let G be an algebraic group and let X be an homogeneous G-space (*i.e.* with a transitive action of the group G).*

(ı) The connected components coincide with the irreducible components of X. These components are homogeneous under G^0. There are translated from each other and are equidimensional of dimension $\dim G - \dim G_x$ for any $x \in X$.

(ıı) If X and G_x are irreducible, then so if G. In particular, in an exact sequence $1 \to H \to G \to K \to 1$, the group G is connected as soon as H and K are connected.

Proof. (ı) Let us write $G = \coprod_i G^0 g_i$. Let $x \in X$, the map $G \to X$ defined by $g \mapsto g \cdot x$ is surjective, therefore we have that X is the union of the irreducible spaces $G^0 g_i x$. If two such spaces intersect, then they agree. We thus have $X = \coprod_{i_j} G^0 g_{i_j} x$. One of these G^0-orbit is closed thus they are all closed. These are the irreducible and the connected components. Furthermore, they are homogeneous under G^0, translated from each other and the last formula follows.

(ıı) If X is irreducible, then we have $X = G^0 x$. Let $g \in G$, then there exists $g_0 \in G^0$ with $gx = g_0 x$ thus $g_0^{-1} g \in G_x$. The group G_x being connected, we have $G_x \subset G^0$ thus $g \in G^0$ and $G = G^0$. $\square$

Remark 5.3.2 Let $G = \mathrm{SL}_2$ act on $X = \mathfrak{sl}_2$ by the adjoint action and let $x = E_{1,2}$. Even if G and X are irreducible, the group G_x is not.

We shall need the following general result ono existence of quotients by closed subgroups. We will prove this result in the next chapter.

Theorem 5.3.3 *Let G be an algebraic group and let H be a closed subgroup.*

(ı) Then the set G/H of orbits under the right action of H has a structure of algebraic varieties such that the map $\pi : G \to G/H$ is a morphism of algebraic varieties satisfying the following universal property: for any morphism $\phi : G \to X$ constant on the classes gH, there exists a unique morphism $\psi : G/H \to X$ making the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & X \\ G/H. \arrow[ur, "\psi"] & \end{tikzcd}$$

(ıı) If furthermore, H is normal in G, then G/H is an algebraic group for the above structure.

(ııı) Any algebraic group morphism $\phi : G \to G'$ induces a bijective morphism of algebraic groups $G/\ker\phi \to \phi(G)$.

Remark 5.3.4 Note that the above morphism is not necessarily an isomorphism as already seen for the map $\mathrm{SL}_p \to \mathrm{PSL}_p$ with $p = \mathrm{char}(k) > 0$.

To know when such a bijective morphism is an isomorphism we shall need the notion of *separable morphisms* which we shall study inthe next chapter. Let us state what we shall need here

Proposition 5.3.5 *Let $\phi : X \to Y$ a G-equivariant morphism between homogeneous G-spaces. Assume that ϕ is bijective and that $d_x\phi$ is surjective for some $x \in X$, then ϕ is an isomorphism.*

5.3.2 Structure Theorem

We start with a structure theorem for nilpotent groups.

Proposition 5.3.6 (Structure Theorem for nilpotent groups) *Let G be a connected solvable group and let $\mathfrak{g}$ be its Lie algebra. We have the equivalence between the following two conditions.*

(ı) The group G is nilpotent.

(ıı) We have the inclusion $G_s \subset Z(G)$.

If these conditions are satisfied, then G_s is a closed connected central subgroup and we have an isomorphism $G_s \times G_u \to G$ induced by the multiplication.

Furthermore, we have $L(G_u) = \mathfrak{g}_n$, $L(G_s) = \mathfrak{g}_s$ and $\mathfrak{g}_s \oplus \mathfrak{g}_n = \mathfrak{g}$.

Proof. (ı)⇒(ıı) Assume that G is nilpotent. We proceed by induction on $\dim G$. If G is abelian, then the result is true. If not, consider $C^n(G)$ with n such that $C^n(G) \neq \{e\}$ but $C^{n+1}(G) = \{e\}$. Therefore $C^n(G)$ is central in G. Let $\pi : G \to G' = G/C^n(G)$. Let $s \in G_s$, the $\pi(s) \in G'_s$ and by induction hypothesis $\pi(s) \in Z(G')$. Let $g \in G$, we have $h = gsg^{-1}s^{-1} \in \ker \pi = C^n(G)$. Thus $sh = hs = gsg^{-1} = z$ is semisimple. We get $zs^{-1} = h = s^{-1}z$. Thus z and s^{-1} are semisimple and commute, their product h is also semisimple. But $h \in C^n(G) \subset D(G)$ is unipotent thus $h = e$ and s is central.

(ıı)⇒(ı) Assume that $G_s \subset Z(G)$. Then by taking the quotient $G/Z(G)$, we may assume that G is unipotent and therefore nilpotent.

Now we embed G in $\mathrm{GL}(V)$. Because all the elements in G_s are central, they commute with each other and we may find a base of V such that $G_s = G \cap \mathrm{D}_n$. Thus G_s is closed. We have a decomposition $V = \oplus_\chi V_\chi$ for χ characters of G_s. Furthermore, since G_s is central, the group G preserves the V_χ. The quotient G/G_s is unipotent and thus there is a base of V_χ such that the elements in G/G_s are upper-triangular matrices. We thus have a morphism $G \to G_s$ obtained by taking the diagonal terms. Thus G_s is connected. Furthermore, the map $G \to G_u$ defined by $g \mapsto g_u$ is also a morphism. We conclude as for the structure theorem for commutative groups. □

Lemma 5.3.7 *Let G be a non trivial nilpotent group, then $Z(G)^0$ is non trivial.*

Proof. If G is abelian, then the result is obvious. If not, then let n with $C^n(G) \neq \{e\}$ and $C^{n+1}(G) = \{e\}$. Then $C^n(G)$ is closed, connected central and non trivial. □

Theorem 5.3.8 (Structure Theorem for solvable groups) *Let G be a connected solvable group and let $\mathfrak{g}$ be its Lie algebra.*

(ı) Then G_u is connected (and also a normal and closed subgroup).

(ıı) If T is a maximal torus in G, then the map $T \times G_u \to G$ induced by multiplication is an isomorphism. In particular $T \simeq G/G_u$. Furthermore, we have $L(G_u) = \mathfrak{g}_n$ and $\mathfrak{g} = L(T) \oplus \mathfrak{g}_n$.

(ııı) All the maximal tori in G are conjuguated under $C^\infty(G)$ Furthermore, any subgroup composed of semisimple elements in contained in a maximal torus.

(ıv) For any family S of commuting semisimpe elements, we have the equalities $N_G(S) = C_G(S) = C_G(S)^0$.

Proof. (ı) Let us write $G' = G/D(G)$. Then G' is commutative and we have $G' = G'_s \times G'_u$. The group G being connected, the groups G' and G'_u are also connected. But $D(G)$ is contained in G_u therefore we have an exact sequence $1 \to D(G) \to G_u \to G'_u \to 1$. As $D(G)$ and G'_u are connected, the same is true for G_u.

(ıı) We start with the following lemmas.

Lemma 5.3.9 *(ı) Let N be a closed connected commutative subgroup of G_u which is normal in G. Let $s \in G_s$. Then the map $\phi : N \to N$ defined by $\phi(n) = sns^{-1}n^{-1}$ is a morphism of algebraic groups. In particular $\phi(N)$ is a closed subgroup of N.*

(ıı) The morphism $C_N(s) \times \phi(N) \to N$ induced by the multiplication is a bijection and $C_N(s) = \ker \phi$ is connected.

Proof. (ı) Let $n, m \in N$, we have

$$\phi(nm) = s(nm)s^{-1}(nm)^{-1} = (sns^{-1})(sms^{-1})(m^{-1})(n^{-1}) = \phi(n)\phi(m),$$

the third equality being true because all the elements in paranthesis are in N and thus commute.

(ii) Let $(c, \phi(n))$ and $(c', \phi(n'))$ be such that $c\phi(n) = c'\phi(n')$. Then $c^{-1}c' = \phi(n)\phi(n')^{-1} \in C_N(s) \cap \phi(N)$. We claim that this intersection is reduced to e. Indeed, if $\phi(n) \in C_N(s)$, then $sns^{-1}n^{-1}s = ssns^{-1}n^{-1}$ thus $(ns^{-1}n^{-1})s = s(ns^{-1}n^{-1})$ and the element $ns^{-1}n^{-1}$ is semisimple and commutes with s. Therefore the product $\phi(n) = sns^{-1}n^{-1}$ is semisimple. But it is in N thus unipotent and $\phi(n) = e$. This proves the injectivity.

The image of the multiplication map is a closed subgroup of dimension $\dim \phi(N) + \dim C_N(s) = \dim \phi(N) + \dim \ker \phi = \dim N$. The group N being connected, this image is N itself and the multiplication map is surjective.

Let us prove that $C_N(S)$ is connected. Decompose $C_N(s) = \coprod_i c_i C_N(s)^0$ in connected components, we have $N = \coprod_i c_i C_N(s)^0 \phi(N)$ and the $c_i C_N(s)^0 \phi(N)$ are orbits for the group $C_N(s)^0\phi(N)$. One of them is closed and there are therefore all closed. In particular, because N is connected we get that there is a unique such orbit *i.e.* $C_N(s) = C_N(s)^0$ is connected. $\square$

Lemma 5.3.10 *Let G be solvable and connected and let $s \in G_s$, then $C_G(s)$ is connected and $G = C_G(s)G_u$.*

Proof. We proceed by induction on $\dim G$. If G is commutative, then the result is obvious ($C_G(s) = G$). Otherwise, let $N = D^n(G) \neq \{e\}$ with $D^{n+1}(G) = \{e\}$. N is a closed connected commutative subgroup of G_u which is normal in G. We may therefore apply the previous lemma to get $N = C_N(s)\phi(N)$. We may also consider the quotient $\pi : G \to G' = G/N$ for which the results holds by induction. Thus we have $C_{G'}(\pi(s))$ is connected and $G' = C_{G'}(\pi(s))G'_u$.

Fact 5.3.11 *We have the equality $\pi(C_G(s)) = C_{G'}(\pi(s))$.*

Proof. Let $g \in C_G(s)$, we obviously have $\pi(g) \in C_{G'}(\pi(s))$. Now let $g \in G$ with $\pi(g) \in C_{G'}(\pi(s))$. We thus have $\pi(sgs^{-1}g^{-1}) = e_{G'}$ *i.e.* $sgs^{-1}g^{-1} \in N$. We may thus write $sgs^{-1}g^{-1} = c\phi(n) = csns^{-1}n^{-1}$ for $c\mathfrak{n}C_N(s)$ and $n \in N$. We thus have $sgs^{-1}g^{-1} = scns^{-1}n^{-1}$ and $gs^{-1}g^{-1} = cns^{-1}n^{-1}$. Note that c is unipotent (because in N) and commutes with $ns^{-1}n^{-1}$ which is semisimple. Therefore this expression is the Jordan decomposition of $gs^{-1}g^{-1}$ which is semisimple. Thus $c = e_G$ and $gs^{-1}g^{-1} = ns^{-1}n^{-1}$. We thus have $n^{-1}g \in C_G(s)$ and $\pi(n^{-1}g) = \pi(n^{-1}\pi(g) = \pi(g)$, proving the fact. $\square$

We thus have an exact sequence $1 \to C_N(s) \to C_G(s) \to C_{G'}(\pi(s)) \to 1$ because $C_G(s) \cap N = C_N(s)$. The extreme terms being connected we get that $C_G(s)$ is connected.

Furthermore, we have $1 \to N \to G \to C_{G'}(\pi(s))G'_u \to 1$. But $\pi : G_u \to G'_u$ is surjective, indeed if $\pi(g) \in G'_u$, then write $g = g_s g_u$ we have $\pi(g_s)\pi(g_u) = \pi(g)$ is the Jordan decomposition thus $\pi(g_s) = e$ or $g_s \in N$ which is unipotent thus $g_s = e$. We thus get $G = C_G(s)G_u$. $\square$

We may now prove that there exists a torus T such that the conditions in (ii) are satisfied. We proceed by induction on $\dim G$. We may assume that G is not nilpotent otherwise we already proved the result. Let thus $s \in G_s$ which is not central thus $C_G(s)$ is a proper subgroup of G. It is connected by the previous lemma thus by induction we get that there exists a torus T of $C_G(s)$ such that $C_G(s) = TC_G(s)_u$. We deduce by the above lemma that $G = C_G(s)G_u = TG_u$ because $C_G(s)_u \subset G_u$.

The morphism $\mu : T \times G_u \to G$ induced by multiplication is therefore surjective. It is injective since $T \cap G_u = \{e_G\}$ since $T \subset G_s$. Let us check that this map is separable or more precisely that there exists $(t, g_u) \in T \times G_u$ such that $d_{(t,g_u)}\mu$ is surjective. This will prove that μ is an isomorphism. We compute this at (e_G, e_G). We have $d_{(e_G,e_G)}\mu(X, Y) = X + Y$.

Fact 5.3.12 *Let T be diagonalisable and G_u unipotent, then $L(T) = L(T)_s$ and $L(G_u) = L(G_u)_n$.*

Proof. We may embed T and G_u in some GL_n such that T is composed of diagonal matrices and G_u is contained in U_n. Then their Lie algebras are composed of diagonal and strictly upper-triangular matrices and the result follows. □

We thus have that X is semisimple and Y is nilpotent. But $X = -Y$ implies that their are both semisimple and nilpotent thus $X = Y = 0$ and $d_{(e_G,e_G)}\mu$ is injective. Furthermore, we have $\dim G \leq \dim T + \dim G_u$ thus in fact it is surjective and μ is an isomorphism. We also get $\mathfrak{g} = L(T) \oplus L(G_u)$. This implies $L(G_u) = \mathfrak{g}_n$.

The composed map $T \to T \times G_u \to G$ is constant on the fibers of the quotient $G \to G/G_u$ thus there is a morphism $T \to G/G_u$ which is a group morphism. This morphism is a bijection and its differential is surjective because $\mathfrak{g} = L(G_u) \oplus L(T)$. It is an isomorphism.

Let us prove the following result.

Proposition 5.3.13 *Let G be a connected solvable group and T be a torus in G such that $G = TG_u$. Let $s \in G_s$, then there exists $g \in C^\infty(G)$ such that $gsg^{-1} \in T$.*

Proof. We procced by induction on $\dim G$. If G is nilpotent, then $T \subset G_s \subset Z(G)$. But the condition $G = TG_u$ implies that $T = G_s$ and the result follows. We may thus assume that G is not nilpotent. Therefore $C^\infty(G)$ is a proper closed normal non trivial subgroup in G. It is unipotent (because it is contained in $D(G)$) and thus nilpotent. Let n such that $C^n(C^\infty(G))$ is non trivial but $C^{n+1}(C^\infty(G)) = \{e\}$. Set $N = C^n(C^\infty(G))$ and consider the quotient $\pi : G \to G/N$. We have $G/N = \pi(T)\pi(G_u) = \pi(T)(G/N)_u$. By induction hypothesis, we get that there is a $g \in G$ with $\pi(g) \in C^\infty(G/N)$ such that $\pi(g)\pi(s)\pi(g)^{-1} \in \pi(T)$. By the following fact, we may assume that $g \in C^\infty(G)$.

Fact 5.3.14 *If $\pi : G \to G'$ is a surjective morphism of group, then $\pi(C^n(G)) = C^n(G')$ and $\pi(D^n(G)) = D^n(G')$ for all n.*

We thus have an $n \in N$ and a $t \in T$ such that $gsg^{-1} = tn$. But N is a closed commutative subgroup of G_u and is normal in G. Therefore, we may write $n = ct^{-1}utu^{-1}$ for $u \in N$ and $c \in C_N(t^{-1})$. We thus have $gsg^{-1} = tct^{-1}utu^{-1}$ thus $gsg^{-1} = cutu^{-1}$ because c commutes with t. But c is unipotent and commutes with utu^{-1} which is semisimple thus the last expression is the Jordan decomposition of gsg^{-1} which is semisimple. Thus $c = e$. We get $s = (g^{-1}u)t(g^{-1}u)^{-1}$. But $g^{-1} \in C^\infty(G)$ and $u \in N \subset C^\infty(G)$ the result follows. □

To finish the proof of (ii) and (iii) we prove the following Theorem.

Theorem 5.3.15 *Let T be a torus such that $G = TG_u$ and let S be a subgroup of G (not nec. closed) with $S \subset G_s$.*

(i) The group S is commutative.

(ii) The group $C_G(S)$ is connected and there exists $g \in C^\infty(G)$ such that $gSg^{-1} \subset T$. In particular all the maximal tori in G are conjugated under $C^\infty(G)$.

(iii) We have the equality $C_G(S) = N_G(S)$.

Proof. (i) We have $D(G) \subset G_u$ therefore G/G_u is commutative. Consider the quotient map $\pi : G \to G/G_u$. Let $\pi|_S$ be its restriction to S, then this restriction is injective since $S \subset G_s$. Thus S is commutative.

(ii) We proceed by induction on $\dim G$. If S is central *i.e.* $S \subset Z(G)$, then $C_G(S) = G$ which is connected. Furthermore, the groups S and T commute. Let H be the group generated by S and T, we have $H \subset G_s$ and H is commutative. The map $\pi|_H : H \to G/G_u$ is therefore injective. But by assumption $\pi(T) \subset \pi(H) \subset \pi(G) = \pi(T)$ thus $H = T$ and $S \subset C_G(S) \subset T$. The result follows in this case.

If S is not central, let $s \in S$ not in the center. Then by the previous proposition, we may assume that $s \in T$. The group $C_G(s)$ is a proper connected subgroup of G containing T and S. By induction we have that $C_{C_G(s)}(S)$ is connected and that there exists $c \in C^\infty(C_G(s)) \subset C^\infty(G)$ such that $cSc^{-1} \subset T$. Note also that $C_G(S) = C_{C_G(s)}(S)$ which is thus connected.

(ııı) Let $g \in N_G(S)$ and $s \in S$. The group G/G_u is abelian thus $\pi(gsg^{-1}s^{-1}) = e$ and $gsg^{-1}s^{-1} \in G_u$. But $s, gsg^{-1} \in S$ are semisimple and commute thus $gsg^{-1}s^{-1}$ is also semisimple therefore trivial and the result follows. $\square$

This theorem concludes the proof. $\square$

Chapter 6

Quotients

6.1 Differentials

6.1.1 Module of Kähler differentials

In this section, we assume that k is any commutative ring. Let A be a k-algebra and let $I = \ker(A \otimes A \to A)$ be the kernel of the multiplication on A. We shall consider $A \otimes A$ as an A-module via the action:

$$a \cdot (b \otimes c) = ab \otimes c = (a \otimes 1)(b \otimes c).$$

The ideal I is obviously an A-submodule for this A-module structure.

Definition 6.1.1 *We define the module $\Omega_{A/k}$ of Kähler differentials of A over k by $\Omega_{A/k} = I/I^2$. We also define $\mathrm{d} : A \to \Omega_{A/k}$ the map sending $a \in A$ to the class in I/I^2 of $1 \otimes a - a \otimes 1$.*

Lemma 6.1.2 *(ı) The ideal I and the module $\Omega_{A/k}$ are spanned by the elements $1 \otimes a - a \otimes 1$ and $\mathrm{d}(a)$ respectively.*

(ıı) The map $\mathrm{d} : A \to \Omega_{A/k}$ is a derivation.

(ııı) If A is of finite type spanned by the x_i (with relations!), then $\Omega_{A/k}$ is an A-module of finite type spanned by the $\mathrm{d}x_i$.

(ıv) The A-module $\Omega_{A/k}$ represents the functor $\mathrm{Der}_k(A, \bullet)$.

Proof. (ı) Let $x = \sum_i a_i \otimes b_i \in I$, then we have $\sum_i a_i b_i = 0$ thus we have $x = \sum_i a_i \otimes b_i - \sum_i a_i b_i \otimes 1 = \sum_i a_i \cdot (1 \otimes b_i - b_i \otimes 1)$. This proves (ı).

(ıı) Remark that $(1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1$ lies in I^2. Let us compute in I/I^2 and use the previous remark in the third equality:

$$\begin{aligned} \mathrm{d}(ab) &= 1 \otimes ab - ab \otimes 1 \\ &= a \cdot (1 \otimes b - b \otimes 1) + (1 \otimes a - a \otimes 1)(1 \otimes b) \\ &= a \cdot (1 \otimes b - b \otimes 1) + (1 \otimes a - a \otimes 1)(b \otimes 1) \\ &= a\mathrm{d}b + b\mathrm{d}a. \end{aligned}$$

(ııı) The algebra A is spanned as a k-module by the monomials $x_1^{n_1} \cdots x_r^{n_r}$. The A-module is therefore spanned by their image $\mathrm{d}(x_1^{n_1} \cdots x_r^{n_r})$. But d being a derivation, we get $\mathrm{d}(x_1^{n_1} \cdots x_r^{n_r}) = \sum_i n_i x_1^{n_1} \cdots x_i^{n_i - 1} \cdots x_r^{n_r} \mathrm{d}(x_i)$ and the result follows.

(ıv) Let M be an A-module then we have a map $\mathrm{Hom}_A(\Omega_{A/k}, M) \to \mathrm{Der}_k(A, M)$ defined by $\phi \mapsto \phi \circ \mathrm{d}$. Let $D \in \mathrm{Der}_k(A, M)$, then $D : A \to M$ and we define $\phi_D : A \otimes A \to M$ by $\phi_D(a \otimes b) = aD(b)$ and extend it by linearity. If $x = (1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1) \in I^2$, then we have $\phi_D(x) =$

$D(ab) + abD(1) - aD(b) - bD(a) = 0$. Therefore ϕ_D restricted to I vanishes on I^2 and defined a morphism $\phi_D : \Omega_{A/k} \to M$. Let us prove that $D \mapsto \phi_D$ is the inverse of $\phi \mapsto \phi \circ \mathrm{d}$.

We have $\phi_D \circ \mathrm{d}(a) = \phi_D(1 \otimes a - a \otimes 1) = D(a) - aD(1) = D(a)$. Conversely, we have $\phi_{\phi \circ \mathrm{d}}(\mathrm{d}(a)) = \phi_{\phi \circ \mathrm{d}}(1 \otimes a - a \otimes 1) = 1\phi(\mathrm{d}(a)) - a\phi\mathrm{d}(1) = \phi(\mathrm{d}(a))$. The result follows. We then have to check that this is functorial but this is easy and left to the reader. □

Let us recall the following classical result.

Lemma 6.1.3 *(Yoneda's Lemma) Let* $\mathbf{C}$ *be a category and let* X, Y *be two objects in* $\mathbf{C}$*. If there is an isomorphism of functors* $\mathrm{Hom}_{\mathbf{C}}(X, \bullet) \simeq \mathrm{Hom}_{\mathbf{C}}(Y, \bullet)$ *then* $X \simeq Y$*.*

Proposition 6.1.4 *If* $A = k[T_1, \cdots, T_r]$ *is a polynomial algebra, then* $\Omega_{A/k}$ *is the free algebra spanned by the* $\mathrm{d}T_i$*.*

Proof. We already know that $\Omega_{A/k}$ is spanned by the $\mathrm{d}T_i$ thus we have a surjective map $A^r \to \Omega_{A/k}$. By Yoneda's Lemma, it is enough to prove that for any A-module M, this map induces an isomorphism $\mathrm{Hom}_{A-\mathrm{mod}}(\Omega_{A/k}, M) \simeq \mathrm{Hom}_{A-\mathrm{mod}}(A^r, M)$. Thus we have to prove that the natural map $\psi : \mathrm{Der}_k(A, M) \to M^r$ defined by $\psi(D) = (D(T_1), \cdots, D(T_r))$ is an isomorphism. But as the computation above shows that a derivation on A is uniquely determined by its values on $T_1, \cdots, T_r$. □

Example 6.1.5 Let k be a field and K be a field extension and x a primitive element *i.e.* $K = k(x)$. Then we will see that the following results are true.

(ı) If x is not algebraic, then $\Omega_{K/k} = K\mathrm{d}x$.

(ıı) If x is algebraic and separable *i.e.* $P_{min}(x)$ is prime with $P'_{min}(x)$, then $\Omega_{K/k} = 0$.

(ııı) If for example $K = k[x]/(x^p - a)$ for some $a \notin k^p$, then $\Omega_{K/k} = K\mathrm{d}x$.

Let us recall the following easy fact.

Fact 6.1.6 *Let* $M \to N \to P$ *be a complex of* A*-modules. Then it is an exact sequence if and only if for all* A*-module* Q*, the sequence* $\mathrm{Hom}(P, Q) \to \mathrm{Hom}(N, Q) \to \mathrm{Hom}(M, Q)$ *is exact.*

Proposition 6.1.7 *Let* $\phi : A \to B$ *be a morphism of* k*-algebra.*

(ı) Then ϕ *induces a morphism of* B*-modules* $\phi_* : B \otimes_A \Omega_{A/k} \to \Omega_{B/k}$ *defined by* $\phi_*(1 \otimes \mathrm{d}a) = \mathrm{d}\phi(a)$*.*

(ıı) We have an exact sequence $B \otimes_A \Omega_{A/k} \to \Omega_{B/k} \to \Omega_{B/A} \to 0$*.*

(ııı) In ϕ *is surjective, then so it* ϕ_* *and we have an exact sequence* $\ker \phi/(\ker \phi)^2 \to B \otimes_A \Omega_{A/k} \to \Omega_{B/k} \to 0$*.*

(ıv) If S *is a multiplicative subset of* A*, then* $S^{-1}A \otimes_A \Omega_{A/k} \to \Omega_{S^{-1}A/k}$ *is an isomorphism.*

Proof. (ı) By the universal property of $\Omega_{A/k}$, if we prove that $D = \mathrm{d} \circ \phi : A \to \Omega_{B/k}$ is a derivation, then there will be an associated A-module morphism $\phi_D : \Omega_{A/k} \to \Omega_{B/k}$ defined by $\phi_D(\mathrm{d}a) = \mathrm{d}\phi(a)$ proving the result. But D is a derivation because d is.

(ıı) By the previous fact, to prove the exactness, we only need to prove that for any B-module M, we have an exact sequence

$$0 \to \mathrm{Hom}_B(\Omega_{B/A}, M) \to \mathrm{Hom}_B(\Omega_{B/k}, M) \to \mathrm{Hom}_B(B \otimes_A \Omega_{A/k}, M)$$

which translates into an exact sequence

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_k(B, M) \to \mathrm{Hom}_A(\Omega_{A/k}, M) = \mathrm{Der}_k(A, M).$$

But we have seen that this is an exact sequence.

(iii) We know that $\Omega_{B/k}$ is spanned by the elements $\mathrm{d}(b)$ for $b \in B$. The map ϕ being surjective, let $a \in A$ with $\phi(a) = b$. We get $\phi_*(1 \otimes \mathrm{d}a) = \mathrm{d}\phi(a) = \mathrm{d}(b)$ and we have the surjectivity.

Let us first construct a map $\ker\phi/(\ker\phi)^2 \to B \otimes_A \Omega_{A/k}$. Consider the linear map $D_0 : A \to B \otimes_A \Omega_{A/k}$ defined by $D_0(a) = 1 \otimes \mathrm{d}(a)$. This is obviously a derivation and we can compute for $a \in A$ and $m \in \ker\phi$:

$$D_0(am) = 1 \otimes (a\mathrm{d}(m) + \mathrm{d}(a)m) = \phi(a) \otimes \mathrm{d}(m) + \phi(m) \otimes \mathrm{d}(a) = a \cdot D_0(m).$$

If furthermore $a \in \ker\phi$, then $D_0(am) = 0$ thus $D_0|_{\ker\phi}$ is A-linear and vanishes on $\ker\phi^2$. This induces a map $D_0 : \ker\phi/(\ker\phi)^2 \to B \otimes_A \Omega_{A/k}$. Furthemore we have $\phi_* \circ D_0(m) = \phi_*(1 \otimes \mathrm{d}(m)) = \mathrm{d}\phi(m) = 0$. Thus we have a complex

$$\ker\phi/(\ker\phi)^2 \to B \otimes_A \Omega_{A/k} \to \Omega_{B/k}.$$

To prove that it is exact we only need to check that for any B-module M we have an exact sequence

$$\mathrm{Hom}_B(\Omega_{B/k}, M) \to \mathrm{Hom}_B(B \otimes_A \Omega_{A/k}, M) \to \mathrm{Hom}_B(\ker\phi/(\ker\phi)^2, M)$$

which translates into an exact sequence

$$\mathrm{Der}_k(B, M) \to \mathrm{Der}_k(A, M) \to \mathrm{Hom}_B(\ker\phi/(\ker\phi)^2, M) = \mathrm{Hom}_A(\ker\phi, M).$$

Furthermore, for $f : B \otimes_A \Omega_{A/k} \to M$ the corresponding element in $\mathrm{Der}_k(A, M)$ is $D_f(a) = f(1 \otimes \mathrm{d}(a))$. The right map above is then given by composition with D_0 *i.e.* $f \circ D_0(m) = f(1 \otimes \mathrm{d}(m)) = D_f(m)$. Therefore the map is simply the restriction to $\ker\phi$. In particular its kernel is the set of derivations $D : A \to M$ with $D|_{\ker\phi} = 0$ *i.e.* $D \in \mathrm{Der}_k(B, M)$ proving the result.

(iv) Use the universal property and the same fact for derivations. □

Corollary 6.1.8 *Let $A = k[T_1, \cdot, T_r]$ and $B = A/(f_1, \cdots, f_m)$. Then we have*

$$\Omega_{B/k} = \left(\bigoplus_{i=1}^{r} B\mathrm{d}T_i\right) \Big/ \left(\sum_{i=1}^{m} B\mathrm{d}f_i\right).$$

Proof. Let $\phi : A \to B$ be the quotient map. By the previous proposition we have an exact sequence

$$\ker\phi/(\ker\phi)^2 \to B \otimes_A \Omega_{A/k} \to \Omega_{B/k} \to 0.$$

We may identify $\Omega_{A/k}$ to $\bigoplus_{i=1}^r A\mathrm{d}T_i$ and we quotient by the image of $\ker\phi/(\ker\phi)^2$ via D_0. The image is spanned by the $D_0(f_i) = (1 \otimes \mathrm{d}f_i)$ and the result follows. □

Proposition 6.1.9 *Let A be a k-algebra, let $B = A[T_1, \cdot, T_r]$, let $\mathfrak{m}$ be an ideal of B and let $C = B/\mathfrak{m}$. Let $(f_j)_{j\in J}$ be a family of elements in B such that their classes in $\mathfrak{m}/\mathfrak{m}^2$ span this space as a C-module.*

(i) We have an identification $\Omega_{B/k} = (B \otimes_A \Omega_{A/k}) \oplus (B\mathrm{d}T_1 \oplus \cdots \oplus B\mathrm{d}T_r)$.

(ii) If δ is the map $\mathfrak{m}/\mathfrak{m}^2 \to C \otimes \Omega_{B/k}$, we have an isomorphism of C modules:

$$\Omega_{C/k} \simeq \left((C \otimes_A \Omega_{A/k}) \oplus (C\mathrm{d}T_1 \oplus \cdots \oplus C\mathrm{d}T_r)\right) / \oplus_{j\in J} C\delta(P_j).$$

Proof. (i) We have an exact sequence $B \otimes_A \Omega_{A/k} \to \Omega_{B/k} \to \Omega_{B/A} \to 0$ and $\Omega_{B/A} = B\mathrm{d}T_1 \oplus \cdots \oplus B\mathrm{d}T_r$. We thus need to prove that the left map is injective and that the sequence splits. For this we only need to check that for any B-module M, the map $\mathrm{Hom}_B(\Omega_{B/k}, M) \to \mathrm{Hom}_B(B \otimes_A \Omega_{A/k}, M) =$

$\mathrm{Hom}_A(\Omega_{A/k}, M)$ is surjective. This is equivalent to have that the map $\mathrm{Der}_k[B, M) \to \mathrm{Der}_k(A, M)$ is surjective. But if $D : A \to M$ is a derivation, we extend it to B by the formula

$$D(\sum_\nu a_\nu T^\nu) = \sum_\nu T^\nu D(a_\nu).$$

This proves ($\imath$).

($\imath\imath$) We have an exact sequence $\mathfrak{m}/\mathfrak{m}^2 \to C \otimes \Omega_{B/k} \to \Omega_{C/k} \to 0$. This proves the result. $\square$

Remark 6.1.10 Note that one can easily compute the map δ above. Indeed, d_B can be identified to $\mathrm{d}_A + \mathrm{d}_{B/A} = \mathrm{d}_A + \sum_i \frac{\partial}{\partial T_i}$. Thus for an element $P = \sum_\nu a_\nu T^\nu$ we get

$$\delta(P) = 1 \otimes \mathrm{d}_B(P) = \sum_\nu \mathrm{d}_A(a_\nu)T^\nu + \sum_i \frac{\partial P}{\partial T_i}\mathrm{d}T_i.$$

6.1.2 Back to tangent spaces

The smoothness of a variety is tested on its tangent space (or on its dual). The Kähler differential will enable us to deal with tangent spaces in family.

Let X be an affine variety. We write $\Omega_{X/k}$ for $\Omega_{k[X]/k}$. Let $x \in X$ and let $\mathfrak{M}_x$ the corresponding maximal ideal. Let $k(x) = k[X]/\mathfrak{M}_x$.

Proposition 6.1.11 *($\imath$) We have an isomorphism $\Omega_{X/k} \otimes k(x) \simeq \mathfrak{M}_x/\mathfrak{M}_x^2 = T_xX^\vee$. For any $f \in k[X]$, the image of* $\mathrm{d}f$ *under this isomorphism is the class of $f - f(x)$* i.e. *the differential $\mathrm{d}_x f$.*

($\imath\imath$) Let $\phi : X \to Y$ be a morphism of affine varieties. Then the following diagram is commutative:

$$\begin{tikzcd}
\Omega_{Y/k} \otimes_{k[Y]} k(x) \arrow[r, "\phi_*^\sharp \otimes 1"] \arrow[d] & \Omega_{X/k} \otimes_{k[X]} k(x) \arrow[d] \\
\mathfrak{M}_{\phi(x)}/\mathfrak{M}_{\phi(x)}^2 \arrow[r, "{}^t(\mathrm{d}_x\phi)"] & \mathfrak{M}_x/\mathfrak{M}_x^2.
\end{tikzcd}$$

Proof. ($\imath$) Consider the quotient map $k[X] \to k(x)$. We have an exact sequence $\mathfrak{M}_x/\mathfrak{M}_x^2 \to k(x) \otimes_{k[X]} \Omega_{k[X]/k} \to \Omega_{k(x)/k} \to 0$. But $k(x) = k$ thus $\Omega_{k(x)/k} = 0$ and we have a surjective map $\mathfrak{M}_x/\mathfrak{M}_x^2 \to k(x) \otimes_{k[X]} \Omega_{k[X]/k}$ which maps the class of $f - f(x)$ to $1 \otimes \mathrm{d}(f - f(x)) = 1 \otimes \mathrm{d}f$. Conversely, the map $f \mapsto \mathrm{d}_x f$ being a derivation from $k[X]$ to $\mathfrak{M}_x/\mathfrak{M}_x^2$, we get a $k[X]$-module morphism $\Omega_{X/k} \to \mathfrak{M}_x/\mathfrak{M}_x^2$ defined by $\mathrm{d}(f) \mapsto \mathrm{d}_x f$ which is the inverse.

($\imath\imath$) By definition we have $\phi_*^\sharp(f) = \mathrm{d}(\phi^\sharp(f))$ while $d_x\phi(f - f(x))\phi^\sharp(f - f(x))$ therefore the above diagram is commutative. $\square$

Now recall the following classical result from commutative algebra. Let X be affine and irreducible and let M be a $k[X]$-module. For I a prime ideal of $k[X]$ we denote by $k[X]_I$ the localisation with respect to $S = k[X] \setminus I$ and by M_I the tensor product $M \otimes_{k[X]} k[X]_I$. We shall essentially use this notation for $I = (0)$ in which case $k[X]_I = k(X)$ and for $I = \mathcal{M}_{X.x}$ in which case we denote M_I by M_x.

Proposition 6.1.12 *Let r be the rank of M* i.e. *$r \dim_{k(X)} M_{k(X)}$, then for all $x \in X$, we have the inequality $\dim_k M_x \geq r$. Furthermore equality holds if and only if M_x is free. This occurs on a non empty open subset of X.*

Corollary 6.1.13 *Let n be the minimum of the dimension of T_xX for $x \in X$. Then n is the rank of $\Omega_{X/k}$ and the minimum is attained on an non empty open subset of X.*

Note also that because $\Omega_{X/k_{k(X)}} = \Omega_{k(X)/k}$ this rank is the dimension $\dim_{k(X)} \Omega_{k(X)/k}$.

6.2 Separable morphisms

6.2.1 Separable and separably generated extensions

Definition 6.2.1 *Let K be a field and let $P \in K[T]$ be irreducible. The polynomial P is said to be separable if P and P' (the derived polynomial) have no common factors.*

Remark 6.2.2 Note that P is not separable if and only if $\mathrm{char}(K) = p > 0$ and $P \in K[T^p]$.

Definition 6.2.3 *Let L/K be an algebraic field extension.*
($\imath$) An element $x \in L$ is called separable if $P_{min}(x)$, its minimal polynomial is separable.
($\imath\imath$) The extension L is called separable if any element $x \in L$ is separable.

Fact 6.2.4 *Let $L = K(x_1, \cdots, x_r)$ be a field. Then L/K is separable if and only if each x_i is separable.*
Furthermore, if $K \subset L' \subset L$ is an intermediate extension, then L/K is separable if and only if L/L' and L'/K are separable.

Proof. Exercise. □

Definition 6.2.5 *Let L/K be a field extension, then L/K is called separably generated if there exists a transcendence basis B of L over K such that the (algebraic) extension $L/K(B)$ is separable. The basis B is called a separable transcendence basis.*

Example 6.2.6 A pure transcendental extension is separably generated. An separable algebraic extension is also separably generated.

Example 6.2.7 Let K be of characteristic $p > 0$. Then $K(T)$ the pure transcendental extension is seprably generated with for example T as separable transcendence basis. But the element T^p is not a separable transcendence basis.

Lemma 6.2.8 *Let $k \subset K \subset L$ be field extensions. Recall that we have an exact sequence:*

$$L \otimes_K \Omega_{K/k} \xrightarrow{\alpha} \Omega_{L/k} \to \Omega_{L/K} \to 0.$$

If L/K is separably generated, then the map α is injective.

Proof. The injectivity of the map α is equivelent to the surjectivity of the map ${}^t\alpha : \mathrm{Hom}_L(\Omega_{L/k}, M) \to \mathrm{Hom}_L(L \otimes \Omega_{K/k}, M) = \mathrm{Hom}_K(\Omega_{K/k}, M)$ for any L-module M. This map is the map $\mathrm{Der}_k(L, M) \to \mathrm{Der}_k(K, M)$ given by restriction. Let D be in $\mathrm{Der}_k(K, M)$, we want to extend it to L.

We therefore only have to deal with a simple extension $L = K(x) = K[T]/(P)$ for some irreducible polynomial P. Let $Q = \sum_i a_i T^i$ be any polynomial in $K[T]$, and let $D(T) \in M$ be a possible value for the extension. We have the equality

$$D(Q(T)) = \sum_i D(a_i)T^i + P'(T)D(T).$$

In particular if we write $P = \sum_i a_i T^i$, the value of $D(T)$ has to satisfy $0 = D(P(T)) = \sum_i D(a_i)T^i + P'(T)D(T)$ or $D(T)P'(T) = -\sum_i D(a_i)T^i$. Because the extension is separable, we have $P'(T) \in L^\times$ thus we may define $D(T) = -(P'(T))^{-1} \sum_i D(a_i)T^i$. □

Lemma 6.2.9 *Let $K = k(x)$, then $\dim_K \Omega_{K/k} \leq 1$ and $\Omega_{K/k} = 0$ if and only if K/k is separable (in particular algebraic).*

Proof. We know that $\Omega_{K/k}$ is spanned by $\mathrm{d}_K x$ over K thus of dimension at most one. If x is not algebraic, we also know that $\dim_K \Omega_{K/k} = 1$. If x is algebraic, then there exists a polynomial $P \in k[T]$ such that $P(x) = 0$. We then get $0 = \mathrm{d}_K(P(x)) = P'(x)\mathrm{d}_K x$. Thus if K/k is separable, then $P'(x)$ is invertible in K and we have $\mathrm{d}_K x = 0$ *i.e.* $\Omega_{K/k} = 0$. If K/k is not separable, then $P'(x) = 0$ and we get $\Omega_{K/k} = K\mathrm{d}_K x$. $\square$

Theorem 6.2.10 *Let $k \subset K \subset L$ be field extensions with L/K of finite type. Then we have the inequalities*

$$\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} + \deg\mathrm{tr}_K L,$$

with equality if L/K is separably generated.

Proof. Let us first prove that it is enough to prove this for a simpe extension $L = K(x)$. Indeed if the statement is true in that case, let $L = K(x_1, \cdots, x_r)$, then we defined $K_i = K_{i-1}(x_i)$ and $K_0 = K$. We thus have $\dim_{K_i} \Omega_{K_i/k} \geq \dim_{K_{i-1}} \Omega_{K_{i-1}/k} + \deg\mathrm{tr}_{K_{i-1}} K_i$. Summing these inequalities we get the desired inequality.

Let us assume $L = K(x)$, if x is transcendent or separable, then the result follows by the previous two lemmas. If x is algebraic not separable, then we have $L = K[T]/(P)$ and we thus have

$$\Omega_{L/k} \simeq \left((L \otimes_K \Omega_{K/k}) \oplus (L\mathrm{d}T)\right)/L\delta(P)$$

proving the inequality.

If L/K is separably generated, then let $(x_1, \cdots, x_r)$ be a separating transcendent basis. We have equality in the above inequalities and $L/K(x_1, \cdots, x_r)$ is separable therefore spanned by a unique separable element x (by the primitive element theorem) and we conclude by the last lemma again. $\square$

Remark 6.2.11 There may be equality even if L/K is not separably generated. For example let $K = k(T^p)$ and $L = k(T)$. Then $\dim_L \Omega_{L/k} = 1$, $\dim_K \Omega_{K/k} = 1$ and $\deg\mathrm{tr}_K L = 0$ thus we have equality while L/K is not separable.

More precisely, in this case we have $L = K[X]/(P)$ with $P(X) = X^p - T^p$. Thus we have

$$\Omega_{L/k} \simeq \left((L \otimes_K \Omega_{K/k}) \oplus (L\mathrm{d}X)\right)/L\delta(P)$$

where if $P = \sum_i a_i X^i$, we have $\delta(P) = \sum_i \mathrm{d}_K(a_i)X^i + P'(X)\mathrm{d}X = -\mathrm{d}_K(T^p) \neq 0$ in $\Omega_{K/k}$.

Corollary 6.2.12 *If K/k is a finitely generated extension, then $\dim_K \Omega_{K/k} \geq \deg\mathrm{tr}_k K$ with equality if and only if K/k is separably generated.*

Proof. By the previous Theorem, we only have to prove that if the equality holds then the extension is separably generated. We proceed by induction on $n = \dim_K \Omega_{K/k}$.

If $n = 0$, then by the previous theorem $\deg\mathrm{tr}_k K = 0$ and K/k is algebraic. Let $x \in K$ and let P be its minimal polynomial. Let $K' = k[T]/(P)$. Then we have $\Omega_{K'/k} = K'\mathrm{d}T/K'\delta(P) = K'[\mathrm{d}T]/(P'(T)\mathrm{d}T)$. But by the previous Theorem, we have $= 0 \dim_K \Omega_{K/k} \geq \dim_{K'} \Omega_{K'/k} + \deg\mathrm{tr}_{K'} K$ thus $\Omega_{K'/k} = 0$ which implies that $\delta(P) \neq 0$ *i.e.* $P'(T) \neq 0$ *i.e.* x is separable.

Let us assume that the result is true in rank $n-1$. Let $x_1, \cdots, x_n$ elements in K such that $\mathrm{d}_K x_1, \cdots, \mathrm{d}_K x_n$ form a basis of $\Omega_{K/k}$. Let $K' = k(x_1, \cdots, x_n)$. The previous Theorem gives the inequality $\dim_K \Omega_{K/k} \geq \dim_{K'} \Omega_{K'/k} + \deg\mathrm{tr}_{K'} K$ but because the $\mathrm{d}_K x_i$ form a basis we have the equalities $\dim_{K'} \Omega_{K'/k} = n$ and $\deg \mathrm{Tr}_{K'} K = 0$. Therefore the extension K'/k is purely transcendental and the extension K/K' is algebraic. Furthermore, we have an exact sequence:

$$K \otimes_{K'} \Omega_{K'/k} \to \Omega_{K/k} \to \Omega_{K/K'} \to 0$$

with the first map surjective because the image contains the elements $\mathrm{d}_K x_i$. Thus we have $\Omega_{K/K'} = 0$ and by the case of rank one K/K' is separable. $\square$

Definition 6.2.13 *Recall that a field k is called perfect if $char(k) = 0$ or $char(k) = p > 0$ and the Frobenius map $F : k \to k$ defined by $F(x) = x^p$ is surjective.*

Example 6.2.14 Algebraically closed fields and finite fields are perfect.

Proposition 6.2.15 *Let k be a perfect field, then any extension K/k of finite type is separably generated. More precisely, write $K = k(x_1, \cdots, x_n)$, then there exists a permutation σ such that $(x_{\sigma(1)}, \cdots, x_{\sigma(x_r)})$ is a separating transcendence basis.*

Proof. We may assume that $\mathrm{char}(k) = p > 0$ otherwise any extension separably generated. Note that if $r = 0$ or if $r = n$, then the extension is purely transcendental and there is nothing to prove. We proceed by induction on n and we may assume that $n \in [1, n-1]$.

We may also assume that $x_1, \cdots, x_r$ is a transcendence basis of K/k. Therefore the elements $x_1, \cdots, x_r, x_{r+1}$ are algebraically dependent. Let $P \in k[X_1, \cdots, X_{r+1}]$ be a non trivial polynomial of minimal total degree such that $P(x_1, \cdots, x_{r+1}) = 0$. The polynomial P has to be irreducible by minimality of the degree. We claim that $P \not\in k[X_1^p, \cdots, X_r^p]$. If is were the case then $P = \sum_\nu a_\nu X^{p\nu}$ by F being sujective, for all ν, there exists $b_\nu \in k$ such that $a_\nu = b_\nu^p$. We deduce that $P = Q^p$ with $Q = \sum_\nu b_\nu X^\nu$. A contradiction to the irreducibility of P.

Therefore, there exists at least one index $i \in [1, r+1]$ such that P is not a polynomial in X_i^p. Let $R(X) = P(x_1, \cdots, x_{i-1}, X, x_{i+1}, \cdots, x_{r+1}) \in K'[X]$ with $K' = k(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$. The polynomial R is separable and vanishes on x_i therefore the extension $K = K'(x_i)/K'$ is separable. By induction hypothesis, there exists a permutation of $[1, n] \setminus \{i\}$ such that $(x_{\sigma(1)}, \cdots, x_{\sigma(x_r)})$ is a separating transcendence basis of K'. Because K/K' is separable, the same hold for K. $\square$

Corollary 6.2.16 *Let $k \subset K \subset L$ be field extensions. Assume that k is perfect, then the following conditions are equivalent.*

(ı) The extension L/K is separably generated.

(ıı) The map $\alpha : L \otimes_K \Omega_{K/k} \to \Omega_{L/k}$ is injective.

Proof. Because k is perfect, we have the equalities $\deg \mathrm{tr}_k L = \dim_L \Omega_{L/k}$ and $\deg \mathrm{tr}_k K = \dim_K \Omega_{K/k}$. Consider now the exact sequence $L \otimes_K \Omega_{K/k} \to \Omega_{L/k} \to \Omega_{L/K} \to 0$. The extension L/K is separably generated if and only if $\deg \mathrm{tr}_K L = \dim_L \Omega_{L/K}$ thus if and only if the map α is injective by dimension count. $\square$

Corollary 6.2.17 *Let k be a perfect field, then for any extension K/k of finite type we have the equality $\dim_L \Omega_{L/K} = \deg \mathrm{tr}_K L$.*

Proof. We only need to prove that any extension K/k is separable. Let $x \in K$ and let P be its minimal polynimial. If P is not separable, then $P \in k[T^p]$ but because k is perfect we get that $P = \sum_i a_i T^{ip} = \sum_i b_i^p T^{ip} = \left(\sum_i b_i T^i\right)^p$ which is not irreducible. A contradiction. $\square$

6.2.2 Smooth and normal varieties

Let us recall the following result from commutative algebra that we shall not prove completly.

Proposition 6.2.18 *Let X be an algebraic variety. For any $x \in X$ we have the inequality $\dim T_x X \geq \dim_x X$. If the equality holds, then $\mathcal{O}_{X,x}$ is a domain and integrally closed.*

Proof. We only prove the inequality. Because k is algebraically closed, it is perfect therefore we have an equality $\mathrm{rk}\Omega_{X/k} = \dim_{k(X)} \Omega_{k(X)/k} = \deg \mathrm{tr}_k k(X) = \dim X$. Thus the inequality is true and the equality holds on an non empty open subset. □

Definition 6.2.19 *Let X be an algebraic variety and let $x \in X$.*

(ı) The variety X is called smooth at x if we have the equality $\dim T_x X = \dim_x X$. The variety X is called smooth if it is smooth at every point.

(ıı) The variety X is called normal at x if the ring $\mathcal{O}_{X,x}$ is an integrally closed domain. The variety X is called normal if X is normal at every point.

Remark 6.2.20 (ı) The above proposition shows that a smooth variety is normal.

(ıı) Recall also that the irreducible component of a variety are in one-to-one correspondence with the minimal prime ideals. In particular, if $\mathcal{O}_{X,x}$ is a domain, then x is contained in a unique irreducible component.

(ııı) As a consequence, if X is smooth or normal at x, then X is contained in a unique irreducible component of X. If X is smooth or normal, then its irreducible components coincide with the connected components.

6.2.3 Separable and birational morphisms

Definition 6.2.21 *A morphism $\phi : X \to Y$ between two varieties is called dominant if $\phi(X)$ is dense in Y.*

Remark 6.2.22 (ı) By Chevalley constructibility Theorem, this is equivalent to the fact that $\phi(X)$ contains a dense open subset of Y.

(ıı) If $\phi : X \to Y$ is dominant and if V is a dense open subset of Y contained in $\phi(X)$, for any affine open subset U of V, the comorphism $\phi^\sharp : \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}(U))$ is injective.

(ııı) In particular for X and Y irreducible, the map $\phi^\sharp$ induces a field extension $k(Y) \to k(X)$.

Definition 6.2.23 *(ı) A morphism $\phi : X \to Y$ between irreducible varieties is called separable if ϕ is dominant and if the induced field extension $k(Y) \to k(X)$ is separably generated.*

(ıı) A morphism $\phi : X \to Y$ between irreducible varieties is called birational if ϕ is dominant and if the induced field extension $k(Y) \to k(X)$ is trivial.

You get back the classical definition of birational morphism via the following proposition.

Proposition 6.2.24 *A morphism $\phi : X \to Y$ is birational if and only if there exists a non empty open subset V of Y such that $\phi|_{\phi^{-1}(V)} : \phi^{-1}(V) \to V$ is an isomorphism.*

Proof. If the second condition holds, then obviously ϕ is birational. Conversely, if ϕ is birational, then ϕ is dominant. Let U be an affine open subset of Y contained in $\phi(X)$ and let W be an affine open subset of $\phi^{-1}(U)$. We have a comorphism $\phi^\sharp : k[U] \to k[W]$ which is injective and by assumption both rings are contained in $k(X) = k(Y) = k(U) = k(W)$. Because $k[W]$ is of finite type, we may pick generators $f_1, \cdots, f_r$ which we can write as $f_i = g_i/h_i$ with $g_i, h_i \in k[U]$. Therefore there exists and element $h \in k[U]$ such that $f_i \in k[U]_h$ for all i. We thus get $k[U]_h = k[W]_h$ and the result follows. $\square$

The following statement shows the usefulness of the notion of separable morphisms.

Theorem 6.2.25 *Let $\phi : X \to Y$ be a morphism between irreducible affine varieties. Assume that $\dim X = \dim Y$ and that ϕ is separable. Then there exists an open subset V of Y contained in $\phi(X)$ such that for any $v \in V$ the fiber $\phi^{-1}(v)$ has $[k(X) : k(Y)]$ elements.*

Proof. Let $K = k(Y)$ and $L = k(X)$. We have $\deg \mathrm{tr}_k K = \deg \mathrm{tr}_k L$. Therefore the extension L/K is algebraic and of finite type (because L/k is so). Thus the extension L/K is finite.

The algebra $K \otimes_{k[Y]} k[X]$ is a K subalgebra of L (we only inverted the elements in $k[Y]$). It is therefore integral and of finite K-dimension. Hence it is a field (the multiplication by any elementy is injective thus surjective). But this field contains $k[X]$ and is contained in $k(X) = L$ thus it is equal to L. In particular, any element $x \in L$ is of the form x'/y with $x' \in k[X]$ and $y \in k[Y]$.

Furthermore, the extension L/K being separable, we may write $L = K(x)$ and by what we proved $x = x'/y$ with $x' \in k[X]$ and $y \in k[Y] \subset K$. Thus we may choose $x \in k[X]$.

The algebra $k[X]$ is of finite type over x. Let us choose $x_1, \cdots, x_r$ some generators. We may write $x_i = P_i(x)$ with $P \in K[T]$. We may thus find $f \in k[Y]$ such that $P \in k[Y]_f[T]$. Let $V = D(f)$. The inverse image of V: $U = \phi^{-1}(V)$ is the affine variety associated to the ring $k[X]_f$. The comorphism is given by $k[V] = k[Y]_f \to k[X]_f = k[U]$. We claim that this induces an isomorphism

$$k[V][x] \simeq k[U].$$

Indeed, it is injective and if $g \in k[U] = k[X]_f$, then g is a polynomial in the x_i *i.e* a polynomial in the $P_i(x)$ and thus in $k[Y]_f[x] = k[V][x]$.

We thus have a sujective morphism of algebras

$$k[V][T] \to k[U]$$

defined by mapping T to x. The kernel of this map may not be a principal ideal but the following trick enable to restrict ourselves to this case.

Let $P \in K[T]$ be the minimal unitary polynomial of x in K. Let $h \in k[Y]$ such that $P \in k[Y]_h[T]$. We define $g = fh$ and set $V' = D(g)$ *i.e.* $k[V'] = k[Y]_g$. Let U' be the open subset in $U \subset X$ such that $k[U'] = k[X]_g$. We claim that the comorphism $\phi^\sharp$ induces an isomorphism

$$k[Y]_g[T]/(P) \simeq k[U'].$$

Indeed, the map is defined by $T \mapsto x$ and surjective by what we already proved. Let $Q \in k[Y]_g[T] = k[V'][T]$ be in the kernel. Then we can proceed to the Euclidean division of Q by P because P is unitary. We get $Q = PS + R$ with $R(x) = 0$ thus $R = 0$ by minimality. This proves the claim.

Now the surjective morphim $k[V'][T] \to k[U']$ induces a closed immersion $U' \to V' \times k$ defined by $u \mapsto (\phi(u), x(u))$ and we may this identify U' with the subvariety

$$\{(v, t) \in V' \times k \ / \ P_v(t) = 0\}$$

of $V' \times k$ where P_v is defined as follows. Write $P(T) = \sum_i a_i T^i$ with $a_i \in k[V']$, then $P_v(T) = \sum_i a_i(v) T^i$. If $\mathrm{disc}(P)$ is the discriminant of P, which lives in $k[V']$ (it is a polynomial in the coefficients of P), then $P_v(T)$ has simple roots if and only if $\mathrm{disc}(P)(v) \neq 0$. Therefore on V'' such that $k[V''] = k[Y]_{g\mathrm{disc}(P)}$ the fiber of ϕ has a constant number of solutions equal to $\deg P$. This is by definition $[L : K]$. $\square$

Corollary 6.2.26 *If $\phi : X \to Y$ is a bijective and separable morphism of irreducible algebraic varieties, then ϕ is birational.*

Proof. We can restrict ourselves to the case where X and Y are affine and in this situation we know that $[k(X) : k(Y)] = 1$ thus ϕ is birational. $\square$

We shall now give some criteria for a morphism to be separable using the differential.

Proposition 6.2.27 *Let $\phi : X \to Y$ be a morphism between irreducible varities. The following conditions are equivalent.*
($\imath$) The morphisme ϕ is separable.
($\imath\imath$) There exists a dense open subset U such that $d_x\phi$ is surjective for all $x \in U$.
($\imath\imath\imath$) There exists a smooth point $x \in X$ such that $d_x\phi$ is surjective.

Proof. The smooth locus is an open subset which has to meet U thus ($\imath\imath$)⇒($\imath\imath\imath$).

Let us prove that ($\imath\imath\imath$)⇒($\imath$). Let $Z = \overline{f(X)}$. Note that the condition $d_x\phi$ surjective is open *i.e.* the locus X_0 where this holds is a dense open. We may thus assume that $\phi(x)$ is smooth in Z. Indeed, the smooth locus $Z^{\rm sm}$ is a dense open therefore the intersection $X^{\rm sm} \cap X_0 \cap \phi^{-1}(Z^{\rm sm})$ is a dense open.

We may also assume that X and Y are affine. Let us denote by $i : Z \to Y$ the inclusion and by $\psi : X \to Z$ the map induced by ϕ. Let $x \in X$ be given by ($\imath\imath\imath$) and let $y = \phi(x) = i(\psi(x))$. We have a commutative diagram:

$$\begin{tikzcd}
k[X] \otimes_{k[Y]} \Omega_{k[Y]/k} \arrow[r] \arrow[d] & k[X] \otimes_{k[Z]} \Omega_{k[Z]/k} \arrow[r] \arrow[d] & \Omega_{X/k} \arrow[d] \\
\mathfrak{M}_{Y,y}/\mathfrak{M}_{Y,y}^2 \arrow[r] & \mathfrak{M}_{Z,y}/\mathfrak{M}_{Z,y}^2 \arrow[r] & \mathfrak{M}_{X,x}/\mathfrak{M}_{X,x}^2.
\end{tikzcd}$$

The transpose ${}^t d_x\phi$ of the differential $d_x\phi$ is the composition of the two horizontal maps of the second row. It is injective. Furthermore, because i is a closed embedding, the first horizontal map of that row, the map ${}^t d_y i$ is surjective. We deduce that ${}^t d_y i$ is bijective and that ${}^t d_x\psi$ is injective. We deduce the inequalities $\dim Y \leq \dim T_{\phi(x)} Y = \dim T_{\phi(x)} Z = \dim Z$. Therefore $Z = Y$ and ϕ is dominant.

Replacing X and Y by an affine open subsets, we may assume that $\Omega_{k[X]/k}$ and $\Omega_{k[Y]/k}$ are free over $k[X]$ and $k[Y]$. Therefore we have a map $k[X] \otimes_{k[Y]} \Omega_{k[Y]/k} \to \Omega_{k[X]/k}$ between free $k[X]$-modules which is injective at some point $x \in X$. It is therefore injective on an open affine subset and in particular the map

$$\alpha : k(X) \otimes_{k(Y)} \Omega_{k(Y)/k} \to \Omega_{k(X)/k}$$

is injective and the extension $k(X)/k(Y)$ is separably generated.

Let us prove the implication ($\imath$)⇒($\imath\imath$). We may again replace X and Y by affine open subsets and we have an injective map $k(X) \otimes_{k(Y)} \Omega_{k(Y)/k} \to \Omega_{k(X)/k}$. This corresponds to the map $k[X] \otimes_{k[Y]} \Omega_{k[Y]/k} \to \Omega_{k[X]/k}$ at the generic point of X and Y. We conclude by the following fact. $\square$

Fact 6.2.28 *Let X be an irreducible affine variety. Let $f : M \to N$ be a morphism of $k[X]$-modules such that $f \otimes k(X) : M_{k(X)} \to N_{k(X)}$ is injective, then the map $f \otimes k(x) : M_x \to N_x$ is injective for x in a dense open subset of X.*

Remark 6.2.29 The smoothness assumption in ($\imath\imath\imath$) is important. Take for example $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ and $Y = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. Then the map ϕ given by inclusion is not separable (not dominant) but the differential of ϕ at the point $(0, 0)$ is surjective.

6.2.4 Application to homogeneous spaces

Lemma 6.2.30 *Let $\phi : X \to Y$ be a G-equivariant morphism between homogeneous G-spaces. If ϕ is birational then it is an isomorphism.*

Proof. Let X_0 be the subset of X where ϕ is an isomorphism. Then X_0 contains a dense open subset but because the map is equivariant and X is homogeneous $X_0 = X$. □

Proposition 6.2.31 *Let ϕ be a G-equivariant morphism between homogeneous G-spaces.*

(ı) If there exists some $x \in X$ such that $d_x\phi$ is surjective then this is true for all $x \in X$. In this case ϕ is separable.

(ıı) If ϕ is separable and bijective, then it is an isomorphism.

(ııı) If $\phi : G \to H$ is a bijective morphism of algebraic groups such that $d_e\phi$ is surjective, then it is an isomorphism.

Proof. (ı) This is obvious by equivariance and homogeneity. The fact that ϕ is separable comes from the former proposition.

(ıı) From (ı) the differential is surjective at any point. Now any irreducible component of X and Y is of the form G^0x and G^0y for some $x \in X$ and $y \in Y$. Let us take x and y such that G^0x is mapped to G^0y. Then this map has to be bijective and separable thus birational. This map is also G^0-equivariant between G^0-homogeneous space so the previous lemma finishes the proof.

(ııı) We only need to prove that it is an isomorphism of varieties. But this follows from (ı) and (ıı). □

Proposition 6.2.32 *Let X be a G-space and let $\phi : G \to X$ be a G-equivariant morphism. Let $x = \phi(e)$ and $G_x = \{g \in G \ / \ gx = \phi(g) = x\}$.*

(ı) We have the inclusion $L(G_x) \subset \ker d_e\phi$.

(ıı) Conversely, let $\psi : G \to Gx$. The equality $L(G_x) = \ker d_e\phi$ holds if and only if ψ is separable.

Proof. (ı) The map ϕ restrited to G_x is constant therefore its differential vanishes and the result follows.

(ıı) The map ψ is equivariant between homogeneous G-spaces therefore it is separable if and only if $d_e\psi$ is surjective. Furthermore, the fibers are all of dimension $\dim G_x$ and Gx is smooth therefore we have $\dim L(G_x) = \dim G_x = \dim G - \dim Gx = \dim G - \dim T_xGx$. On the other hand, we have $\dim \ker d_e\phi = \dim G - \dim \mathrm{im}(d_e\phi) = \dim G - \dim(d_e\psi)$. Thus $d_e\psi$ is surjective if and only if these dimensions agree and the result follows. □

Corollary 6.2.33 *Let $\phi : G \to H$ be a bijective morphism of algebraic groups. Then if $d_e\phi$ is injective then ϕ is an isomorphism.*

Proof. By the above corollary, if $d_e\phi$ is injective then the orbit map ψ (here $\psi = \phi$) is separable. Therefore ϕ is separable, bijective and G-equivariant between homogeneous G-spaces thus an isomorphism. □

6.2.5 Flat morphisms

We recall in this section some basic properties of flatness.

Definition 6.2.34 *Let $\phi : X \to Y$ be a morphism. Then ϕ is called flat if for any $x \in X$, the $\mathcal{O}_{Y,\phi(x)}$-module $\mathcal{O}_{X,x}$ is flat.*

Theorem 6.2.35 *Let $\phi : X \to Y$ be a flat morphism, then f is universally open (*i.e. *for any variety Z, the map $X \times Z \to Y \times Z$ is open).*

Theorem 6.2.36 (Generic flatness) *Let $\phi : X \to Y$ be a dominant morphism between irreducible varieties. Then there exists a dense open subset $V \in Y$ such that $\phi : \phi^{-1}(V) \to V$ is flat.*

Corollary 6.2.37 *Let $\phi : X \to Y$ be a G-equivariant morphism between homogeneous G-spaces. Then ϕ is universally open.*

Theorem 6.2.38 *Let $\phi : X \to Y$ be a dominant morphism between irreducible varieties. Let $r = \dim X - \dim Y$. Then there exists a dense open subset U in X such that.*

($\imath$) The restriction $\phi|_U$ is universally open.

($\imath\imath$) If Z is a closed subvariety of Y and W an irreducible component of $\phi^{-1}(Z)$ meeting U, then $\dim W = \dim Z + r$.

Corollary 6.2.39 *Let $\phi : X \to Y$ be a G-equivariant morphism between homogeneous G-spaces. Let $r = \dim X - \dim Y$. Then subvariety Z of Y and for any irreducible component W of $\phi^{-1}(Z)$, we have $\dim W = \dim Z + r$.*

6.3 Quotients

6.3.1 Chevalley's semiinvariants

Let G be an algebraic group and let H be a closed subgroup. Let $\mathfrak{g} = L(G)$ and $\mathfrak{h} = L(H)$. Let $X^*(H)$ be the groups of characters of H.

Theorem 6.3.1 (Chevalley's Theorem) *There exists a finite dimensional representation V of G and a vector subspace $U \subset V$ of dimension 1 such that $G_U = H$ and $\mathrm{Stab}_{\mathfrak{g}}(U) = \mathfrak{h}$.*

If furthermore we have $X^*(H) = \{1\}$, then for any $u \in U$ we have $G_u = H$ and $\mathrm{Stab}_{\mathfrak{g}}(u) = \mathfrak{h}$.

Proof. Let I be the ideal of H in G and let $f_1, \cdots, f_r$ be generators of I. We let G act on $k[G]$ via right multiplication *i.e.* $\rho(g)(f)(g') = f(g'g)$. The derived action is given by the action of $\mathfrak{g}$ seen as $\mathrm{Der}(k[G])^{\lambda(G)}$. Let W be the $\rho(G)$-submodule of $k[G]$ spanned by the x_i (it has to be finite dimensional) and let $E = W \cap I$.

Lemma 6.3.2 *We have the equalities $H = G_E$ and $\mathfrak{h} = \mathrm{Stab}_{\mathfrak{g}}(E)$.*

Proof. Recall that we have the equalities $H = G_I$ and $\mathfrak{h} = \mathrm{Stab}_{\mathfrak{g}}(I)$.

For $g \in G$ or for $\xi \in \mathfrak{g}$, we have $\rho(g)(W) \subset W$ and $d\rho(\xi)(W) \subset W$. Therefore, if g or ξ stabilise I it stabilises also E. Conversely if g or ξ stabilise E then it stabilises I because I is spanned by E. □

Let us also recall the following easy result from linear algebra.

Lemma 6.3.3 *Let $E \subset W$ be vector spaces of finite dimension. Let $d = \dim E$ and consider the inclusion $D = \Lambda^d E \subset \Lambda^d W$ where D has dimension 1.*

Let $g \in GL(W)$ and $X \in \mathfrak{gl}(W)$ and consider their induced actions on $\Lambda^d(W)$. Then we have the equivalences:

$$g(E) = E \Leftrightarrow g(D) = D \quad \text{and} \quad X(E) \subset E \Leftrightarrow X(D) = D.$$

We then set $U = \Lambda^{\dim W \cap I} W \cap I$ to get the first part of the result. Now H acts on U by a character thus trivially if $X^*(H) = \{1\}$. The result follows. □

With the notation of the last statement, we have the following proposition.

Proposition 6.3.4 *Let $u \in U$ and let $[u] \in \mathbb{P}(V)$ its class. Then we have the equalities $H = G_{[u]}$, $\mathfrak{h} = \mathrm{Stab}_{\mathfrak{g}}([u])$ and the morphism $\phi : G \to G[u]$ is separable.*

If furthermore, we have the equality $X^*(H) = \{1\}$, then the morphism $\psi : G \to Gu$ is separable.

Proof. The equalities on stabilisers follow from the previous statement. Consider the differential $d_e\psi$ of ψ at e. We have $d_e\psi(\xi) = \xi(u)$. Therefore we have $d_e\phi(\xi) = d_u\pi(d_e\psi(\xi))$ where $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ is the projection map. The map $d_u\pi$ is the projection map from V with respect to ku. In particular $\ker d_e\phi = d_e\psi^{-1}(ku) = \mathrm{Stab}_{\mathfrak{g}}(U) = \mathfrak{h} = L(G_U)$. This implies that ϕ is separable. The same proof works if $X^*(H) = 1$. □

Corollary 6.3.5 *For any closed subgroup H of an algebraic group G, there exists a structure of variety on G/H such that the map $G \to G/H$ is a separable morphism.*

Proof. Take $U \subset V$ as in Chevalley's Theorem and use the previous Proposition for separability. □

We first prove that the quotient G/H is unique. To avoid using Zariski's main Theorem we first do not prove the more general universal property of the quotient.

Proposition 6.3.6 *Let $\phi : G \to X$ be any G-equivariant morphism between homogeneous G-spaces with $\phi(h) = \phi(1)$ for all $h \in H$. Then there exists a G-equivariant morphism $\psi : G/H \to X$ such that $\phi = \psi \circ \pi$ where $\pi : G \to G/H$ is the quotient map.*

Proof. Let $\phi : G \to X$ be such a morphism. The morphism ψ if it exists has to be unique because of the surjectivity of π. Consider the map $\theta : G \to G/H \times X$ define by $\pi \times \phi$. Let W be its image, because θ is G-equivariant, the set W is open in its image and thus a locally closed subvariety of $G/H \times X$. We have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "p"] & G/H \times X \arrow[r] \arrow[d] & X \\
G/H \arrow[r, equal] & G/H \arrow[r, equal] & G/H. &
\end{tikzcd}$$

The composition $p \circ \theta$ is π therefore surjective. But p is also injective since for $\bar{g} \in G/H$, the fiber of p is the set of all $x = (\bar{g}, \phi(g))$ with $g \in G$. But if $\bar{g} = \bar{g}'$, then $g' = gh$ for $h \in H$ and $\phi(g') = \phi(gh) = \phi(g)\phi(h) = \phi(g)\phi(1) = \phi(g)$. Therefore p is bijective.

Furthermore, since π is separable, the map $d_e\pi$ is surjective and thus, since we have $d_e\pi = d_e\theta \circ \mathrm{d}_{\theta(e)}p$ we get that $d_{\theta(e)}p$ is surjective. The map p being equivariant and bijective between homogeneous spaces, it is an isomorphism. We define ψ as the composition of the inverse of p and the projection to X. □

Corollary 6.3.7 *The variety G/H is quasi-projective and quasi-affine if $X^*(H) = 1$.*

Proof. Use the construction via Chevalley's Theorem. $\square$

Let us now prove the classical universal property. For this we need a version of Zariski's main Theorem.

Theorem 6.3.8 (Zariski's Main Theorem) *Let $\phi : X \to Y$ be a bijective morphism of varieties and assume that Y is normal. Then ϕ is an isomorphism.*

Example 6.3.9 This statement is not empty as shows the map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x, y]/(x^3 - y^2)$ given by $x \mapsto t^2$ and $y \mapsto t^2$.

Theorem 6.3.10 *($\imath$) The morphism $\pi : G \to G/H$ is separable and flat. For any open subset U in G/H, we have*

$$\pi^\sharp(k[U]) = k[\pi^{-1}(U)]^H := \{f \in k[\pi^{-1}(U)] \ / f \text{ is constant on the classes } gH\}.$$

In particular, we have $k[U] = \{g : U \to k \ / \ g \circ \pi \in k[\pi^{-1}(U)]\}$.

($\imath\imath$) The variety G/H satisfies the following universal property. For any morphism $\phi : G \to X$ such that ϕ is constant on the classes gH, there exists a morphism $\psi : G/H \to X$ such that $\phi = \psi \circ \psi$.

($\imath\imath\imath$) If G is connected, then π induces an isomorphism $k(G/H) \simeq k(G)^H$.

Proof. ($\imath$) The map π is separable by what we already proved. Furthermore by the above discution on flat morphisms, it is also flat.

Let V be an open subset of G/H. As π is surjective, then $\pi^\sharp$ defines an injection $k[V] \subset k[\pi^{-1}(V)]$. We want to prove that this inclusion is an equality.

Let us first prove that we may assume that V is irreducible. Indeed, let $x_1, \cdots, X_r$ be the irreducible components of G/H. These are connected components and writting $V_i = V \cap X_i$, we have a disjoint union $V = \coprod_i V_i$. We also have the disjoint union $\pi^{-1}(V) = \coprod_i \pi^{-1}(V_i)$. We thus have the equality

$$k[\pi^{-1}(V)] = \bigoplus_i k[\pi^{-1}(V_i)].$$

Therefore if the result is true for V irreducible it is true in general.

If V is irreducible, let $U = \pi^{-1}(V)$ and let $f \in k[U]^H$. Let $W \subset U \times \mathbb{A}^1_k$ be the graph of f and let $\phi : U \times \mathbb{A}^1_k \to V \times \mathbb{A}^1_k$ be given by $\phi = (\pi, \mathrm{Id})$. The graph W is closed and π being universally open, $\phi(W^c)$ is open. We claim that $\phi(W^c) = \phi(W)^c$ and therefore $W' = \phi(W)$ is closed. Indeed, because ϕ is surjective, we have the inclusion $\phi(W)^c \subset \phi(W^c)$ while if (v, a) lies in the intersection, $\phi(W^c) \cap \phi(W)$, we have $(\pi(u), a) = (\pi(u'), f(u'))$ with $u, u' \in U$ and $a \neq f(u)$. But we then have $h \in H$ with $u' = uh$ and $a = f(u') = f(uh) = f(u)$ a contradiction.

The variety W' is closed in $V \times \mathbb{A}^1_k$. Let p be the projection $p : W' \to V$. We have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\mathrm{Id}\times f"] \arrow[d, "\pi"'] & W \arrow[r, "\phi"] & W' \arrow[d, "p"] \\ V \arrow[rr, equal] & & V. \end{tikzcd}$$

We claim that p is bijective. Indeed, the above diagram proves its surjectivity. Furthermore, if $\pi(u) = p(\pi(u), f(u)) = p(\pi(u'), f(u')) = \pi(u')$, then $u' = uh$ for $h \in H$ and $f(u') = f(uh) = f(u)$

proving the injectivity. Furthermore, because the map π is separable and equivariant between G-homogeneous spaces, the differential $d_u\pi$ is surjective for any $u \in U$ therefore, for any $w' = (\pi(u), f(u))$ in V we have $d_u\pi = d_{w'}p \circ d_u(\pi \times f)$ is surjective and thus $d_{w'}p$ is surjective.

Let W'_1 be an irreducible component of W', then the restriction of p to W'_1 is separable and bijective on its image thus birational on its image. This image is a dense open subset V_1 of V. If W'_2 is another irreducible component of W' then its image V_2 by p is also dense and open in V. The intersection $V_1 \cap V_2$ is dense and open in V thus $p^{-1}(V_1 \cap V_2)$ is dense and open in W'_1 and W'_2 therefore $W'_1 = W'_2$. This means that W' is irreducible. Therefore the map $p : W' \to V$ is bijective and separable thus birational. By Zariski main theorem (V being open in G/H which is homogeneous thus smooth, it is smooth thus normal), the morphism p is an isomorphism.

Note that we have $f = q \circ (\pi \times f)$ and defined $f_V = q \circ p^{-1} : V \to \mathbb{A}^1_k$. We have $f' \in k[V]$ and $\pi^\sharp(f') = f' \circ \pi = q \circ p^{-1} \circ \pi = q \circ (\pi \times f) = f$. This proves (i).

(ii) Let $\phi : G \to X$ be a morphism constant on the classe gH for all $g \in G$. We may define a map $\psi : G/H \to X$ by $\psi(gH) = \phi(g)$. Let us prove that this map is a morphism. Let W be open in X, let $U = \phi^{-1}(W)$ and $V = \pi(U)$ which is also open because π is open. Let $f \in k[W]$, we want to prove that $f \circ \psi$ lies in $k[V]$. But we have the equality $f \circ \psi \circ \pi = f \circ \phi$ thus $\pi^\sharp(f \circ \psi) \in k[U]$. It is oviously invariant under H thus $\pi^\sharp(f \circ \psi) \in k[U]^H = \pi^\sharp(k[V])$. As $\pi^\sharp$ is injective we deduce that $f \circ \psi \in k[V]$.

(iii) Let us first define the action of H on $k(G)$. We already know that H acts on $k[G]$. Let $f \in k(G)$, then there exists an open affine $U \subset G$ such that $f \in k[U]$. Then we may define a function $h \cdot f$ by $h \cdot f(g) = f(gh)$. This is a regular function on Uh^{-1} and we define $h \cdot f \in k(G)$ to be the class of this function in $k(G)$. This obviously does not depend on the choice of U.

Let $f \in k(G/H)$, then there exists an open affine V in G/H such that $f \in k[V]$. The function $\pi^\sharp(f)$ lies in $k[\pi^{-1}(V)]^H$ thus its class in $k(G)$ lies in $k(G)^H$.

Conversely, if f lies in $k(G)^H$, then there exists U open affine with $f \in k[U]$. Because f is invariant under H, the function f is defined on the open set $U_H = \cup_{h \in H} Uh^{-1}$. Let $V_H = \pi(U_H)$. This is an open subset in G/H and we have $U_H = \pi^{-1}(V_H)$ (because U_H is invariant under H). Thus we have $f \in k[U_H]^H = \pi^\sharp(k[V_H])$ and the result follows. □

Theorem 6.3.11 *Let H be a normal closed subgroup of an algebraic group G. Then the variety G/H is an algebraic group.*

Proof. Let $U \subset V$ as in Chevalley's semiimvariant Theorem *i.e.* the vector space V is finite dimensional, the subspace U is of dimension 1, the group G acts linearly on V and we have the equalities $H = G_D$, $\mathfrak{h} = \mathrm{Stab}_{\mathfrak{g}}(D)$ where $\mathfrak{h}$ and $\mathfrak{g}$ are the Lie algebras of G and H. Then H acts on U via a character $\chi \in X^*(H)$. Furthermore, the group G acts on $X^*(H)$ by $g\chi(h) = \chi(g^{-1}hg)$ and there exists a finite dimensional subspace $W \subset X^*(H)$ stable under G and containing χ such that the action is rational. We then have a direct sum

$$E = \oplus_{g \in G} V_{g\chi} \subset V$$

and in particular the orbit of χ under G is finite. We claim that the space E is stable under G. Indeed, let $g \in G$ and $v \in V_{g'\chi}$, then $hg \cdot v = gg^{-1}hg \cdot v = g((g'\chi)(g^{-1}hg)v) = g((gg')\chi(h)v) = (gg'\chi)(h)gv$ and $gv \in V_{gg'\chi}$. Let $\rho : G \to \mathrm{GL}(E)$ be the induced representation. Let us compose this representation with the adjoint representation $\psi = \mathrm{Ad}_{\mathrm{GL}(E)} \circ \rho : G \to \mathrm{GL}(\mathfrak{gl}(E))$. For $u \in \mathfrak{gl}(E)$, we have

$$\psi(g)(u) = \rho(g)u\rho(g^{-1}).$$

Let $A = \oplus_{g\chi} \mathfrak{gl}(V_{g\chi})$. This is a subalgebra of $\mathfrak{gl}(E)$ and because G permutes the spaces $V_{g\chi}$, this algebra is stable under the action of $\psi(G)$. Let $\phi : G \to \mathrm{GL}(A)$ be the induced representation. We claim that $\ker \phi = H$ and $\ker d_e\phi = \mathfrak{h}$.

If $h \in H$, then $\rho(h)$ acts by a scalar on any $V_{g\chi}$ and thus $\phi(h) = \psi(h) = \mathrm{Id}$. Therefore $H \subset \ker \phi$. This in turn implies $\mathfrak{h} \subset \ker d_e\phi$. Conversely, if $\phi(g) = \mathrm{Id}$. Then $\rho(g)$ commutes with any element in A. In particular it commutes with the projection $u_\chi \oplus_{g\chi} V_{g\chi} \to V_\chi$. We get for $v \in V_\chi$ the equality $\rho(g)(v) = \rho(g)u_\chi(v) = u_\chi\rho(g)(v)$. If $\rho(g)(v) \notin V_\chi$, then $\rho(g)(v) = 0$ a contradiction since $\rho(g)$ is bijective. Thus $\rho(g)(V_\chi) = V_\chi$. Furthermore $\rho(g)$ commutes with any element in $\mathfrak{gl}(V_\chi)$ thus $\rho(g)|_{V_\chi} = \lambda_{g,\chi}\mathrm{Id}_{V_\chi}$. In particular $\rho(g)$ stabilises the subspace $U \subset V_\chi \subset V$ thus $g \in H$.

The same proof works on the Lie algebra level once we remarked that for $\eta \in \ker d_e\phi$, we have for all $u \in A$ the equalities $0 = d_e\phi(\eta)(u) = d_eAd(d_e\rho(\eta))(u) = \mathrm{ad}\,(d_e\rho(\eta))(u) = [d_e\rho(\eta), u]$ thus by the same argument $\eta \in \mathfrak{h}$.

Now we have a morphism $\phi : G \to \phi(G) \subset \mathrm{GL}(A)$. Thus $\phi(G)$ is a closed algebraic group and we have the commutative diagram:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & \phi(G) \\ G/H. \arrow[ur, "\psi"] & \end{tikzcd}$$

The map ψ exists and is bijective because $H = \ker \phi$ while the map ϕ is separable because $L(\ker \phi) = L(H) = \mathfrak{h} = \ker d_e\phi$. This implies that ψ is separable, being bijective between homogeneous spaces it is an isomorphism. $\square$

Chapter 7

Borel subgroups

7.1 Borel fixed point Theorem

7.1.1 Reminder on complete varieties

Definition 7.1.1 *(ı) Let $\phi : X \to Y$ be a morphism, the ϕ is called proper if ϕ is universally closed* i.e. *for any Z, the morphism $\phi \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is closed.*

(ıı) A variety X over k is called proper or complete if the morphism $X \to \mathrm{Spec}(k)$ is proper.

Example 7.1.2 The variety $\mathcal{A}^1_k$ is not proper. The point $\mathrm{Spec}(k)$ is proper.

Theorem 7.1.3 *The projective spaces are proper varieties.*

Proposition 7.1.4 *Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms. If ϕ and ψ are proper, then so is $\psi \circ \phi$.*

Proof. Exercise. □

Proposition 7.1.5 *Let X be a proper variety.*

(ı) If Y is closed in X, then Y is proper.

(ıı) If Y is proper, then so is $X \times Y$.

(ııı) If $\phi : X \to Y$ is a surjective morphism, then Y is proper.

(ıv) If $\phi : X \to Y$ is a morphism, then $\phi(X)$ is closed in Y and proper.

(v) If X is connected, then $k[X] = 1$.

Proof. Exercise. □

Corollary 7.1.6 *(ı) Any projective variety is proper.*

(ıı) Any proper quasi-projective variety is projective.

Corollary 7.1.7 *If X is affine and proper, then $X = \mathrm{Spec}(k)$.*

Proof. Indeed, we have $k[X] = k$. □

Remark 7.1.8 There exists proper non projective varieties.

Corollary 7.1.9 *Let $\phi : X \to Y$ be a G-equivariant morphism between G-homogeneous spaces. Assume that ϕ is bijective, then if Y is propre, so is X.*

Proof. Let Z be a variety and consider the diagram

$$\begin{tikzcd} X \times Z \arrow[r, "\phi\times\mathrm{Id}_Z"] \arrow[d, "p_X"'] & Y \times Z \arrow[d, "p_Y"] \\ Z \arrow[r, equal] & Z. \end{tikzcd}$$

Let W be a closed subset in $X \times Z$ and let $W' = p_X(W)$ be its image under the left vertical map. We have the equality $W' = p_X(W) = (\phi \times \mathrm{Id}_Z) \circ p_Y(W)$ and because p_Y is closed we only need to prove that $\phi \times \mathrm{Id}_Z(W)$ is closed *i.e.* $\phi \times \mathrm{Id}_Z$ is a closed morphism. But ϕ is universally open (because G-equivariant between hommogeneous G-spaces) thus $\phi \times \mathrm{Id}_Z$ is open. It is bijective thus a homeomorphism. Therefore it is closed. □

7.1.2 Borel fixed point Theorem

Lemma 7.1.10 *Let X be a variety and G acting on X. Then $X^G = \{x \in X \ / \ gx = x \text{ for all } g \in G\}$ is closed in X.*

Proof. Let $g \in G$, then the set $X^g = \{x \in X \ / \ gx = x\}$ is the inverse image of the diagonal Δ_X in $X \times X$ under the morphism $X \to X \times X$ defined by $x \mapsto (x, gx)$. Therefore it is closed. The set X^G is the intersection of all X^g and thus is also closed. □

Theorem 7.1.11 *Let G be a connected solvable group acting on X a non empty proper variety. Then G has a fixed point in X.*

Proof. We proceed by induction on $\dim G$. For $\dim G = 0$, this is obvious since $G = \{e\}$. Otherwise, the group $D(G)$ is a proper subgroup in G therefore $X^{D(G)}$ is non empty. This subset is closed thus proper. We claim that it is G-stable. Indeed, for $x \in X^{D(G)}$, $g \in G$ and $g' \in D(G)$, we have $g'gx = gg^{-1}g'gx = gx$ because $D(G)$ is normal and thus we have the inclusion $g^{-1}g'g \in D(G)$.

Let Gx be a minimal orbit of G in $X^{D(G)}$. It has to be closed therefore proper. Let G_x be the stabiliser of x. We have a bijective morphism $G/G_x \to Gx$ between G-equivariant homogeneous G-spaces. Therefore as Gx is proper, so is G/G_x. But G_x contains $D(G)$ therefore G_x is a normal subgroup in G and the quotient G/G_x is affine. Being connected, proper and affine the quotient G/G_x is a point and so is the orbit Gx. Therefore x is a fixed point for the action of G. □

We may recover the Lie-Kolchin's Theorem from the above result.

Theorem 7.1.12 *Let G be a connected solvable group and let $\rho : G \to \mathrm{GL}(V)$ be a rational representation. Then there exists a basis of V such that $\rho(G) \subset \mathrm{B}_n$.*

Proof. As usual, by induction on $\dim V$, we only need to prove that there exists a one dimensional subspace of V stable under the action of G. This is exivalent to the existence of a fixed point in $\mathbb{P}(V)$ and follows from the former statement. □

7.2 Cartan subgroups

7.2.1 Borel pairs

Definition 7.2.1 *Any maximal closed solvable connected subgroup of G is called a Borel subgroup of G.*

Theorem 7.2.2 *Let G be a connected algebraic group. Then all Borel subgroups are conjugated and if B is a Borel subgroup, then G/B is projective.*

Proof. Let S be a Borel subgroup of maximal dimension. By Chevalley's Theorem, there exists a representation V of G together with a line $V_1 \subset V$ such that $S = G_{V_1}$. We claim that we may assume V to be faithful. Indeed, let W be a faithful representation of G and consider the representation $V \oplus W$. Then $G_{V_1} = S$ also for this representation.

So we assume V to be faithful. By Lie-Kolchin's Theorem, there exists a complete flag $V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n = V$ stable under S. We have $S \subset G_{V_\bullet} \subset G_{V_1} = S$ thus $S = G_{V_\bullet}$. We thus have a bijective morphism

$$G/S \to GV_\bullet \subset \mathcal{F}$$

where $\mathcal{F}$ is the variety of all flags. Let us prove that $GV_\bullet$ is a minimal orbit therefore closed. Indeed, let $V'_\bullet$ be another complete flag in V and let $G_{V'_\bullet}$ be its stabiliser. The elements in $G_{V'_\bullet}$ are upper triangular matrices for a basis compatible with $V'_\bullet$ thus $G_{V'_\bullet}$ is connected. By assumption, we have $\dim G_{V'_\bullet} = \dim G^0_{V'_\bullet} \leq \dim S$. Therefore $\dim GV'_\bullet \geq \dim GV_\bullet$ proving the minimality.

But $\mathcal{F}$ is a closed subset in the product of all grassmannians therefore it is projective. In particular the orbit $GV_\bullet$ is proper. We deduce that G/S is proper. Being quasi-projective, it is projective.

Let B be any Borel subgroup. Then it acts on G/S by left multiplication. It has a fixed point gS *i.e.* $Bg \subset gS$. Thus $B \subset gSg^{-1}$. By maximality we must have equality. □

Definition 7.2.3 *A couple (B, T) with B a Borel subgroup and T a maximal torus of G contained in B is called a Borel pair.*

Corollary 7.2.4 *(ı) Any maximal torus T of G is contained in a Borel subgroup B. Furthermore the Borel pairs are conjugated.*

(ıı) The maximal closed connected unipotent subgroups of G are all connected and of the form B_u for some Borel subgroup B of G.

Proof. (ı) Let T be a maximal torus. It is closed connected and solvable therefore contained in a maximal such group: a Borel subgroup. It is a maximal torus of B. Because any two Borel subgroups are conjugated and any two maximal tori in B are conjugated, the result follows.

(ıı) Let U be unipoetne maximal. It is closed connected and solvable therefore contained in a maximal such group: a Borel subgroup. It is a maximal unipotent subgroup of B. But B_u is such a group thus $U = B_u$. There are conjugated because Borel subgroups are and that $(gBg^{-1})_u = gB_ug^{-1}$. □

Definition 7.2.5 *A closed subgroup P of G is called a parabolic subgroup if G/P is complete (and therefore projective).*

Proposition 7.2.6 *Let P be a closed subgroup of G. The following conditions are equivalent.*

(ı) The subgroup P is a parabolic subgroup of G.

(ıı) The subgroup P contains a Borel subgroup.

Proof. If P contains a Borel subgroup B, then we have a surjective morphism $G/B \to G/P$ thus G/P is proper since G/B is. Conversely, if G/P is proper, then any Borel subgroup B has a fixed point gP in G/P thus $Bg \subset gP$ and $g^{-1}Bg \subset P$. □

Corollary 7.2.7 *A closed subgroup B in G is a Borel subgroup if and only if it is a connected solvable parabolic subgroup.*

Theorem 7.2.8 *Let $\phi : G \to G'$ be a surjective morphism of algebraic groups. Let H be a closed subgroup of G. If H is a parabolic subgroup, a Borel subgroup a maximal torus or a maximal unipotent subgroup, then so if $\phi(H)$. Furthermore, any such subgroup is obtained in that way.*

Proof. Because the map ϕ is surjective, the morphism ϕ realises G' as a homogeneous G-space. The morphism $G/H \to G'/\phi(H)$ induced by ϕ is surjective thus if H is a parabolic subgroup, so is $\phi(H)$.

If H is a Borel subgroup, then $\phi(H)$ is connected and solvable thus a Borel subgroup.

If H is a maximal unipotent subgroup, then $H = B_u$ for some Borel subgroup and $\phi(H) = \phi(B_u) \subset \phi(B)_u$. Furthermore if $\phi(g) \in \phi(B)_u$, then there exists $b \in B$ such that $\phi(g) = \phi(b)$. Write $b = b_s b_u$ the Jordan decomposition. We get $\phi(g) = \phi(b)_s \phi(b)_u$ which is the Jordan decomposition of $\phi(g)$ thus $\phi(b_s) = e$ and $\phi(g) = \phi(b_u)$ *i.e* we have the equality $\phi(H) = \phi(B)_u$ is a maximal unipotent subgroup.

If H is a maximal torus, let B be a Borel subgroup containing H. Then $\phi(H)$ is again a torus of $\phi(B)$ a Borel subgroup in G'. Furthermore, we have $B = HB_u$ thus $\phi(B) = \phi(H)\phi(B)_u$ thus $\phi(H)$ is a maximal torus of $\phi(B)$ and thus a maximal torus of G'.

If H' is a Borel subgroup, a maximal unipotent subgroup or a maximal torus of G' and H is of the same type. Then $\phi(H)$ is of the same type and there exists $g' = \phi(g)$ such that $H' = g'\phi(H){g'}^{-1} = \phi(gHg^{-1})$. If H' is a parabolic subgroup, then H' contains a Borel subgroup $\phi(B)$ with B a Borel subgroup of G. Then $H = \phi^{-1}(H')$ contains B and is therefore a parabolic subgroup of G with $\phi(H) = H'$. □

7.2.2 Centraliser of Tori, Cartan subgroups

Lemma 7.2.9 *(ı) Let G be a connected algebraic group and let B be a Borel subgroup of G. Let $\phi \in \mathrm{Aut}(G)$ with $\phi|_B = \mathrm{Id}_B$, then $\phi = \mathrm{Id}_G$.*

(ıı) As a consequence, if $g \in G$ centralises B, then $g \in Z(G)$ i.e. *$C_G(B) \subset Z(G)$.*

(ııı) In particular $Z(B) \subset Z(G)$.

Proof. Let $\phi : G \to G$ be such an automorphism. It is constant on B therefore it can be factorised through the quotient G/B *i.e.* there exists a morphism $\psi : G/B \to G$ such that $\phi = \psi \circ \pi$ with $\pi : G \to G/B$ the quotient map. But G/B is proper thus $\psi(G/B)$ is proper and closed in G therefore affine. This implies that $\psi(G/B)$ is apoint and the result follows. □

Proposition 7.2.10 *Let G be an algebraic group and let B be a Borel subgroup of G. If B is nilpotent, then $G^0 = B$.*

Proof. We may assume that G is connected. We proceed by induction on $\dim G$. If $B = \{e\}$, then $G = G/B$ is affine and proper thus $G = \{e\}$. If not, let n be such that $C^n(B)$ is non trivial but $C^{n+1}(B) = \{e\}$. The group $C^n(B)$ is central in B therefore it is central in G. We may thus look at the quotients $B/C^n(B) \subset G/C^n(B)$. By induction we have $G/C^n(B) = B/C^n(B)$ and the result follows. □

Corollary 7.2.11 *Let G be a connected group of dimension at most 2, then G is solvable.*

Proof. Let B be a Borel subgroup of G. We want to prove that $G = B$. Let us write $B = TB_u$ with T a maximal torus of G. If $\dim B = 1$, then $B = T$ or $B = B_u$ thus B nilpotent and the result follows from the previous proposition. If $\dim B = 2$, then $B = G$ because G is connected therefore irreducible. □

Corollary 7.2.12 *Let G be a connected algebraic group.*
(ı) If $G = G_s$, then G is a torus.
(ıı) If G_u is a subgroup, then G is solvable.
(ııı) If G_s is a subgroup, then G is nilpotent.

Proof. (ı) Let B be a Borel subgroup of G. Then $B = TB_u = T$ thus B is nilpotent and $G = B = T$.

(ıı) The subgroup G_u is normal since for $g \in G$ and $g_u \in G_u$ we have $gg_ug^{-1} \in G_u$. We may consider the quotient G/G_u whose elements are all semisimple thus G/G_u is a torus. Therefore G is an extension of T and G_u both of which are solvable thus G is solvable.

(ııı) Let B be a Borel subgroup. The subgroup $B_s = B \cap G_s$ is commutative by the structure Theorem on solvable groups. Thus we may embed B in GL_n such that $B_s = \mathrm{D}_n \cap B$ therefore B_s is a closed subgroup of B. This subgroup is normal in B (because the conjugate of a semisimple element is again semisimple) and thus it is central: $B = N_B(B_s) = C_B(B_s)$. This implies by the characterisation of nilpotent groups that B is nilpotent. By the above proposition $G = B$. □

Proposition 7.2.13 *Let T be a maximal torus in G, then $C = C_G(T)^0$ is nilpotent and $C = N_G(C)^0$.*

Proof. Let $g \in C_s$ an element which is semisimple. Then $gt = tg$ for all $t \in T$. Let H be the closed subgroup spanned by T and g. Then H is commutative all its elements are semisimple therefore it is a torus and $T \subset H$ thus $T = H$ and $g \in T$. This proves that $C_s = T$ is a subgroup thus C is nilpotent.

Another proof: let B be a Borel subgroup of C, then T is a maximal torus of B and is central in B thus B is nilpotent and thus C is also nilpotent.

We know that $C = N_G(T)^0$. Let us prove the inclusion $N_G(C) \subset N_G(T)$. Note that C being nilpotent, then C_s is a closed subgroup containing T and thus equal to T. But C_s is stable under conjugation thus if $g \in N_G(C)$, then $gTg^{-1} = gC_sg^{-1} \subset C \cap G_s = C_s = T$ proving the result. □

7.2.3 Cartan subgroups

Definition 7.2.14 *Let G an algebraic group and let T be a maximal torus. The group $C = C_G(T)^0$ is called a Cartan subgroup of G.*

Remark 7.2.15 We shall prove later that $C_G(T)$ is connected therefore is a Cartan subgroup.

Lemma 7.2.16 *Let G be a connected algebraic group and let H be a closed subgroup. Let us set*

$$X = \bigcup_{g \in G} gHg^{-1}.$$

(ı) The subset X contains a dense open subset of its closure $\overline{X}$.
(ıı) If G/H is proper, then X is closed.
(ııı) If $N_G(H)/H$ is finite and there exists an element $g \in G$ which is contained in a finite number of conjugates of H, then $\overline{X} = G$.

Proof. (i) Let $M = \{(x,y) \in G \times G \ / \ y \in xHx^{-1}\}$. This is a closed irreducible subvariety in $G \times G$. Indeed, it is the image of $G \times H$ under the isomorphism $G \times G \to G \times G$ given by $(x,y) \mapsto (x, xyx^{-1})$. The variety X is the image of the second projection and the result follows since this image is constructible by (one of the many) Chevalley's Theorem.

(ii) If G/H is proper we simply factor the above map through $G/H \times G$. Indeed, the relation $y \in xHx^{-1}$ only depends on the class of x in G/H. More precisely, we defined $N = \{(xH, y) \in G/H \times G \ / \ y \in xHx^{-1}\}$. We have a projection $\psi : G \times G \to G/H \times G$ whose restriction maps M to N. We have $M = \psi^{-1}(N) = \psi^{-1}(\psi(M))$. In particular, because ψ is open (the quotient map is universally open) we get that $N = \psi(M)$ is closed. But G/H is proper so the projection $p : G/H \times G \to G$ is closed and $X = p(N)$ is closed.

(iii) Let q be the projection of N into G/H and p the projection to G. The map q is surjective with fibers isomorphic to H. Therefore $\dim N = \dim G$. On the other hand, let $g \in G$ an element contained in finitely many conjugates of H, say $g \in x_iHx_i^{-1}$ for $i \in [1,n]$. We consider the fiber $p^{-1}(g) = \{xH \in G/H \ / \ g \in xHx^{-1}\}$. For xH in the fiber we have $xHx^{-1} = x_iHx_i^{-1}$ thus $x^{-1}x_i \in N_G(H)$ thus xH and x_iH are equal modulo an element in $N_G(H)/H$. Therefore $p^{-1}(g)$ is finite thus $\dim p(N) = \dim N = \dim G$ and G being connected we have the equalities $\overline{X} = \overline{p(N)} = G$. □

Theorem 7.2.17 *Let G be a connected algebraic group.*

*(i) The union of all Cartan subgroups (*i.e. $\cup_{T \text{ max. torus}} C_G(T)^0$*) contains a dense open subset of G.*

(ii) The group G is equal to the union of all Borel subgroups.

(iii) Any semisimple elements is contained in a maximal torus.

(iv) Any unipotent elements element of G is contained in a maximal connected unipotent subgroup.

Proof. (i) Let T be a maximal torus and let $C = C_G(T)^0$. We know that $C = N_G(C)^0$ thus $N_G(C)/C$ is finite. We also know that there exists $t \in T$ such that $C_G(T) = C_G(t)$. Let us prove that t is in finitely many conjugate of C. If $t \in xCx^{-1}$, then $x^{-1}tx \in C$ thus $x^{-1}tx \in C_s = T$ (because C is nilpotent therefore C_s is a subgroup and thus the unique maximal torus of C). Therefore $C \subset C_G(x^{-1}tx) = x^{-1}C_G(t)x = x^{-1}C_G(T)x$. We get $C = x^{-1}Cx$. So t is contained in only one conjugate of C: the group C itself. By the previous lemma we get that the union of Cartan subgroups is dense and therefore contains a dense open.

(ii) The group C being connected and nilpotent, it is contained in some Borel subgroup B of G. Therefore the union of all Borel subgroups is dense but because G/B is proper it is also closed by the previous lemma and the result follows.

(iii) Let s be a semisimple element in G. It is in a Borel subgroup B and by the structure theorem of Borel subgroups it is in a maximal torus of B which is also a maximal torus of G.

(iv) Let u be unipotent, it is contained in some Borel B and thus in B_u, this is the result. □

Corollary 7.2.18 *Let G be connected and assume that there exists a normal Borel subgroup, then $G = B$* i.e. *the group G is solvable.*

Proof. First proof, the quotient G/B is affine and proper. It is connected thus it is a point.

Second proof, the group G is the union of the conjugates of B, there is a unique such conjugate B itself. □

Corollary 7.2.19 *Let G be connected, then we have the equality $Z(G) = Z(B)$ for any Borel subgroup B.*

Proof. We already know the inclusion $Z(B) \subset Z(G)$. Let $g \in Z(G)$, then there exists a Borel subgroup B such that $g \in B$ and thus $g \in Z(B)$. Furthermore, if xBx^{-1} is another Borel subgroup, then $g = xgx^{-1} \in xBx^{-1}$ and the result follows. $\square$

Lemma 7.2.20 *Let G be connected and S be a connected solvable subgroup. Let $x \in C_G(S)$. Then there exists a Borel subgroup containing S and x.*

Proof. Let B be a Borel subgroup containing x. In particular the variaty $(G/B)^x$ contains eB. Let S act on G/B, it stabilises $(G/B)^x$ which is proper thus there is a fixed point gB. We have $Sg \subset gB$ thus $S \subset gBg^{-1}$ and $xgB = gB$ thus $x \in gBg^{-1}$. $\square$

Theorem 7.2.21 *Let G be connected and S be a torus in G.*

(ı) Then $C_G(S)$ is connected.

(ıı) If B is a Borel subgroup of G containing S, then $B \cap C_G(S)$ is a Borel subgroup of $C_G(S)$.

(ııı) Furthermore any Borel subgroup of $C_G(S)$ is obtained in this way.

Proof. (ı) Let $x \in C_G(S)$, then x and S are contained in some Borel subgroup B. Then $x \in C_B(S)$ which is connected by the structure Theorem on connected solvable groups. Therefore $x \in C_G(S)^0$ and $C_G(S) = C_G(S)^0$.

(ıı) Set $C = C_G(S)$. It is enough to prove that $C/C \cap B$ is proper therefore it is enough to prove that $C(eB) \subset G/B$ is closed. Because the map $\pi : G \to G/B$ is open, it is enough to prove that $\pi^{-1}(C(eB)) = CB$ is closed. Note that this variety is irreducible as the image of $C \times B$ by multiplication.

For $y = cb \in CB$ with $c \in C$ and $b \in B$, we have $y^{-1}Sy = b^{-1}c^{-1}Scb = b^{-1}Sb \subset B$ because $S \subset B$. Therefore for any $y \in \overline{CB}$ we also have $y^{-1}Sy \subset B$.

Let T be a maximal torus of B containing S and let $\phi : B \to B/B_u$ be the quotient map. It realises an isomorphism from T to B/B_u. We may consider the morphism $\psi : \overline{CB} \times S \to B/B_u$ defined by $(y, s) \mapsto \phi(y^{-1}sy)$. By the rigidity lemma we have that ψ does not depend on y (we need $\overline{CB}$ to be affine, we need that S and $B/B)u$ are diagonalisable and that ψ_y is a group morphism).

Now let $y \in \overline{CB}$, we have $y^{-1}Sy$ is a torus in B thus there exists $u \in C^\infty(B) \subset B_u$ such that $u^{-1}y^{-1}Syu \subset T$. Furthermore, for any $s \in S$ we have $\psi(u^{-1}y^{-1}syu) = \psi(yu, s) = \psi(s) = \pi(s)$ (for this note that because CB is stable by right multiplication by elements in B, so is $\overline{CB}$ thus $yu \in \overline{CB}$). But π is injective on T and $u^{-1}y^{-1}syu$ and s are in T thus $s = u^{-1}y^{-1}syu$ for all $s \in S$ thus $yu \in C$ and $y \in CB$. Thus CB is closed proving the result.

(ııı) Let B' be a Borel subgroup of $C = C_G(S)$. Let B be a Borel subgroup of G containing S. Then there exists $c \in C$ such that $B' = c(B \cap C)c^{-1}$. But $cCc^{-1} = C$ and $B = cBc^{-1} \cap C$. This is what we wanted. $\square$

Corollary 7.2.22 *Let G be a connected group and T a maximal torus. Let $C = C_G(T)$.*

(ı) The group C is connected, nilpotent and equal to $N_G(C)^0$ (thus the quotient $N_G(C)/C$ is finite).

(ıı) Any Borel subgroup B containing T contains C.

Proof. (ı) The previous theorem implies the connectedness and we already proved that C is nilpotent and equal to $N_G(C)^0$.

(ıı) If B contains T, then $B \cap C$ is a Borel subgroup of C and is nilpotent as a subgroup of C. Thus we must have $C = C \cap B$. $\square$

7.3 Normalisers of Borel subgroups

Theorem 7.3.1 (Chevalley) *Let G be a connected group.*

($\imath$) For any Borel subgroup, we have the equality $B = N_G(B)$.

($\imath\imath$) For any parabolic subgroup, we have the equalities $N_G(P) = P = P^0$.

($\imath\imath\imath$) For any Borel subgroup we have the equality $B = N_G(B_u)$.

Proof. ($\imath$) We proceed by induction on $\dim G$. If $\dim G \leq 2$, then G is solvable thus $G = B$ and the result follows.

Set $N = N_G(B)$ and let $n \in N$. Let T be a maximal torus of G contained in B. Then nTn^{-1} is again a maximal torus contained in B. Therefore there exists $b \in B$ with $bnT(bn)^{-1} = T$. Replacing n by nb we may assume that $n \in N_G(T)$.

Consider the morphism $\psi : T \to T$ defined by $\psi(t) = ntn^{-1}t^{-1}$. This is a morphism of algebraic groups. Let $S = (\ker \psi)^0$ which is a subtorus of T. Then n lies in $C_G(S)$.

Assume first that S is not trivial. Then n normalises $B \cap C_G(S)$ which is a Borel subgroup of S thus if $C_G(S) \neq G$, we get by induction that n lies in $B \cap C_G(S)$ thus $n \in B$. If $C_G(S) = G$, then S is central in G thus the quotient G/S is an algebraic group and B/S is a Borel subgroup. The element nS is in $N_{B/S}(G/S)$ and by induction again we have $nS \subset B$ thus $n \in B$.

Assume now that S is trivial. Then ψ is surjective (because its image is a closed connected subgroup of the same dimension as T). Let V be a representation of G such that $N = G_U$ for some subspace of dimension 1 in V. Then N acts on U via a character $\chi \in X^*(N)$. This character has to be trivial on B_u because it maps unipotent elements to unipotent elements in $\mathbb{G}_m$. It also has to be trivial on T because any element of T is a commutator. Therefore B acts trivially on U and if u is a non trivial vector in U, the morphism $G \to V$ defined by $g \mapsto gu$ factors through G/B. But G/B is proper thus the image is proper and closed in V thus affine. Therefore the image is constant and G acts trivially on u. We get $B = G = N$.

($\imath\imath$) Let P be a parabolic subgroup and B a Borel subgroup contained in P. We have $B \subset P^0$ because B is connected. Let $n \in N_G(P)$, then xBx^{-1} is again a Borel subgroup of P^0. Thus there exists $p \in P^0$ with $xBx^{-1} = pBp^{-1}$. Therefore $p^{-1}x \in N_G(B) = B$ thus $x \in P^0$ and the result follows.

($\imath\imath\imath$) Let $U = B_u$ and $N = N_G(U)$. We have $B \subset N$ thus B is a Borel subgroup of N^0 (it has to be maximal!). Therefore, any unipotent element in N^0 is conjugated to an element in U. But U being normal in N^0, we have $U = (N^0)_u$. Therefore N^0/U is a torus (connected and all elements are semisimple). Thus N^0 is solvable. Thus $N^0 = B$. Furhtermore, because N normalises N^0 we get $N \subset N_G(B) = B$ proving the result. □

Corollary 7.3.2 *Let G be connected, let B be a Borel subgroup and let P and Q be two parabolic subgroups containing B and conjugated in G. Then $P = Q$.*

Proof. We have $Q = gPg^{-1}$ thus B and gBg^{-1} are Borel subgroups of Q. Therefore there exists $qw \in Q$ with $qBq^{-1} = gBg^{-1}$. We get $qg^{-1} \in N_G(B) = B$ thus $g \in Q$ and $P = Q$. □

7.4 Reductive and semisimple algebraic groups

7.4.1 Radical and unipotent radical

Definition 7.4.1 *Let G be an affine algebraic group.*

($\imath$) We define the radical of G to be the maximal closed connected solvable normal subgroup of G. We denote it by $R(G)$.

(ii) We define the unipotent radical of G to be the maximal closed connected unipotent normal subgroup of G. We denote it by $R_u(G)$.

Let us denote by $\mathcal{B}$ the set of all Borel subgroups of G.

Proposition 7.4.2 *We have the equalities*

$$R(G) = \left(\bigcap_{B\in\mathcal{B}} B\right)^0 \text{ and } R_u(G) = \left(\bigcap_{B\in\mathcal{B}} B_u\right)^0 = R(G)_u.$$

Proof. The above intersection is obviously a closed connected solvable group of G. Furthermore since any two Borel subgroups are conjugated it is also normal. Therefore the intersection is contained in $R(G)$. Conversely, the group $R(G)$ being solvable and connected, it is contained in all Borel subgroup thus in the above intersection. Note that any automorphism of G maps a Borel subgroup to a Borel subgroup thus the group $R(G)$ is even characteristic.

The same argument give the second equality. For the last one, because $R(G)$ is characteristic and solvable, we have that $R(G)_u$ is a normal subgroup of $R(G)$ and thus also normal of G. Being unipotent it is contained in $R_u(G)$. Now if U is a normal unipotent connected subgroup of G, it is contained in $R(G)$ and thus in $R(G)_u$. □

7.4.2 Reductive and semisimple algebraic groups

Definition 7.4.3 *An algebraic group G is called reductive if $R_u(G) = \{e\}$ and semisimple if $R(G) = \{e\}$.*

Lemma 7.4.4 *Let $1 \to H \to G \to K \to 1$ be an exact sequence of algebraic groups. The group G is unipotent if and only if H and K are also unipotent.*

Proof. Exercise. □

Proposition 7.4.5 *The quotient $G/R_u(G)$ is reductive and the quotient $G/R(G)$ is semisimple.*

Proof. Let $\pi : G \to G/R(G)$ be the quotient map and let H be a connected closed normal solvable subgroup of $G/R(G)$. Then $\pi^{-1}(H)$ is also closed connected sovable and normal therefore contained in $R(G)$. The result follows. For the unipotent radical, the same proof works using the previous lemma. □

Chapter 8

Geometry of the variety of Borel subgroups

8.1 The variety of Borel subgroups

Definition 8.1.1 *Let G be a connected algebraic group, we denote by $\mathfrak{B}$ the set of all Borel subgroups of G. This variety is called the flag variety of G.*

Proposition 8.1.2 *The set $\mathfrak{B}$ can be endowed with a structure of variety such that it becomes isomorphic to G/B for any Borel subgroup B. This variety is therefore irreducible, proper, smooth and homogeneous under G.*

Proof. Let us fisrt prove that $\mathfrak{B}$ is in bijection with G/B. We have a natural map $\phi_B : G/B \to \mathfrak{B}$ defined by $gB \mapsto gBg^{-1}$. This map is surjective since any two Borel subgroups are conjugated. Futhermore, if $gBg^{-1} = xBx^{-1}$, then gx^{-1} lies in $N_G(B) = B$ thus $gB = xB$ and the map is injective.

If B' is another Borel subgroup and let $g \in G$ with $B' = gBg^{-1}$. Then we have the following commutative diagram:

$$\begin{tikzcd} G/B \arrow[r, "\mathrm{Int}(g)"] \arrow[d, "\phi_B"'] & G/B' \arrow[d, "\phi_{B'}"] \\ \mathfrak{B} \arrow[r, "\mathrm{Int}(g)"] & \mathfrak{B} \end{tikzcd}$$

proving that the structure of varieties does not depend on the choice of the Borel. □

Lemma 8.1.3 *Let S be a subset of G. Then $\mathfrak{B}^S$ is closed and given by $\mathfrak{B}^S = \{B \in \mathfrak{B} \ / \ B \supset S\}$.*

Proof. The action of an element $s \in G$ on $\mathfrak{B}$ is given by $s \cdot B = sBs^{-1}$. Therefore, we have the equivalences: $s \cdot B = B \Leftrightarrow s \in N_G(B) = B$. The result follows. □

Definition 8.1.4 *The Weyl group of an algebraic group G with respect to a torus T of G is the finite group $W(G,T) = N_G(C)/C$ where $C = C_G(T)$ is the associated Cartan subgroup. Note that we also have $W(G,T) = N_G(T)/C$.*

Theorem 8.1.5 *Let G be connected and T be a maximal torus. Then $W(G,T)$, the Weyl group acts simply transitively on $\mathfrak{B}^T$. In particular $|\mathfrak{B}^T| = |W(G,T)|$ is finite.*

Proof. Let $C = C_G(T)$. Let $n \in N_G(T)$ and $B \in \mathfrak{B}^T$. Then $n \cdot B = nBn^{-1}$ contains $nTn^{-1} = T$ thus $n \cdot B$ is invariant under the action of T. Thus $N_G(T)$ acts on $\mathfrak{B}^T$.

On the other hand, for $B \in \mathfrak{B}^T$, we have $T \subset B$ thus $C \subset B$. In particular C acts trivially on B and thus on $\mathfrak{B}^T$. Therefore the action of $N_G(T)$ factors through an action of $W(G,T)$.

Let B and $B' = g \cdot B$ be elements in $\mathfrak{B}^T$. Then T and $g^{-1}Tg$ are tori of B' thus there exists $b \in B$ with $b^{-1}Tb = g^{-1}Tg$ and then $n = gb^{-1} \in N_G(T)$. Thus $B' = g \cdot B = nb \cdot B = n \cdot B$. Therefore the action of $W(G,T)$ is transitive on $\mathfrak{B}^T$.

Finally let $n \in N_G(T)$ such that $n \cdot B = B$ for all $B \in \mathfrak{B}^T$. We thus have $nBn^{-1} = B$ and thus $n \in N_G(B) = B$. In particular $n \in N_B(T) = C_B(T) = C_B(T)^0$. Thus n lies in $N_G(T)^0 = C_G(T)$ and the result follows. □

Let $\phi : G \to G'$ be a surjective morphism of algebraic groups. We know that the assignation $B \mapsto \phi(B)$ defines a surjective map $\phi_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}'$ with $\mathfrak{B}'$ the flag variety of G'.

Proposition 8.1.6 *(ı) With the above notation, assume that* $\ker\phi$ *is contained in a Borel subgroup. Then* $\ker\phi$ *is contained in all Borel subgroups and* $\phi_{\mathfrak{B}}$ *is bijective.*

(ıı) Let T be a maximal torus of G and $T' = \phi(T)$ a maximal torus of G'.

(a) Then ϕ induces a group morphism $W(G,T) \to W(G',T')$.

(b) We have a commutative diagram

$$\begin{array}{ccc} W(G,T) & \longrightarrow & W(G',T') \\ \downarrow & & \downarrow \\ \mathfrak{B}^T & \longrightarrow & \mathfrak{B}'^{T'} \end{array}$$

where the vertical maps are given by the action on a Borel subgroup $B \in \mathfrak{B}^T$ and $B' = \phi(B) \in \mathfrak{B}'^{T'}$.

(c) The map between Weyl groups is surjective.

(d) If $\ker\phi$ *is contained in a Borel subgroup, then the map $W(G,T) \to W(G',T')$ is an isomorphism of finite groups.*

Proof. (ı) If $\ker\phi$ is contained in a Borel subgroup, then since it is normal it is contained in any Borel subgroup and we have $B = \phi^{-1}(\phi(B))$ for all $B \in \mathfrak{B}$ proving the injectivity.

(ıı).(a) Let $n \in N_G(T)$ and consider $\phi(n)$. Then $\phi(n)T\phi(n)^{-1} = \phi(nTn^{-1}) = \phi(T) = T'$ thus $\phi(n) \in N_{G'}(T')$. Furthermore, if $n \in C_G(T)$, then the same computation gives $\phi(n) \in C_{G'}(T')$ thus the map $n \mapsto \phi(n)$ induces the desired morphism.

(b) First note that for $B \in \mathfrak{B}^T$, then $T \subset B$ thus $T' \subset \phi(B)$ thus $\phi(B) \in \mathfrak{B}'^{T'}$. Let us check the commutativity. We have to check the equality $\phi(n)\cdot\phi(B) = \phi(n\cdot B)$ which is simply $\phi(n)\phi(B)\phi(n)^{-1} = \phi(nBn^{-1})$ and follows from the fact that ϕ is a group morphism.

(c) Let $\phi(B_0) \in \mathfrak{B}'^{T'}$. Then $\phi(T') \subset \phi(B_0)$. Therefore $T \subset \phi^{-1}(\phi(B_0)) = P$ which is a parabolic subgroup containing B_0. The torus T is therefore a maximal torus of P and thus is contained in a Borel subgroup B of P. But B_0 is also a Borel subgroup of P thus B and B_0 are conjugate in P thus B is a Borel subgroup of G. This prove the surjectivity.

(d) By (c) the map is surjective and by (ı) it is injective. □

Remark 8.1.7 We proved the following fact: if P is a parabolic subgroup of G and B a Borel subgroup of P, then B is also a Borel subgroup of G.

8.2 Action of a torus on a projective space

Lemma 8.2.1 *Let M be a $\mathbb{Z}$-module and $(M_i)_{i\in[1,n]}$ be proper submodules such that M/M_i is torsion free for all $i \in [1,n]$. Then*

$$M \neq \bigcup_{i=1}^{n} M_i.$$

Proof. Assume that the equality holds. We may assume that for all i we have $M_i \not\subset \cup_{j\neq i} M_j$ (otherwise simply remove M_i of the list). Let $m_i \in M_i$ and not in M_j for $j \neq i$ (choose such an element for any i). Because M/M_1 is torsion free, for all $k \in \mathbb{Z}$ we have $m_1 + km_2 \notin M_1 \cup M_2$. Therefore there exists a pair (k,r) with $r > k$ such that $m_1 + km_2$ and $m_1 + rm_2$ are in the subspace M_i with $i \geq 3$. Thus $(r-k)m_2 \in M_i$ and because M/M_i is torsion free, we have $m_2 \in M_i$ a contradiction. $\square$

Recall that for T a torus, we defined a bilinear form $X^*(T) \times X_*(T) \to \mathbb{Z}$ by $(\chi,\phi) \mapsto \langle\chi,\phi\rangle$ where $\langle\chi,\phi\rangle$ is defined by $\chi \circ \phi(z) = z^{\langle\chi,\phi\rangle}$. Recall also the proposition.

Proposition 8.2.2 *The above bilinear map is a parfect pairing.*

Proof. This is an explicit check. If $T = \mathbb{G}_m^r$, there is a group isomorphism $X_*(T) = \mathbb{Z}^r$ given by $(\nu_1, \cdots, \nu_r) \mapsto (z \mapsto (z^{\nu_1}, \cdots, z^{\nu_r}))$ and another isomorphism $X^*(T) = \mathbb{Z}^r$ given by $(\nu_1, \cdots, \nu_r) \mapsto ((x_1, \cdots, x_r) \mapsto x_1^{\nu_1} \cdots x_r^{\nu_r})$. The pairing is then given by $((a_i),(b_i)) \mapsto \sum_i (a_i b_i)$ which is easily checked to be a perfect pairing. $\square$

Lemma 8.2.3 *Let T be a torus and V be a representation of T. There exists $\phi \in X_*(T)$ such that $\mathbb{P}(V)^T = \mathbb{P}(V)^{\phi(\mathbb{G}_m)}$. More precisely, there exists $(\chi_i)_{i\in[1,s]}$ such that the previous equality holds for all $\phi \in X_*(T)$ with $\langle\phi,\chi_i\rangle \neq 0$ for all $i \in [1,s]$.*

Proof. Let $(\chi_i)_{i\in[1,r]}$ be the weights of T in V *i.e* the characters χ such that the eigenspace V_χ is non trivial. We then have $V = \oplus_{i=1}^r V_{\chi_i}$. Let $M = X_*(T)$ and $M_{i,j} = \{\phi \in X_*(T) \ / \ \langle\chi_i,\phi\rangle = \langle\chi_j,\phi\rangle\}$. The quotient $M/M_{i,j}$ is torsion free since $\mathbb{Z}$ is. By the previous lemma we get an element $\phi \in X_*(T)$ with the $\langle\chi_i,\phi\rangle$ all distinct. We easily get that a line in V is stable under $\phi(\mathbb{G}_m)$ if and only if it is contained in one of the V_{χ_i} which is exactely the fixed locus for T. $\square$

Lemma 8.2.4 *Let V be a representation of $\mathbb{G}_m$ and let $v \in V$. Let $[v]$ be its class in $\mathbb{P}(V)$.*
($\imath$) Then $[v]$ is a fixed point if and only if v is an eigenvector for $\mathbb{G}_m$.
($\imath\imath$) If $[v]$ is not fixed, write

$$v = \sum_{i=r}^{s} v_i$$

with $v_i \in V_i$ and $r < s$ (the space V_i is the eigenspace for the eigenvalue $i \in \mathbb{Z}$. In this case the morphism $\sigma : \mathbb{G}_m \to \mathbb{P}(V)$ defined by $\sigma(z) = z \cdot v$ extends to a morphism $\tilde{\sigma} : \mathbb{P}^1 \to \mathbb{P}(V)$ with $\tilde{\sigma}(0) = [v_r]$ and $\tilde{\sigma}(\infty) = [v_s]$.

We have $\tilde{\sigma}(\mathbb{P}^1) = \overline{\mathbb{G}_m[v]} = \mathbb{G}_m[v] \cup \{[v_r]\} \cup \{[v_s]\}$ and $[v_r]$ and $[v_s]$ are the only fixed points of $\mathbb{G}_m$ in this orbit closure.

Proof. This is quite obvious by writing down eigenbasis of the action. $\square$

Lemma 8.2.5 *Let H be an hyperplane in $\mathbb{P}(V)$ and X an irreducible closed subvariety in $\mathbb{P}(V)$ of dimension $d \geq 1$ not contained in H. Then $X \cap H$ is non empty and equidimensional of dimension $d-1$.*

Proof. The dimension assertion is a consequence of Krull Hauptidealsatz. If $X \cap H$ was empty then X would be a closed subvariety if $\mathbb{P}(V) \setminus H$ thus affine but also proper since it is closed in $\mathbb{P}(V)$ therefore a point. A contradiction to the dimension. □

Proposition 8.2.6 *Let T be a torus and V a representation of T. Let X be a closed irreducible subvariety of $\mathbb{P}(V)$ stable under T. Then we have the inequality*

$$|X^T| \geq \dim(X) + 1.$$

Proof. We may replace T by $\mathbb{G}_m$. Let $d = \dim X$ and $n = \dim V$. We proceed by induction on $d + n$ and we may assume that $d \geq 1$. We choose a basis (e_i) of eigenvectors of V with eigenvalue (m_i) such that this sequence of eigenvalues is non decreasing. Let $W = \langle e_2, \cdots, e_n \rangle$ and $H = \mathbb{P}(W)$. Then W and H are T-stable. By induction, we may assume that X is not contained in H. Then $\mathbb{G}_m$ will stabilise all the irreducible components of $X \cap H$ (because $\mathbb{G}_m$ is connected thus irreducible). By induction $\mathbb{G}_m$ has at least d fixed points in $X \cap H$. But X is not contained in H thus there exists $[v] \in X$ with $[v]$ not in H. Write $v = \sum a_i e_i$ with $a_1 \neq 0$. Then either $[v]$ is a fixed point and we are done, or $\lim_{t \to 0} t[v] = [v_1 + v']$ with v' of eigenvalue m_1. This is a fixed point outside H and in X because X is closed. □

Corollary 8.2.7 *Let G connected and T be a maximal torus.*

(ı) For P a parabolic subgroup we have the inequality $|(G/P)^T| \geq \dim G/P + 1$.

(ıı) We have the equivalence $W(G,T) = \{e\} \Leftrightarrow G$ is solvable.

(ııı) We have the equivalence $|W(G,T)| = 2 \Leftrightarrow \dim \mathfrak{B} = 1$. In this case $\mathfrak{B} = \mathbb{P}^1$.

(ıv) The group G is spanned by the Borel subgroups containing T.

Proof. (ı) By Chevalley's Theorem, the variety G/P is a closed subvariety of some $\mathbb{P}(V)$. The result follows from the previous proposition.

(ıı) We know that if G is solvable then the Weyl group is trivial. Conversely, if the Weyl group is trivial, then by the previous proposition $\dim \mathfrak{B} = 0$ thus $G = B$ and G is solvable.

(ııı) In this case $\dim \mathfrak{B} = 1$. Conversely, if this dimension is equal to 1, then $\mathfrak{B}^T \neq \mathfrak{B}$ (because $\mathfrak{B}^T$ is finite) thus by one of the above lemma, there is an orbit isomorphic to $\mathbb{P}^1$ with only two fixed points.

(ıv) We proceed by induction on $\dim G$. It is true for G of dimension at most two since in that case $G = B$. Let P be the subgroup spanned by the Borel subgroups. It is closed thus this is a parabolic subgroup. If P is proper in G, then $\dim G/P \geq 1$ thus $|(G/P)^T| \geq 2$. Let Q be another fixed point. We have $Q = gPg^{-1}$ and Q contains T. By induction hypothesis, Q is spanned by its Borel subgroups containing T. But we have seen that these Borels are also Borel subgroups of G. Therefore in P. This implies $Q \subset P$ and $Q = P$ a contradiction. □

8.3 Cartan subgroups of a reductive group

We start with a result on unipotent groups.

Theorem 8.3.1 (Kostant-Rosenlicht) *Let G be a unipotent group and let X be an affine variety with a G-action. Then any orbit in X is closed.*

Proof. Let O be such an orbit, it is dense in its closure $Y = \overline{O}$. Let $Z = Y \setminus O$. This is a closed subset of X and we denote by I its ideal in $k[X]$. It is contained in $k[Y]$. Note that because Z is stable under G, the ideal I is also stable under G. By Lie-Kolchin Theorem, there exists $f \in I^G$ a non zero invariant. But O is dense in Y so if f' is in $k[Y]^G$, then f' is constant on O and thus also constant on Y. Thus $I^G \subset k[Y]^G = k$. In particular f is a non-zero constant and $I = k[Y]$ thus $Z = \emptyset$, the result follows. □

Let G be a connected algebraic group and let T be a maximal torus of G. We define the subgroups $I(T)$ and $I_u(T)$ of G by

$$I(T) = \left(\bigcap_{B \in \mathfrak{B}^T} B\right)^0 \text{ and } I_u(T) = \left(\bigcap_{B \in \mathfrak{B}^T} B_u\right)^0.$$

Proposition 8.3.2 *(ı) The group $I(T)$ is solvable and connected.*

(ıı) We have the equalities $I_u(T) = I(T)_u = R_u(I(T))$.

(ııı) The group T is a maximal torus of $I(T)$ and we have $I(T) = TI_u(T)$.

Proof. (ı) This is obvious.

(ıı) The group $I_u(T)$ is unipotent thus contained in $I(T)_u$. Let $g \in I(T)_u$, then $g \in B_u$ for all $B \in \mathfrak{B}^T$ thus because $I(T)_u$ is connected we get the converse inclusion.

The group $I(T)$ is solvable thus there is a unique Borel subgroup, the group $I(T)$ itself and $R_u(I(T)) = I(T)_u$.

(ııı) Clearly T is a maximal torus and because $I(T)$ is solvable, the result follows by (ıı). □

We want to prove the following result.

Theorem 8.3.3 (Chevalley) *We have the equality $I_u(T) = R_u(G)$.*

Note first that $R_u(G)$ is a normal subgroup of $I_u(G)$. Thus there is only one inclusion to prove.

Let us first deduce some important consequences of this results. Quotienting by $R_u(G)$, we get that if G is reductive we have the equalities $I_u(T) = \{e\}$ and $I(T) = T$.

Corollary 8.3.4 *Let G be a reductive group.*

(ı) If T is a maximal torus then $C_G(T) = T$.

(ıı) The center $Z(G)$ is the intersection of maximal tori.

(ııı) If S is a torus, then $C_G(S)$ is reductive and connected.

Proof. (ı) Let B containing T, then B contains $C_G(T)$ thus $C_G(T) \subset I(T)$ and the result follows.

(ıı) The group $Z(G)$ is contained in $C_G(T)$ for all T thus $Z(G)$ is contained in all the maximal tori. Conversely, if g lies in the intersection of all maximal tori, then g commutes with all elements in Cartan subgroups. But these elements form a dense open thus g lies in the center.

(ııı) We already proved the connectedness. The Borel subgroups of $C_G(S)$ are of the form $B \cap C_G(S)$ for B be a Borel subgroup containing S. Let T be a maximal torus containing S. We get

$$R_u(C_G(T)) = \left(\bigcap_{B \in \mathfrak{B}^S} (B \cap C_G(S))_u\right)^0 \subset \left(\bigcap_{B \in \mathfrak{B}^T} B_u\right)^0 = I_u(T)$$

and the result follows. □

To simplify notation we set $J = I_u(T)$.

Definition 8.3.5 *Let $B \in \mathfrak{B}^T$, we define $\mathfrak{B}(B) = \{B' \in \mathfrak{B} \ / \ B \in \overline{T \cdot B'}\}$.*

Theorem 8.3.6 (Luna) *For $B \in \mathfrak{B}^T$, the variety $\mathfrak{B}(B)$ is stable under J, open and affine in $\mathfrak{B}$.*

Proof. Let B be a Borel subgroup and W be a representation with a line $L \subset W$ such that $B = G_L$ and $\mathfrak{b} = \mathrm{Stab}_{\mathfrak{g}}(L)$. Let V be the subspace of W spanned by $G \cdot L$. Then $\mathfrak{B} = G/B \simeq G \cdot L$ is a closed subvariety of $\mathbb{P}(V)$ not contained in any hyperplane.

Let $\phi \in X_*(T)$ be such that $\mathfrak{B}^T = \mathfrak{B}^{\phi(\mathbb{G}_m)}$. An element $B \in \mathfrak{B}^T$ is of the form $[v(B)]$ for some eigenvector $v(B)$ of the action of $\mathbb{G}_m$ on V. Let us denote by $m(B) \in \mathbb{Z}$ its weight *i.e.* its eigenvalue.

We know that the elements $[v(B)]$ for $B \in \mathfrak{B}^T$ are in the same orbit under the group $N_G(T)$ thus there are in the same orbit under the action of G. In particular, for any $B \in \mathfrak{B}$ the orbit $G \cdot [v(B)]$ spans V.

Choose $B_0 \in \mathfrak{B}^T$ such that $m(B_0)$ is minimal. Let $e_0 = v(B_0)$ and choose a basis $(e_0, \cdots, e_n)$ of V composed of eigenvectors. Let m_i be the weight of e_i. We may assume that $m_1 \leq \cdots \leq m_n$. Let $(e_o^*, \cdots, e_n^*)$ be the dual basis.

Lemma 8.3.7 *We have $m_0 < m_1$.*

Proof. Remark that because $\mathfrak{B}$ is not contained in any hyperplane, there must be a vector $v \in V$ with $[v] \in \mathfrak{B}$ such that $e_i^*(v) \neq 0$ for all i (the condition being open if there is one v for each i there is one v for all i simultaneously). Consider the action $\phi(z) \cdot [v]$, by assumption $[v]$ is not stable and because $\mathfrak{B}$ is closed it contains the limit when z goes to 0. This will be an element $B = [v(B)]$ in $\mathfrak{B}^T$.

If $m_1 < m_0$, then the weight of $v(B)$ is strictly smaller than m_0 this is a contradiction to the minimality of m_0.

If $m_1 = m_0$, then let $Z = \{z \in k \ / \ \exists v \in V \text{ with } e_0^*(v) = 1,\ e_1^*(v) = z \text{ and } [v] \in \mathfrak{B}\}$. Let $\mathfrak{B}_0$ be the open subset of $\mathfrak{B}$ of the elements $[v]$ such that $e_0^*(v) \neq 0$. This is non empty otherwise $\mathfrak{B}$ would be contained in an hyperplane. We then can see $\mathfrak{B}_0$ as a subset of V by replacing $[v] \in \mathfrak{B}_0$ with $v/e_0^*(v)$.

The variety Z is the image of the morphism $e_1^* : \mathfrak{B}_0 \to k$. In particular Z is irreducible. If Z is finite, then Z is one point $\{z\}$ and $\mathfrak{B}_0$ would be contained in the hyperplane $e_1^*(v) = z$. Thus $\mathfrak{B}$ would be contained in this hyperplane, a contradiction. Thus Z is infinite. For $z \in Z$, let $[v_z] \in \mathfrak{B}$ be such that $e_0^*(v_z) = 1$ and $e_1^*(v_z) = z$. The closure of the orbit $\mathbb{G}_m \cdot [v(z)]$ is contained in $\mathfrak{B}^{\mathbb{G}_m} = \mathfrak{B}^T$ and contains an element of the form $[e_0 + ze_1 + w_z]$ with w of weight $m_0 = m_1$ not in the span of e_0 and e_1. In particular we get infinitely many elements in $\mathfrak{B}^T$. A contradiction. □

Let us prove the following proposition.

Proposition 8.3.8 *Set $\mathfrak{B}(\phi, B_0) = \{[v(B)] = B \in \mathfrak{B} \ / \ e_0^*(v(B)) \neq 0\}$. This is an affine open subset of $\mathfrak{B}$, stable under T such that $\mathfrak{B}(\phi, B_0) = \mathfrak{B}(B_0)$ and is stable under $I(T)$ (and in particular under $I_u(T)$).*

Proof. This is obviously an affine open subset. For $t \in T$ and $[v(B)] \in \mathfrak{B}(\phi, B_0)$, we have $e_0^*(t \cdot v(B)) = t^{m_0} e_0^*(v(B))$ thus $t \cdot [v(B)]$ is again in $\mathfrak{B}(\phi, B_0)$.

Let $[v(B)] \in \mathfrak{B}(\phi, B_0)$. Then in the closure of the orbit under $\phi(\mathbb{G}_m)$ of this element there is $[e_0]$ because $m_0 < m_i$ for all $i > 0$. Thus $[v(B)] \in \mathfrak{B}(B_0)$. Conversely if $[v(B)] \in \mathfrak{B}(B_0)$, then the closure of the orbit under T of this element contains $[e_0]$. This implies that $e_0^*(v(B)) \neq 0$ because it is already the case at the limit. Thus $[v(B)] \in \mathfrak{B}(\phi, B_0)$.

Let $e_0^\perp$ be the hyperplane in $V^\vee$ of linear form vanishing on e_0. The group G acts on $V^\vee$ and thus on $\mathbb{P}(V^\vee)$.

Lemma 8.3.9 *(ı) Any orbit of G in $\mathbb{P}(V^\vee)$ meets the open subset $\mathbb{P}(V^\vee) \setminus \mathbb{P}(e_0^\perp)$.*

(ıı) The orbit $G \cdot [e_0^*]$ is closed in $\mathbb{P}(V^\vee)$.

Proof. (i) Let $f \in V^\vee$ a non trivial linear form. If $G \cdot f \subset e_0^\perp$, then $0 = g \cdot f(e_0) = f(g^{-1}e_0)$ thus f would vanish on $G \cdot [e_0] = \mathfrak{B}$ which spans V thus f would be trivial. This proves (i).

(ii) Let us first compute the action of $\phi(\mathbb{G}_m)$ on e_i^*. We have $z \cdot e_i(v) = e_i(\phi(z)^{-1} \cdot v) = z^{-m_i} e_i(v)$ thus the weight is $-m_i$. Thus e_0^* has maximal weight. In particular, for $f \in \mathbb{P}(V^\vee) \setminus \mathbb{P}(e_0^\perp)$, the closure of $G \cdot f$ contains the point $[e_0^*]$. By (i), any orbit closure contains the point $[e_0^*]$. Therefore the orbit of $[e_0^*]$ is contained in all orbit closure and thus is a (and even the unique) minimal orbit thus closed. □

Let us prove that $\mathfrak{B}(\phi, B_0)$ is stable under $I(T)$. Let P be the stabiliser of e_0^*. This is a parabolic subgroup since the orbit is closed and thus proper. As e_0^* is a weight vector for T, the class $[e_0^*]$ is stable under T and $T \subset P$. Thus there exists a Borel subgroup B containing T and contained in P. In particular $I(T)$ is contained in B and thus in P. Thus $[e_0^*]$ is fixed by $I(T)$ and therefore $\mathfrak{B}(\phi, B_0)$ is stable under $I(T)$.

Note that because $I_u(T)$ is unipotent it even fixes the vector e_0^*. □

Let us finish the proof of the theorem. Let $B \in \mathfrak{B}^T$. Then there exists $n \in N_G(T)$ such that $B = n \cdot B_0$. Then $\mathfrak{B}(B) = \{B' \in \mathfrak{B} \ / \ n \cdot B_0 \in \overline{T \cdot B'}\} = \{B' \in \mathfrak{B} \ / \ B_0 \in \overline{n^{-1}T \cdot B'}\} = \{B' \in \mathfrak{B} \ / \ B_0 \in \overline{T \cdot n^{-1} \cdot B'}\} = n \cdot \{B'' \in \mathfrak{B} \ / \ B_0 \in \overline{T \cdot B''}\} = n \cdot \mathfrak{B}(\phi, B_0)$. The later being affine, the proof is complete. □

Proposition 8.3.10 (Luna) *The group $I_u(T)$ acts trivially on $\mathfrak{B}$.*

Proof. The group T being solvable and connected, the only closed orbits of T in $\mathfrak{B}$ are fixed points. Indeed if X is a non empty closed orbit. Then X is irreducible and proper. Thus contains a fixed point and thus is reduced to the fixed point.

We claim that the varieties $\mathfrak{B}(B)$ for $B \in \mathfrak{B}^T$ cover $\mathfrak{B}$. Indeed, if $B' \in \mathfrak{B}$, then there exists a closed T-orbit *i.e.* a T-fixed point B in the closure of its T-orbit thus $B \in \overline{T \cdot B'}$ and $B \in \mathfrak{B}^T$ thus $B' \in \mathfrak{B}(B)$.

Let $B' \in \mathfrak{B}$. Then because $I_u(T)$ is solvable and connected, there is a $I_u(T)$-fixed point B'' in the closure of the orbit $I_u(T) \cdot B'$. This point contained in some $\mathfrak{B}(B)$ for $B \in \mathfrak{B}^T$. The subset $Z(B) = \mathfrak{B} \setminus \mathfrak{B}(B)$ is closed and $I_u(T)$-stable. If this closed subset meets $I_u(T) \cdot B'$ then it has to contain $\overline{I_u(T) \cdot B'}$ and thus to contain B''. A contradiction. Thus $I_u(T) \cdot B'$ is contained in $\mathfrak{B}(B)$. By Kostant-Rosenlicht Theorem and because $I_u(T)$ is unipotent this orbit is closed in $\mathfrak{B}(B)$. But B'' lies in the closure of the orbit and in $\mathfrak{B}(B)$ thus B'' lies in the orbit. As it was a fixed point, the orbit is trivial. □

Corollary 8.3.11 (Chevalley's Theorem) *We have the equality $I_u(T) = R_u(G)$.*

Proof. The group $I_u(T)$ acts trivially on $\mathfrak{B}$ thus it is contained in all Borel subgroup B and even in B_u since it is unipotent. As it is connected the result follows. □

Chapter 9

Structure of reductive groups

9.1 First definitions and results

9.1.1 Examples

Let us start to give some example of reductive and semisimple groups.

Lemma 9.1.1 *Let V be a faithful representation of G. Assume that V is a simple representation, then G is reductive.*

Proof. Let $R_u = R_u(G)$. Because R_u is normal the subspace V^{R_u} is a subrepresentation. It is non empty because R_u is unipotent therefore it is equal to V. In particular R_u acts trivially on V and because the representation is faithful R_u is trivial. □

Corollary 9.1.2 *The groups $\mathrm{GL}(V)$, $\mathrm{SL}(V)$, $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$ (with $\dim V$ even in this last case) are reductive groups.*

Proof. In all cases, the standard representation V is faithful and simple. □

9.1.2 Root datum

Recall first the definition of a root system.

Definition 9.1.3 *A root system is a pair $(R, V(R))$ of a vector space $V(R)$ and a finite set R such that the following conditions are satisfied.*

- *$0 \notin R$ and R spans $V(R)$.*
- *For any $\alpha \in R$, there exists a linear form $\alpha^\vee \in V(R)^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = \alpha^\vee(\alpha) = 2$ and $s_\alpha \in \mathrm{End}_k(V(R))$ defined by $s_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha$ preserves R.*
- *For any $\alpha, \beta \in R$ we have $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.*

Definition 9.1.4 *A root datum is a quadruple $(M, M^\vee, R, R^\vee)$ satisfying the following conditions:*

($\imath$) The sets M and $M^\vee$ are free $\mathbb{Z}$-modules of finite rank with $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote by $\langle m, f \rangle = f(m)$ for $m \in M$ and $f \in M^\vee$ the pairing.

(ii) The sets R and $R^\vee$ are finite subsets of M and $M^\vee$ respectively with a bijection $\vee : R \to R^\vee$ denoted by $\alpha \mapsto \alpha^\vee$ such that the following conditions hold: $\langle \alpha, \alpha^\vee \rangle = 2$ and

$$s_\alpha(R) = R, \quad s_{\alpha^\vee}(R^\vee) = R^\vee$$

where $\sigma_\alpha(m) = m - \langle m, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee}(f) = f - \langle \alpha, f \rangle \alpha^\vee$.

The root datum is called reduced if for $\alpha, \beta \in R$, the condition $\mathbb{Z}\alpha = \mathbb{Z}\beta \Rightarrow \beta = \pm\alpha$ is satisfied.

Remark 9.1.5 (i) Over algebraically closed fields, only reduced root data are useful.

(ii) A root is always a non zero element.

(iii) We shall denote by V and $V^\vee$ the tensor products $V = M \otimes_\mathbb{Z} \mathbb{R}$ and $V^\vee = M^\vee \otimes_\mathbb{Z} \mathbb{R}$. Then R is a root system in the subspace $V(R)$ it spans in V and $R^\vee$ is the dual root system in $V^\vee(R^\vee) \simeq (V(R))^\vee$.

Definition 9.1.6 *Let V be a vector space with $R \subset V$ a root system, let $R^\vee$ be the dual root system in $V^\vee$.*

(i) We denote by $Q(R)$ the subgroup of V spanned by R. This is a lattice and we call it the root lattice. We define in the same way the coroot lattice $Q(R^\vee)$ in $V^\vee$.

(ii) We denote by $P(R)$ the subgroup of V defined by

$$P(R) = \{v \in V \ / \ \langle v, \alpha^\vee \rangle \in \mathbb{Z}, \quad \textit{for all } \alpha^\vee \in R^\vee\}.$$

This is called the weight lattice. We define in the same way $P(R^\vee)$ which is the coweight lattice.

Definition 9.1.7 *The Weyl group of a root datum is the subgroup of* $\mathrm{GL}(M \otimes_\mathbb{Z} \mathbb{R})$ *generated by the reflections s_α for $\alpha \in R$.*

Definition 9.1.8 *A root datum is called semisimple if $Q(R)$ has the same rank as M (which is also the dimension of V).*

Fact 9.1.9 *The root datum is semisimple if and only if the dual root datum $(M^\vee, M, R^\vee, R)$ is semisimple* i.e. *if and only if* $\mathrm{rk} Q(R^\vee) = \mathrm{rk}(M^\vee)$

Proof. If the root datum is semisimple, then $V(R) = V$ and by properties of dual root systems, we have $V^\vee = V^\vee(R^\vee)$ which is therefore semisimple. □

Proposition 9.1.10 *Let $(M, M^\vee, R, R^\vee)$ be a semisimple root datum. Then this datum equivalent to the root system (R, V) together with the finite Z-submodule $M/Q(R)$ of $P(R)/Q(R)$.*

Proof. If we have a semisimple root datum then we have the inclusions $Q(R) \subset M \subset P(R)$ and these Z-module have the same rank equal to $\dim V$. In particular $P(R)/Q(R)$ is finite.

Conversely, if we have the root system R, then $Q(R)$ and $P(R)$ are well defined. Therefore if we have the finite submodule $M/Q(R)$ we recover M by taking the inverse image of this module by the surjective map $P(R) \to P(R)/Q(R)$. □

9.2 Centraliser of semisimple elements

Theorem 9.2.1 *Let G' be an affine algebraic group and assume that G is a closed connected subgroup of G'. Let $\mathfrak{g} = L(G)$ and let S be an abelian (not nec. closed) subgroup of G' with $S \subset G'_s \cap N_{G'}(G)$. Let*

$$G^S = \{g \in G \ / \ sgs^{-1} = g \textit{ for all } s \in S\}$$

$$\mathfrak{g}^S = \{\eta \in \mathfrak{g} \ / \ (\mathrm{Ad}\, s)(\eta) = \eta \textit{ for all } s \in S\}.$$

Then we have $L(G^S) = \mathfrak{g}^S$.

Let us first state a corollary of this result.

Corollary 9.2.2 *Let G be a connected algebraic group, let $\mathfrak{g} = L(G)$ and let S be a closed diagonalisable subgroup of G. Then we have the equality*

$$\mathfrak{g}^S = L(C_G(S)).$$

Corollary 9.2.3 *Let T be a maximal torus of a reductive group G act on $\mathfrak{g} = L(G)$ via the adjoint representation. Let $\mathfrak{h} = L(T)$ then we have the equality $\mathfrak{h} = \mathfrak{g}^T$.*

Proof. We start with the following proposition which deals with the case of a unique element.

Proposition 9.2.4 *Let G and G' as above and let $\mathfrak{g}$ and $\mathfrak{g}'$ be their Lie algebras. Let s be a semisimple element in G' normalising G. Then we have the equality*

$$\mathfrak{g}^s = L(G^s).$$

Proof. Embedding G' in some $\mathrm{GL}(V)$, we may assume that $G' = \mathrm{GL}(V)$. We may also assume that s is a diagonal element $s = \mathrm{diag}(x_1, \cdots, x_1, \cdots, x_r, \cdots, x_r)$ where x_i appears n_i times. They we easily compute the following equalities

$$(G')^s = \prod_{i=1}^r \mathrm{GL}_{n_i} \ \text{ and } L((G')^s) = \prod_{i=1}^r \mathfrak{gl}_{n_i} = (\mathfrak{g}')^s.$$

Furthermore, $\mathrm{Ad}\, s$ is a semisimple element in $\mathfrak{gl}(\mathfrak{g}')$ thus we have a decomposition $\mathfrak{g}' = (\mathfrak{g}')^s \oplus \mathfrak{r}$ where $\mathfrak{r}$ is the direct sum of the eigenspaces of $\mathrm{ad}\, s$ with eigenvalue different from 1. Let $\mathrm{Ad}_{\mathfrak{g}}\, s$ be the restriction of $\mathrm{Ad}\, s$ to $\mathfrak{g}$.

Consider now the morphism $\phi : G' \to G'$ defined by $\phi(x) = xsx^{-1}s^{-1}$. On the one hand, its image X is a locally closed subset of G' and it the translate by s^{-1} of the conjugacy class of s. Therefore $\dim X = \dim(G'/(G')^s) = \dim G' - \dim(G')^s$. On the other hand, ϕ is the following composition

$$G' \xrightarrow{\mathrm{Id}\times\theta} G' \xrightarrow{\mu} G'$$

where $\theta(x) = sx^{-1}s^{-1}$. We have $\theta = (\mathrm{Ad}\, s) \circ i$ where $i(x) = x^{-1}$ therefore we get

$$d_e\phi = \mathrm{Id} + d_e\theta = \mathrm{Id} - \mathrm{Ad}\, s.$$

In particular, we get that $\ker d_e\phi = (\mathfrak{g}')^s = L((G')^s)$. Therefore the image of $d_e\phi$ is of dimension $\dim G' - \dim(G')^s = \dim X$ and thus $d_e\phi$ is surjective onto T_eX. In particular we get $T_eX = \mathfrak{r}$.

Let $\psi = \phi|_G$, then we have $d_e\psi = \mathrm{Id}_{\mathfrak{g}} - \mathrm{Ad}_{\mathfrak{g}}\, s$. But ψ is constant on G^s thus $L(G^s) \subset \ker d_e\psi = \mathfrak{g}^s$. We are left to prove the inequality $\dim \mathfrak{g}^s \leq \dim G^s$.

We proceed as before: let Y be the image of ψ. It is a locally closed subvariety in G of dimension $\dim Y = \dim G - \dim G^s$. Furthermore, since s normalises G, we have $Y \subset G \cap X$ thus $T_eY \subset \mathfrak{g} \cap T_eX = \mathfrak{g} \cap \mathfrak{x}$. Thus we have the inequality $\dim Y \leq \dim \mathfrak{g} \cap \mathfrak{x}$.

On the other hand, we have the decomposition in $(\mathrm{Ad}\, s)$-eigenspaces $\mathfrak{g}' = (\mathfrak{g}')^s \oplus \mathfrak{x}$ which induces a decomposition $\mathfrak{g} = \mathfrak{g}^s \oplus (\mathfrak{g} \cap \mathfrak{x})$ thus $\dim \mathfrak{g}^s = \dim \mathfrak{g} - \dim(\mathfrak{g} \cap \mathfrak{x}) \leq \dim \mathfrak{g} - \dim Y = \dim G^s$. The result follows. □

We prove the theorem by induction on $\dim G$. If $G = G^S$, then $\mathrm{Ad}\, s = \mathrm{Id}_G$ and we are done. Otherwise, let $s \in S$ with G^s proper in G. By the previous proposition we get that $L((G^s)^0) = \mathfrak{g}^s$. The group $(G^s)^0$ is normalised by S (because S is abelian) thus by induction we have $\mathfrak{g}^S = (\mathfrak{g}^s)^S = L((G^s)^0)^S = L(((G^s)^0)^S) \subset L(G^S)$.

On the other hand, $\mathrm{Int}(s)$ acts trivially on G^S thus its differential $\mathrm{Ad}\, s$ acts trivially on $L(G^S)$ and we get the converse inclusion $L(G^S) \subset \mathfrak{g}^S$. □

9.3 Structure theorem for reductive groups

Let T be a maximal torus of G reductive and let it act on $\mathfrak{g}$ via the adjoint representation. Then we have a decomposition in eigenspaces as follows:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $\mathfrak{h} = L(T)$, where R is a subset of $X(T) \setminus \{0\}$ and where

$$\mathfrak{g}_\alpha = \{\eta \in \mathfrak{g} \ / \ t \cdot \eta = \alpha(t)\eta \ \text{ for all } t \in T\}.$$

We want to prove the following result.

Theorem 9.3.1 *Let G be a reductive groupe and let T be a maximal torus. Let $W(G,T)$ be the Weyl group of G and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h}$ be the Lie algebra of T and R the set of non trivial characters of T appearing in $\mathfrak{g}$. Then we have $\mathfrak{g}^T = \mathfrak{h}$ and a decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

such that the following properties hold.

(ı) $(X^*(T), X_*(T), R, R^\vee)$ is a root datum and the group $W(G,T)$ is isomorphic to the Weyl group of this root datum.

(ıı) For any $\alpha \in R$, we have $\dim \mathfrak{g}_\alpha = 1$ and there exists a unique closed connected unipotent subgroup U_α, normalised by T such that $L(U_\alpha) = \mathfrak{g}_\alpha$.

(ııı) The group G is spanned by T and the U_α for $\alpha \in R$.

(ıv) The Borel subgroups containing T are in one-to-one correspondence with W, in one-to-one correspondence with the Weyl chambers, in one-to-one correspondence with the basis of R.

This proof will need several steps. Let us start with a definition.

Definition 9.3.2 *Let G be an algebraic group.*

(ı) The rank $\mathrm{rk}(G)$ *of G is the dimension of a maximal torus T.*

(ıı) The reductive rank $\mathrm{rk}_r(G)$ *of G is the rank of $G/R_u(G)$.*

(ııı) The semisimple rank $\mathrm{rk}_{ss}(G)$ *of G is the rank of $G/R(G)$.*

Fact 9.3.3 *Let G with $\mathrm{rk}(G) = 0$, then G is unipotent.*

Proof. Indeed, let B be a Borel subgroup. The $B = TB_u = B_u$ thus B is unipotent therefore nilpotent thus $G = B$ and the result. $\square$

Proposition 9.3.4 *Let G be a reductive group.*
($\imath$) Then $R(G) = Z(G)^0$ is a torus and $\mathrm{rk}_{ss}(G) = \mathrm{rk}(G) - \dim Z(G)^0$.
($\imath\imath$) The group $Z(G) \cap D(G)$ is finite.
($\imath\imath\imath$) The group $D(G)$ is semisimple and $\mathrm{rk}(D(G)) \leq \mathrm{rk}_{ss}(G)$.

Proof. ($\imath$) We already know that $Z(G)$ is contained in the intersection of all maximal tori thus in the intersection of all Borel subgroup. This gives the inclusion $Z(G)^0 \subset R(G)$.

Conversely, because G is reductive, we that $R_u(G) = R(G)_u$ is trivial therefore $R(G)$ is a torus. Let T be a maximal torus containing $R(G)$, then because $R(G)$ is normal we have $R(G) \subset gTg^{-1}$ for any g thus $R(G)$ is contained in the intersection of maximal tori and thus in $Z(G)$.

($\imath\imath$) It is enough to prove that $Z(G)^0 \cap D(G)$ is finite since then we have $(Z(G) \cap D(G))^0 \subset Z(G)^0 \cap D(G)$ is finite and the result follows.

But $S = Z(G)^0$ is a torus. Let us choose a faithful representation $G \to \mathrm{GL}(V)$, then there exists characters $(\chi_i)_i$ of S such that $V = \oplus_i V_{\chi_i}$. As G commutes with S, we get the inclusions

$$G \subset \prod_i \mathrm{GL}(V_{\chi_i}) \text{ and } D(G) \subset \prod_i \mathrm{SL}(V_{\chi_i}).$$

The result follows from this presentation.

($\imath\imath\imath$) Let $R = R(D(G))$. This is a characteristic subgroup of $D(G)$ and since $D(G)$ is normal in G it is normal in G. Furthermore, it is closed connected and solvable thus $R(D(G)) \subset R(G) = Z(G)^0$. In particular $R(D(G)) \subset Z(G) \cap D(G)$ and is finite, being connected, it is trivial and $D(G)$ is semisimple. This also proves that the restriction map $D(G) \to G/R(G)$ is finite onto its image proving the rank inequality. $\square$

9.4 Semisimple groups of rank one

9.4.1 Rank one and PGL_2

We start with a simple result on $\mathbb{P}^1$.

Proposition 9.4.1 *The automorphism group of $\mathbb{P}^1$ is $\mathrm{PGL}_2 = \mathrm{GL}_2/Z(\mathrm{GL}_2)$. Furthermore we have*

$$Z(\mathrm{GL}_2) = \{\lambda \mathrm{Id} \ / \ \lambda \in k^\times\} \simeq \mathbb{G}_m.$$

Proof. The decription of the center is well known and the fact that the quotient acts on $\mathbb{P}^1$ is obvious. Furthermore it obviously acts faithfully therefore PGL_2 is a subgroup of the automorphism group. Let V be the open set in $(\mathbb{P}^1)^3$ defined as follows:

$$V = \{(x, y, z) \in (\mathbb{P}^1)^3 \ / \ x, y \text{ and } z \text{ are distinct } \}.$$

The group PGL_2 acts on V and consider the morphism $\phi : \mathrm{PGL}_2 \to V$ given by the orbit of $(0, 1, \infty)$. We claim that ϕ is an isomorphism. The map is obviously injective since $(0, 1, \infty)$ form a projective basis of $\mathbb{P}^1$ and it is well known that this map is also surjective (Exercise!). Thus ϕ is bijective and we have to check that $d_e\phi$ is surjective to prove that it is an isomorphism. We have three natural subgroups in PGL_2 which are the images of the following subgroups in GL_2:

$$U_{-\alpha} = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \ / \ a \in k \right\}, \ \ U_\alpha = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \ / \ a \in k \right\} \ \text{ and } \ T = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \ / \ a \in k^\times \right\}.$$

We only need to compute the differential of the action of these three subgroups. The action of the first one on $(0, 1, \infty) = ([1:0], [1:1], [0:1])$ is $a \cdot (0, 1, \infty) = (a, a+1, \infty)$ thus the image of its differential is $(1, 1, 0)$ in $k^3 = T_{(0,1,\infty)}(\mathbb{P}^1)^3$. Similarly we have that the image of the differential for the second one is $(0, 1, 1)$. For the last one $a \cdot (0, 1, \infty) = (a, a^{-1}, \infty)$ thus the image is $(0, 1, 0)$ and the differential is surjective.

Now let G be an algerbaic group acting on $\mathbb{P}^1$. We only need to prove that the acting factors through a morphism $G \to \mathrm{PGL}_2$. Define a morphism $\psi : G \to V$ by $g \mapsto g \cdot (0, 1, \infty)$ and compose it with ϕ^{-1}. We get a morphism $\Phi : G \to \mathrm{PGL}_2$. This is a group morphism and we need to check that G acts on $\mathbb{P}^1$ via Φ. But we have $\Phi(g)^{-1}g \cdot (0, 1, \infty) = \cdot(0, 1, \infty)$ thus it is enough to prove that an automortphism of $\mathbb{P}^1$ fixing $(0, 1, \infty)$ is trivial. But in $k(\mathbb{P}^1) = k(x)$, the function x is the unique function with a zero of order 1 in 0, a pole of order 1 in ∞ and no other pole. Furthermore we have $x(1) = 1$. The function $\Phi(g)^{-1}g \cdot x$ must have the same property thus $\Phi(g)^{-1}g \cdot x = x$. Thus $\Phi(g)^{-1}g$ is the identity on a dense open set of $\mathbb{P}^1$ and thus is the identity. □

Theorem 9.4.2 *Let G be a connected reductive group of rank 1. Assume that G is not solvable and let T be a maximal torus, $W(G, T)$ the Weyl group $\mathfrak{g} = L(G)$ and $\mathfrak{h} = L(T)$.*

(ı) Then we have $|W(G, T)| = 2$, $\mathfrak{B} \simeq \mathbb{P}^1$, $\dim G = 3$, $D(G) = G$ and G is semisimple.

(ıı) There exists $\alpha \in X(T)$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$. There exists a closed connected unipotent subgroup U_α (resp. $U_{-\alpha}$) whose Lie algebra is $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$). The groups $B = TU_\alpha$ and $B^- = TU_{-\alpha}$ are the Borel subgroups containing T. Their Lie algebras are $\mathfrak{h} \oplus \mathfrak{g}_\alpha$ and $\mathfrak{h} \oplus \mathfrak{g}_{-\alpha}$.

(ııı) Let $n \in N_G(T) \setminus T$, then the orbit morphism $\psi_n : U_\alpha \to G/B$ defined by $\psi(u) = un \cdot B$ is an isomorphism onto its image.

(ıv) We have a surjective morphism $G \to \mathrm{PGL}_2$ with finite kernel.

Proof. (ı) The group G is not solvable thus $W = W(G, T)$ is not trivial. But the rank of G being one, the maximal torus is $\mathbb{G}_m$. Furthermore we have a group morphism $W \to \mathrm{Aut}(T)$ defined by $n \mapsto (t \mapsto ntn^{-1})$. This map is injective since $C_G(T) = T$. But the group $\mathrm{Aut}(T)$ is $\{\pm 1\}$ (the only group morphisms are the character thus given by $t \mapsto t^m$ and these morphisms are isomorphisms only for $m = \pm 1$). Therefore $|W| = 2$ and we deduce that $\mathfrak{B} \simeq \mathbb{P}^1$.

We have an action of G on $\mathfrak{B} \simeq \mathbb{P}^1$ thus we get a group morphism $\phi : G \to \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$. The kernel of the morphism is

$$\ker \phi = \bigcap_{B \in \mathfrak{B}} B$$

thus $\ker(\phi)^0 = R(G)$. Because G is reductive, we have that $R(G)$ is a torus and thus either $R(G) = \{e\}$ or $R(G) = T$ (because T is of dimension 1). The last case is not possible: otherwise T would act trivially on $\mathfrak{B}$ (or otherwise $T = R(G) = Z(G)^0$ would be central and thus B would be nilpotent and $G = B$ also). Therefore $R(G) = \{e\}$ and G is semisimple. Furthermore $\ker \phi$ is finite thus G has dimension at most 3. Since a group of dimension at most 2 is solvable we get that $\dim G = 3$. This proves also (ıv).

Now $D(G)$ is not trivial (otherwise G is abelian thus solvable) it is not of dimension less than 2 otherwise $D(G)$ and $G/D(G)$ would be of dimension less than 2 and solvable thus $\dim D(G) = 3$ and G and $D(G)$ being connected we get $G = D(G)$.

(ıı) We know that there is an eigenspace decomposition of $\mathfrak{g}$ under the adjoint action of T and that $\mathfrak{g}^T = \mathfrak{h}$ is of dimension 1. Let B be a Borel subgroup of G containing T. Then $B = TB_u$. Let T normalises B_u and thus T acts on $L(B_u)$. The group B_u is of dimension 1 (since $\dim B = \dim G - \dim \mathfrak{B} = 2$ and $\dim B = \dim T + \dim B_u$) and its Lie algebra must be an eigenspace for T. Futhermore $L(B) = \mathfrak{h} \oplus L(B_u)$ thus this eigenspace is associated to a non trivial character α of the torus. Write $\mathfrak{g}_\alpha = L(B_u)$.

Let $n \in N_G(T)\backslash T$, then $B^- = n \cdot B$ lies in $\mathfrak{B}^T$ and is different from B. The same argument as above shows that $L(B_u^-)$ is an eigenspace for T. Let us compute its eigenvalue. We have $B_u^- = nB_un^{-1}$ thus if we take $t \in T$ and $b_u \in B_u$ we have $\mathrm{Int}(t)(nb_un^{-1}) = tnb_un^{-1}t^{-1} = n(n^{-1}tnb_un^{-1}t^{-1}n)n^{-1}$. But n acts on T as its only non trivial automorphisms thus $ntn^{-1} = t^{-1}$ thus $\mathrm{Int}(t)(nb_un^{-1}) = nt^{-1}b_utn^{-1} = n(\mathrm{Int}(t^{-1})(b_u))n^{-1}$. We thus have the following commutative diagram

$$\begin{array}{ccc} B_u^- & \xrightarrow{\mathrm{Int}(t)} & B_u^- \\ {\scriptstyle \mathrm{Int}(n)}\downarrow & & \downarrow{\scriptstyle \mathrm{Int}(n)} \\ B_u & \xrightarrow{\mathrm{Int}(t^{-1})} & B_u. \end{array}$$

Taking the differentials we get the diagram

$$\begin{array}{ccc} L(B_u^-) & \xrightarrow{\mathrm{Ad}\ (t)} & L(B_u^-) \\ {\scriptstyle \mathrm{Ad}\ (n)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}\ (n)} \\ L(B_u) & \xrightarrow{\mathrm{Ad}\ (t^{-1})} & L(B_u). \end{array}$$

Thus for $\eta \in L(B_u^-)$, we have $\mathrm{Ad}(t)\cdot\eta = \mathrm{Ad}(n)(\mathrm{Ad}(t^{-1})(\mathrm{Ad}(n)^{-1}(\eta))) = \alpha(t^{-1})\mathrm{Ad}(n)(\mathrm{Ad}(n)^{-1}(\eta)) = -\alpha(t)\eta$. Thus $L(B_u^-)$ is associated to the eigenvalue $-\alpha$. This proves (ıı).

(ııı) Consider the quotient map $\pi : G \to G/n \cdot B = G/nBn^{-1}$. Its restriction is the morphism $\pi_U : U \to Un \cdot B$. The quotient map being separable, we have $\ker d_e\pi = L(n \cdot B) = L(B^-) = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha}$. Therefore we get that $\ker d_e\pi_U = L(U) \cap (\mathfrak{h} \oplus \mathfrak{g}_{-\alpha}) = \mathfrak{g}_\alpha \cap (\mathfrak{h} \oplus \mathfrak{g}_{-\alpha}) = 0$. In particular this map is separable since $\mathfrak{B}$ is of dimension 1 and the image of $d_e\pi_U$ also. But π_U is injective since it is a U-equivariant map between U-homogeneous spaces and an element in the fiber $\pi_U^{-1}(n \cdot B)$ lies in U and in $n \cdot B = B^-$ thus in $B \cap B^- = T$. Since U is unipotent $U \cap T = \{e\}$ and the injectivity follows. This proves the isomorphism. □

Remark 9.4.3 (ı) More precisely one can prove that we have $G \simeq \mathrm{SL}_2$ or $G \simeq \mathrm{PGL}_2$.

(ıı) One easily checks, using the action of PGL_2 that the orbit $Un \cdot B \in \mathbb{P}^1$ is isomorphic to $\mathbb{A}^1_k$. In particular $U \simeq \mathbb{A}^1_k$ and it is easy with this to prove that $U \simeq \mathbb{G}_a$. This also follows from the (unproved) structure Theorem for one-dimensional algebraic groups.

9.4.2 Groups of semisimple rank one

Proposition 9.4.4 *Let G be a reductive group of semisimple rank one. Let T be a maximal torus, $\mathfrak{g} = L(G)$, $\mathfrak{h} = L(T)$ and $W = W(G,T)$ the Weyl group. Let $G' = G/R(G) = G/Z(G)^0$ and let T' be the image of T in G'.*

(ı) The torus T' is of dimension one and we have the inclusion $\mathbb{Z} = X^*(T') \subset X(T)$.

(ıı) There exists $\alpha \in X^*(T)$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ with $\dim \mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha} = 1$.

(ııı) The group $D(G)$ is semisimple of rank one and $G = D(G)Z(G)^0$.

(iv) There exists a unique close connected subgroup U_α (resp. $U_{-\alpha}$) normalised by T whose Lie algebra is $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$). It is unipotent. The groups $B_\alpha = TU_\alpha$ and $B_{-\alpha} = TU_{-\alpha}$ are the Borel subgroups containing T. Their Lie algebras are $\mathfrak{b}_\alpha = \mathfrak{h} \oplus \mathfrak{g}_\alpha$ and $\mathfrak{b}_{-\alpha} = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha}$.

(v) Let T_1 the unique maximal torus of $D(G)$ contained in T. There exists a unique $\alpha^\vee \in X_*(T_1) \subset X_*(T)$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and if s_α is the unique non trivial element in W then we have

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha \quad \textit{and} \quad s_\alpha(\phi) = \phi - \langle \alpha, \phi \rangle \alpha^\vee,$$

for all $\chi \in X^*(T)$ and $\phi \in X_*(T)$.

Proof. (i) We know that the image of a maximal torus by a surjective morphism of algebraic groups is again a maximal torus and because G was of semisimple rank one this T' is of dimension 1. We get obviously the inclusion.

(ii) We have the decomposition

$$\mathfrak{g} = \mathfrak{h} \bigoplus_\alpha \mathfrak{g}_\alpha.$$

But the group $R(G) = Z(G)^0$ is central thus acts on $\mathfrak{g}$ with trivial eigenvalue thus $L(R(G)) \subset \mathfrak{h}$ and we get that $\mathfrak{g}' = L(G') = L(G)/L(R(G))$ and in particular

$$\mathfrak{g}' = (\mathfrak{h}/L(R(G))) \bigoplus_\alpha \mathfrak{g}_\alpha.$$

In particular because G' is of rank one there are only two non trivial eigenvalues: α and $-\alpha$ with eigenspaces of dimension 1. This proves (ii).

(iii) The morphism $G \to G'$ is surjective thus the map $D(G) \to D(G')$ is surjective. But G' is semisimple thus $G' = D(G')$ and the map $D(G) \to G'$ is surjective. Furthermore the intersection $D(G) \cap R(G)$ is finite thus G' is of dimension 3 and not solvable otherwise G' would be trivial. Its rank is smaller than the semisimple rank of G *i.e.* its rank is one.

The group generated by $D(G)$ and $R(G)$ is closed and equal to $D(G)R(G)$ since the latter is central. Since the intersection of these groups is finite its dimension is $\dim D(G) + \dim R(G) = \dim G' + \dim R(G) = \dim G$. Since G is connected $G = D(G)R(G)$.

(iv) We know that U_α and $U_{-\alpha}$ do exist in $D(G)$ and these groups satisfy the conditions. Their image U'_α and $U'_{-\alpha}$ in G' also satisfy this condition.

Furthermore, since $G \to G'$ is a surjective morphism whose kernel is contained in any Borel subgroup, then there is a bijection between Borel subgroups containing T and Borel subgroups containing T'. In particular there are only two Borel subgroup B_α and $B_{-\alpha}$ containing T. Their image in G' are the Borels of G' containing T' which are $T'U'_\alpha$ and $T'U'_{-\alpha}$ thus $B_\alpha = TU_\alpha$ and $B_{-\alpha} = TU_{-\alpha}$. Their Lie algebras are $\mathfrak{b}_\alpha = \mathfrak{h} \oplus \mathfrak{g}_\alpha$ and $\mathfrak{b}_{-\alpha} = \mathfrak{h} \oplus \mathfrak{g}_{-\alpha}$.

We now prove the uniqueness. If H was such a subgroup, then since $L(H) = \mathfrak{g}_\alpha$, its dimension is 1. It it is semisimple, then by rigidity of tori, since it is normalised by T it is centralised by T. This implies that the weight of T on $L(H)$ is trivial, a contradiction. Thus H is unipotent normalised by T thus HT is a solvable connected thus contained in a Borel subgroup containing T thus in B_α thus $H = U_\alpha$.

(v) Let T_1 be a maximal torus in $D(G)$. It is contained in some maximal torus of G and by conjugation $gT_1g^{-1} \subset T$. Since $D(G)$ is normal, gT_1g^{-1} is again a maximal torus in $D(G)$ thus there are maximal tori T_1 of $D(G)$ contained in T. Let T_1 be such a maximal torus, then T_1 is equal to $(D(G) \cap T)^0$ (it is obviously contained in it and equal by maximality). Thus T_1 is unique.

Let ϕ be a generator of $X_*(T_1) \simeq Z$. Let α_{T_1} be the restriction of α on T_1. This restriction is non trivial since $\mathfrak{g}_\alpha$ is also contained in the Lie algebra of $D(G)$. In particular the integer $\langle \alpha_{T_1}, \phi \rangle$ is non zero. We may then define $\alpha^\vee = 2\phi/\langle \alpha, \phi \rangle$. We have $\langle \alpha, \alpha^\vee \rangle = 2$.

Let $s = s_\alpha$ be the non trivial element in W the Weyl group, let $n \in N_G(T)$ be a representative. Let $\psi \in X_*(T)$ and look at $s(\psi) - \psi : \mathbb{G}_m \to T$ defined by

$$t \mapsto n\psi(t)n^{-1}\psi(t)^{-1}.$$

This takes values in $T \cap D(G)$ thus in T_1 (by connectedness). Thus there exists a morphism $f \in \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z})$ such that $s(\psi) - \psi = -f(\psi)\phi$. Note that α is trivial on $R(G) = Z(G)^0$ because this group is central thus acts trivially by the adjoint action. Therefore the value of α is completely determined by the value of the induced character α' on T'. Let us compute $\langle \alpha, s(\psi)\rangle$. By the previous argument and if we denote the composed map

$$\mathbb{G}_m \xrightarrow{\psi} T \longrightarrow T'$$

by ϕ' we have $\langle \alpha, s(\psi)\rangle = \langle \alpha', s(\psi')\rangle$ but s is induced by the unique non trivial automorphism of T' (which is of dimension 1) thus $s(\psi') = -\psi'$ and

$$\langle \alpha, s(\psi)\rangle = \langle \alpha', s(\psi')\rangle = \langle \alpha', -\psi'\rangle = -\langle \alpha, \psi\rangle.$$

We get $\langle \alpha, s(\psi) - \psi\rangle = -2\langle \alpha, \psi\rangle$ and thus $f(\psi) = 2\langle \alpha, \psi\rangle / \langle \alpha, \phi\rangle$. This gives

$$s(\psi) = \psi - 2\frac{\langle \alpha, \psi\rangle}{\langle \alpha, \phi\rangle}\alpha = \psi - \langle \alpha, \psi\rangle\alpha^\vee.$$

Let $\chi \in X^*(T)$ and $\psi \in X_*(T)$. Note that we hav by definition $\langle s(\chi), \psi\rangle = \langle \chi, s(\psi)\rangle$. In particular we get

$$\langle s(\chi), \psi\rangle = \langle \chi, s(\psi)\rangle = \langle \chi, \psi - \langle \alpha, \psi\rangle\alpha^\vee\rangle = \langle \chi - \langle \chi, \alpha^\vee\rangle\alpha, \psi\rangle.$$

The result follows because the pairing is perfect. □

9.5 Structure Theorem

9.5.1 Root datum of a reductive group

Let G be a connected reductive group, let T be a maximal torus and let $L(G) = \mathfrak{g}$, $L(T) = \mathfrak{h}$ be their Lie algebras. There is a decomposition of the Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where R is a subset of $X^*(T)$.

Definition 9.5.1 *For any $\alpha \in R$, we define the subtorus S_α of T by $S_\alpha = (\ker \alpha)^0$. We define the reductive subgroup Z_α of G by $Z_\alpha = C_G(S_\alpha)$.*

Lemma 9.5.2 *(ı) Let S be a codimension 1 subtorus in T and let $\pi : X^*(T) \to X^*(S)$ the induced application. Then there exists $\chi \in X(T)$ such that $\ker \pi = \mathbb{Z}\chi$.*

(ıı) Let $\alpha \in X^*(T)$ and $n \in \mathbb{Z} \setminus \{0\}$. Then $(\ker \alpha)^0 = (\ker \alpha^n)^0$.

(ııı) Let $\alpha, \beta \in X^*(T) \setminus \{0\}$, then $(\ker \alpha)^0 = (\ker \beta)^0$ if and only if there exists non zero integers m and n such that $m\alpha = n\beta$.

Proof. (i) Let $T' = T/S$, this is a one dimensional torus thus $X^*(T') = \mathbb{Z}$ and we may choose χ a generator of this group. This character induces a character still denoted by χ on T which is trivial on S thus in the kernel. If χ' is a character in the kernel, then it is trivial on S and induces a character on T' which is a multiple of χ. The result follows.

(ii) We have the obvious inclusion $(\ker\alpha)^0 \subset (\ker\alpha^n)^0$. The restriction of α to $(\ker\alpha^n)^0$ is torsion but this group being connected there is no torsion in its character group thus α is trivial and the result follows.

(iii) If $m\alpha = n\beta$ then the result follows from (ii). Conversely, we know that if $S = (\ker\alpha)^0 = (\ker\beta)^0$ and with notation as in (i), we have that $\alpha, \beta \in \mathbb{Z}\chi$ and the result follows. □

Proposition 9.5.3 *(i) For any $\alpha \in R$, the group Z_α is connected reductive and of semisimple rank one. Furthermore we have*

$$L(Z_\alpha) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}, \quad \dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1 \quad \textit{and} \quad \mathbb{Q}\alpha \cap R = \{\alpha, -\alpha\}.$$

(ii) Let s_α be the only non trivial element in $W(Z_\alpha, T) \subset W(G,T)$, then there exists a unique $\alpha^\vee \in X_*(T)$ such that $\langle \alpha, \alpha^\vee\rangle = 2$ and we have

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee\rangle\alpha \quad \textit{and} \quad s_\alpha(\phi) = \phi - \langle \alpha, \phi\rangle\alpha^\vee,$$

for all $\chi \in X^*(T)$ and $\phi \in X_*(T)$.

(iii) Set $R^\vee = \{\alpha^\vee \in X_*(T) \ / \ \alpha \in R\}$, then $(X^*(T), X_*(T), R, R^\vee)$ is a reduced root datum.

(iv) Let $W(R)$ the Weyl group of R and $W'(G,T)$ the subgroup of $W(G,T)$ spanned by the s_α for $\alpha \in R$, then $W'(G,T) = W(R)$.

Proof. (i) We already know that Z_α is connected and reductive. Furthermore S_α is in the center of Z_α thus $S_\alpha \subset R(Z_\alpha)$ and we get that $T/R(Z_\alpha)$ which is a maximal torus of $Z_\alpha/R(Z_\alpha)$ is of dimension at most 1.

If it was of dimension 0, then we would have the equality $R(Z_\alpha) = T$ (indeed Z_α being reductive the group $R(Z_\alpha)$ is a torus). In particular T would be central in Z_α thus $Z_\alpha \subset C_G(T) = T$ and thus $Z_\alpha = T$. But in that case we would have $L(Z_\alpha) = \mathfrak{h}$ and also since S_α is a torus $L(Z_\alpha) = L(G^{S_\alpha}) = \mathfrak{g}^{S_\alpha}$. But S_α acts trivially on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ a contradiction. This also give the equality

$$L(Z_\alpha) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

and because it is of semisimple rank one $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha} = 1$. Furthermore, for $\beta \in R \cap \mathbb{Q}\alpha$, we have $S_\alpha = S_\beta$ by the previous Lemma thus $\mathfrak{g}_\beta \subset L(Z_\alpha)$ and $\beta = \pm\alpha$.

(ii) There is an inclusion of Weyl group because of the obvious inclusion $N_{Z_\alpha}(T) \subset N_G(T)$ and the equality $C_{Z_\alpha}(T) = T = C_G(T)$. Note that if $\alpha^\vee$ is such a cocharacter, then if $\phi \in X_*(T)$ satisfies $\langle \alpha, \phi\rangle \neq 0$, then $\alpha^\vee = (\phi - s_\alpha(\phi))/\langle \alpha, \phi\rangle$. The existence comes from the previous Proposition, we even know that $\alpha^\vee$ lies in $X_*(T_1)$ where T_1 is the unique maximal torus of $D(Z_\alpha)$ contained in T.

(iii) Let $n \in N_G(T)$. Then n acts on T by conjugation and thus acts on $X^*(T)$. Furthermore n acts on $\mathfrak{g}$ by $\mathrm{Ad}\,(n)$. Furthermore for $t \in T$, we have $\mathrm{Ad}\,(n) \circ \mathrm{Ad}\,(t) = \mathrm{Ad}\,(ntn^{-1}) \circ \mathrm{Ad}\,(n)$ thus $\mathrm{Ad}\,(n)(\mathfrak{g}_\alpha) = \mathfrak{g}_{n\cdot\alpha}$. Therefore n maps R to R and the Weyl group respects R.

We also have $nZ_\alpha n^{-1} = nC_G(S_\alpha)n^{-1} = C_G(nS_\alpha n^{-1}) = C_G(S_{n\cdot\alpha})$. And if w is the corresponding element in the Weyl group, we have $ws_\alpha w^{-1} = s_{w(\alpha)}$. This gives for $\chi \in X^*(T)$:

$$\chi - \langle w^{-1}(\chi), \alpha^\vee\rangle w(\alpha) = ws_\alpha w^{-1}(\chi) = s_{w(\alpha)}(\chi) = \chi - \langle w(\alpha)^\vee, \chi\rangle w(\alpha)$$

and thus $\langle \chi, w(\alpha^\vee)\rangle = \langle w^{-1}(\chi), \alpha^\vee\rangle = \langle \chi, w(\alpha)^\vee\rangle$ proving the equality $w(\alpha^\vee) = w(\alpha)^\vee$ and $R^\vee$ is also stable. Thus we have a root datum and we have already seen that it is reduced.

(ıv) The group $W(G,T)$ acts on $X^*(T)$ thus we have a natural surjective map $W'(G,T)$ to $W(R)$ (by definition the Weyl group $W(R)$ is spanned by the reflections with respect to the roots). We only have to prove that the representation of $W(G,T)$ on $X^*(T)$ is faithful but this is clear (write $T = (\mathbb{G}_m)^r$). ☐

9.5.2 Weyl group

Our next goal is to prove the equality $W'(G,T) = W(G,T)$.

Definition 9.5.4 *(ı) Let $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $V^\vee = X_*(T) \otimes_Z \mathbb{R}$ its dual. For $\alpha \in R$ we define the hyperplane $H_\alpha \subset V^\vee$ by*

$$H_\alpha = \{f \in V^* \ / \ \langle \alpha, f \rangle = 0\}.$$

(ıı) The Weyl chambers in $V^\vee$ are the connected components of

$$V^\vee \setminus \bigcup_{\alpha \in R} H_\alpha.$$

Theorem 9.5.5 *The theory of root systems tells us that $W(R)$ acts simply transitively on the Weyl chambers.*

Definition 9.5.6 *A cocharacter $\phi \in X_*(T)$ is called regular if $\langle \alpha, \phi \rangle \neq 0$ for all $\alpha \in R$.*

Fact 9.5.7 *The regular cocharacters are those which are contained in a Weyl chamber.*

Proposition 9.5.8 *Let ϕ be a regular cocharacter, then $\mathfrak{B}^T = \mathfrak{B}^{\phi(\mathbb{G}_m)}$.*

Proof. Let X be a connected component of $\mathfrak{B}^{\phi(\mathbb{G}_m)}$. Then X is T-stable (since T commutes with the image of the cocharacter) and projective thus it contains a T-fixed point B. We have an isomorphism $G/B \simeq \mathfrak{B}$ mapping e to B. In particular, the tangent space of $\mathfrak{B}$ at B identifies with the quotient $\mathfrak{g}/\mathfrak{b}$. But T is contained in B thus $\mathfrak{h} \subset \mathfrak{b}$ and $T_B\mathfrak{B} = \oplus_{\alpha \in R(B)} \mathfrak{g}_\alpha$ for some subset $R(B)$ of R. Note that the weights of $\phi(\mathbb{G}_m)$ on this tangent space are the integers $\langle \alpha, \phi \rangle$ for $\alpha \in R(B)$ and by assumption these intergers are non zero.

If X was positive dimensional, then T_BX would be positive dimensional and $\phi(\mathbb{G}_m)$ would have a trivial weight. A contradiction to the previous statement. Thus $\mathfrak{B}^{\phi(\mathbb{G}_m)}$ is finite and all its points are T-stable thus in $\mathfrak{B}^T$. ☐

Lemma 9.5.9 *Let $B \in \mathfrak{B}^T$ and $\alpha \in R$. Then B contains exactly one of the two groups U_α and $U_{-\alpha}$.*

Proof. Let $S_\alpha = (\ker \alpha)^0$ and let $Z_\alpha = C_G(S_\alpha)$. Then Z_α contains exactly two Borel subgroups containing T: the groups TU_a and $TU_{-\alpha}$. But for B containing T, the intersection $B \cap Z_\alpha$ is a Borel subgroup of Z_α containing T. The result follows. ☐

Let $B \in \mathfrak{B}^T$ and consider a representation W of G with a line L such that $G_L = B$ and $\mathrm{Stab}_{\mathfrak{g}}(L) = \mathfrak{b}$. Then G/B is a closed subvariety of $\mathbb{P}(W)$ and we take V tp be the span of $G \cdot L$ in W. The variety G/B is again a closed subvariety of $\mathbb{P}(V)$.

Let ϕ be a regular cocharacter and set $\psi = -\phi$ which is also regular. Recall that we proved that if $B(\phi) \in \mathfrak{B}^T$ is the Borel subgroup such that the weight m_0 of $\psi(\mathbb{G}_m)$ on $v_\phi \in V$ such that $[v_\psi] = B(\psi)$ is the smallest possible, then completing $v_\phi = e_0$ in (e_i) an eigenbasis for $\psi(\mathbb{G}_m)$ with m_i the weight of e_i we may choose an order such that we have

$$m_0 < m_1 \leq \cdots \leq m_n.$$

The weights of ϕ are opposite to the weights of ψ thus v_ϕ is of highest weight for $\phi(\mathbb{G}_m)$.

Lemma 9.5.10 *We have the equality* $\mathfrak{b}(\phi) = L(B(\phi)) = \mathfrak{h} \oplus \bigoplus_{\langle\alpha,\phi\rangle>0} \mathfrak{g}_\alpha$.

Proof. Recall that $T_{B(\phi)}\mathfrak{B} = \mathfrak{g}/\mathfrak{b}(\phi)$. This space is contained in $T_{[v_\phi]}\mathbb{P}(V) = V/kv_\phi$. Furthermore, the vector space kv_ϕ being $\phi(\mathbb{G}_m)$-invariant the quotient is again $\phi(\mathbb{G}_m)$-invariant and the weights are the $-m_i - (-m_0) = m_0 - m_i$ and therefore negative. On the other hand $\mathfrak{g}/\mathfrak{b}(\phi)$ is the image of the $\mathfrak{g}_\alpha$ for $\mathfrak{g}_\alpha$ not in $\mathfrak{b}(\phi)$ thus if the weight of $\mathfrak{g}_\alpha$ is positive *i.e.* if $\langle\phi,\alpha\rangle > 0$ then $\mathfrak{g}_\alpha$ has to be contained in $\mathfrak{b}(\phi)$. As B contains U_α or $U_{-\alpha}$ the result follows. □

Lemma 9.5.11 *Let H be a closed connected subgroup of G and (H_i) finitely many closed connected subgroups of H. Assume that $L(H) = \sum_i L(H_i)$, then H is spanned by the K_i.*

Proof. Let K be the subgroup of G spanned by the H_i. It is a closed subgroup of G contained in H. Futhermore, $\sum_i L(K_i) \subset L(K) \subset L(H)$ thus we have equality and $\dim H = \dim K$. Thus $K = H$ because H is connected. □

Corollary 9.5.12 *(ı) The group G is spanned by the groups T and U_α for $\alpha \in R$.*

(ıı) The group $B(\phi)$ only depends on the Weyl chamber $C(\phi)$ containing ϕ i.e. *for any $\phi' \in C(\phi)$ we have $B(\phi) = B(\phi')$.*

Proof. (ı) This is a direct application of the decomposition of the Lie algebra $\mathfrak{g}$ and the previous lemma.

(ıı) Obviously $\mathfrak{b}(\phi)$ does only depend on $C(\phi)$ therefore $\mathfrak{b}(\phi') = \mathfrak{b}(\phi)$. Futhermore $B(\phi)$ and $B(\phi')$ contain T and the U_α for $\langle\alpha,\phi\rangle > 0$ thus they are equal to the group spanned by these subgroups and the result follows. □

Definition 9.5.13 *(ı) Let $\mathfrak{C}$ we the set of Weyl chambers. Define the map $\mathfrak{C} \to \mathfrak{B}^T$ by $C \mapsto B(\phi)$ for $\phi \in C$. We shall also write $B(C) = B(\phi)$.*

(ıı) For $B \in \mathfrak{B}^T$, denote by $R^+(B)$ the following set of roots:

$$R^+(B) = \{\alpha \in R \ / \ U_\alpha \subset B\} = \{\alpha \in R \ / \ \mathfrak{g}_\alpha \subset \mathfrak{b}\}.$$

(ııı) For $B \in \mathfrak{B}^T$ define the subset $C(B)$ of $V^\vee$ defined by

$$C(B) = \{f \in V^\vee \ / \ \langle\alpha, f\rangle > 0 \ \alpha \in R^+(B)\}.$$

Remark 9.5.14 Note that $C(B(C)) = C$ and that in general $C(B)$ is empty or a chamber.

Theorem 9.5.15 *(ı) The map $C \mapsto B(C)$ is bijective. Its inverse is $B \mapsto C(B)$.*

(ıı) The Weyl group $W(G,T)$ is generated by the s_α for $\alpha \in R$ and is therefore isomorphic to $W(R)$.

Proof. Let $B \in \mathfrak{B}^T$, let $C \in \mathfrak{C}$ and $\phi \in C$. There exists $n \in N_G(T)$ such that $B = nB(\phi)n^{-1}$. Let $w = \bar{n}$ the class of n in the Weyl group $W(G,T)$. We have

$$\mathfrak{b} = L(B) = L(nB(\phi)n^{-1} = \mathfrak{h} \oplus \bigoplus_{\langle\alpha,\phi\rangle>0} \mathfrak{g}_{w(\alpha)} = \mathfrak{h} \oplus \bigoplus_{\langle\beta,w^{-1}(\phi)\rangle>0} \mathfrak{g}_\beta = \mathfrak{b}_{w^{-1}(\phi)}.$$

Thus $B = B(w^{-1}(\phi))$ and the map $C \mapsto B(C)$ is surjective. Furthermore, we have that $C(B)$ is the chamber $w^{-1}(C)$ proving that the map is injective. Because $\mathfrak{B}^T$ is in bijection with $W(G,T)$ which contains $W(R)$ which is in bijection with the Weyl chambers and is contained in $W(G,T)$, we get the equality $W(G,T) = W(R)$ and the result follows. □

Corollary 9.5.16 *The structure Theorem for reductive groups holds.*

Proof. We only need to prove that for any $\alpha \in R$, there exists a unique closed connected subgroup H in G with $L(H) = \mathfrak{g}_\alpha$. Let $S_\alpha = \ker \alpha$ and $Z_\alpha = C_G(S_\alpha)$. We have $L(H) = \mathfrak{g}_\alpha = \mathfrak{g}_\alpha^{S_\alpha} = L(H)^{S_\alpha} = L(C_H(S_\alpha) = H \cap Z_\alpha$. But H is of dimension 1 as well as $H \cap Z_\alpha$ thus $H \subset Z_\alpha$ and the result follows from the unicity in Z_α. □

9.5.3 Subgroups normalised by T

Definition 9.5.17 *Note that for any root α the group U_α is unipotent of dimension 1 thus is isomorphic to $\mathbb{G}_a$. We fix an isomorphism $u_\alpha : \mathbb{G}_a \to U_\alpha$.*

Lemma 9.5.18 *The group T acts on U_α via the formula*

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$$

for $t \in T$ and $x \in \mathbb{G}_m$.

Proof. Consider the action of T on $\mathbb{G}_m$ defined by $t \cdot x = u_\alpha^{-1}(tu_\alpha(x)t^{-1})$. This action is linear since conjugation respects the group structure. There is therefore a character $\chi \in X^*(T)$ such that $t \cdot x = \chi(t)x$ thus $tu_\alpha(x)t^{-1} = u_\alpha(\chi(t)x)$. Deriving this action, we get an action on the Lie algebra by the character χ. But T acts on the Lie algebra $\mathfrak{g}_\alpha$ via α and thus $\chi = \alpha$. □

Proposition 9.5.19 *Let H be a closed connected subgroup normalised by T.*

(ı) Then for any $\alpha \in R$ we have an equivalence between

(a) $U_\alpha \subset H$ and

(b) $\mathfrak{g}_\alpha \subset L(H)$.

(ıı) Let E be the set of roots satisfying the above conditions, then we have the equality

$$L(H) = L(T \cap H) \oplus \bigoplus_{\alpha \in E} \mathfrak{g}_\alpha.$$

Furthermore the group H is spanned by $T \cap H$ and the U_α for $\alpha \in E$.

Proof. (ı) The implication $(a) \Rightarrow (b)$ is obvious. Conversely, if $\mathfrak{g}_\alpha \subset L(H)$, then $\mathfrak{g}_\alpha \subset L(H)^{S_\alpha}$ with $S_\alpha = \ker \alpha$ thus $\mathfrak{g}_\alpha \subset L(C_H(S_\alpha)) = L(H \cap Z_\alpha)^0$ where $Z_\alpha = C_G(S_\alpha)$. Thus $\mathfrak{g}_a$ is contained in $L(K)$ where K is the subgroup of Z_α spanned by T and $(H \cap Z_\alpha)^0$. The group K is closed and connected and because T normalises $(H \cap Z_\alpha)^0$ we have

$$K = T(H \cap Z_\alpha)^0.$$

Assume that $U_\alpha \subset K$ and let us prove that in that case $U_\alpha \subset H$. Indeed, let $x \in \mathbb{G}_a$, then there exists $t \in T$ and $h \in (H \cap Z_\alpha)^0$ with

$$u_\alpha(x) = th.$$

Let $z \in \mathbb{G}_m$ and conjugate the previous relation by $\alpha^\vee(z)$. We get for $h' = \alpha^\vee(z)h\alpha^\vee(z)^{-1} \in (H \cap Z_\alpha)^0$ the equality

$$th' = \alpha^\vee(z)th\alpha^\vee(z)^{-1} = u_\alpha(\alpha(\alpha^\vee(z))x) = u_\alpha(z^2x).$$

We deduce the equality $h^{-1}h' = u_\alpha((z^2-1)x)$ which is thus in $(H \cap Z_\alpha)^0$. This is true for all x and z thus $U_\alpha \subset (H \cap Z_\alpha)^0 \subset H$.

Let us prove that $U_\alpha \subset K$. But $\dim Z_\alpha = \dim T + 2$ and K is connnected containing T and $(H \cap Z_\alpha)^0$. Thus its Lie algebra contains $\mathfrak{h} \oplus \mathfrak{g}_\alpha$ and K is of dimension $\dim T + 1$ of $\dim T + 2$. In the former case we are done since $H = Z_\alpha \supset U_\alpha$. If $\dim K = \dim T + 1$, let B be a Borel subgroup of K containing T. If $B = T$, then B is nilpotent and $K = B = T$ a contradiction. Thus $B = K$ and is a closed connected solvable subgroup of Z_α. Because Z_α is not solvable and B is of codimension 1, B is a Borel subgroup of Z_α and because its Lie algebra contains $\mathfrak{g}_\alpha$ it is B_α. The result follows.

($\imath\imath$) Because $L(H)$ is a T-module, it has to be the direct sum of some $\mathfrak{g}_\alpha$ and $L(H)^T = L(C_H(T)) = L(H \cap C_G(T)) = L(H \cap T)$. The direct sum decomposition follows as well the last statement. $\square$

Let T be a maximal torus of G reductive and let B be a Borel subgroup containing T. This B defines a Weyl chamber C (called dominant) and a set of positive roots $R^+ = \{\alpha \in R \ / \ \langle \alpha, \phi \rangle > 0\}$ for some (any) $\phi \in C$. Let $U = R_u(B) = B_u$ and $\mathfrak{n} = L(U)$. Let us denote by $\beta_1, \cdots, \beta_N\}$ the elements in R^+.

Theorem 9.5.20 (Structure Theorem for B and U) *($\imath$) The T-equivariant morpihsm*

$$\Phi : \prod_{i=1}^N U_{\beta_i} \to U$$

defined by multiplication is an isomorphism of varieties.

($\imath\imath$) Let U' be a closed subgroup of U normalised by T and let $\alpha_1, \cdots, \alpha_r$ be the weights of T in $L(U') = \mathfrak{n}'$. Then U' is connected, we have the inclusion $U_{\alpha_i} \subset U'$ and the morphism

$$\Phi' : \prod_{i=1}^r U_{\alpha_i} \to U'$$

defined by multiplication is an isomorphism of varieties.

($\imath\imath\imath$) The morphism $T \times U \to B$ defined by multiplication is an isomrphism therefore any element $b \in B$ can be writen uniquely as

$$b = t u_{\beta_1}(x_1) \cdots u_{\beta_N}(x_N) = u_{\beta_1}(\beta_1(t)x_1) \cdots u_{\beta_N}(\beta_N(t)x_N)$$

for some $t \in T$ and $x_i \in \mathbb{G}_a$.

Proof. ($\imath$) Let $V = \prod_{i=1}^N U_{\beta_i}$ which is isomorphic to $\mathbb{G}_a^N$ thus to k^n and therefore to T_eU. Futhermore we have a T-action on these two varieties and the T-actions coincide. Note that the map Φ induces an isomorphism $d_e\Phi$. Let $\phi \in C$ and recall that the weights of ϕ on T_eV and T_eU are positive. The result will be a consequence of the following general result.

Proposition 9.5.21 *Let X be an affine connected variety with a $\mathbb{G}_m$-action.*

($\imath$) Assume that there exists a fixed point $x \in X^{\mathbb{G}_m}$, then T_xX is a $\mathbb{G}_m$-representation.

($\imath\imath$) Assume furthermore that the weights of $\mathbb{G}_m$ on T_xX are non zero and of the same sign. Then $A = k[X]$ is a graded ring

$$A = k \oplus \bigoplus_{n>0} A_n$$

and X can be identified with a closed T-stable subvariety of T_xX. The point x is the unique T-fixed point in X.

($\imath\imath\imath$) Assume in addition that X is smooth in x, then X is isomorphic to T_xX as $\mathbb{G}_m$-variety.

Proof. (i) Let $A = k[X]$ and $\mathfrak{M}$ the maximal ideal corresponding to x. Then A is a $\mathbb{G}_m$-module and $\mathfrak{M}$ is stable un der this action (because x is T-fixed). The same is true for $\mathfrak{M}^2$ and thus for the quotient $\mathfrak{M}/\mathfrak{M}^2$ and the dual $(\mathfrak{M}/\mathfrak{M}^2)^\vee = T_xX$.

(ii) Replacing the action by the opposite action given by $t \bullet v = t^{-1} \cdot v$ we may assume that the weight of $\mathbb{G}_m$ on T_xX are negative. Then the weights of $\mathbb{G}_m$ on the irrelevant ideal B^+ of $B = k[T_xX] = S(\mathfrak{M}/\mathfrak{M}^2)$ are positive. Note that the graded ring

$$\mathrm{gr}_{\mathfrak{M}}(A) = \bigoplus_{n\geq 1} \mathfrak{M}^n/\mathfrak{M}^{n+1}$$

is a quotient of $B = S(\mathfrak{M}/\mathfrak{M}^2)$ thus the weights of

$$\bigoplus_{n\geq 1} \mathfrak{M}^n/\mathfrak{M}^{n+1}$$

are positive. Let us prove that the same is true for $\mathfrak{M}$.

If X is irreducible, the A is a domain and Krull intersection's Theorem implies the equality

$$\bigcap_{n\geq 1} \mathfrak{M}^n = 0.$$

Let $f \in \mathfrak{M}$ be a weight vector of weight $i \in \mathbb{Z}$. There exists n such that $f \in \mathfrak{M}^n \setminus \mathfrak{M}^{n+1}$ and the image of ϕ in $\mathfrak{M}^n/\mathfrak{M}^{n+1}$ is non zero and still of weight i which therefore has to be positive.

We finish the proof of the Theorem in this case. We have $A = k \oplus \mathfrak{M} = k \oplus \bigoplus_{n\geq 0} A_n$ with $A_n = \{f \in A \ / \ t \cdot f = t^n f\}$. This proves the grading result. Let E be a $\mathbb{G}_m$-stable complement of $\mathfrak{M}^2$ in $\mathfrak{M}$. We have the following result.

Lemma 9.5.22 (Graded Nakayama's Lemma) *Let $A = \bigoplus_{n\geq 0} A_n$ be a graded commutative k-algebra with $A_0 = k$. Let $\mathfrak{M} = \bigoplus_{n>0} A_n$ be the irrelevant ideal and E a graded complement of $\mathfrak{M}^2$ in $\mathfrak{M}$. Then E spans A as an algebra and $\mathfrak{M}$ as an ideal.*

Proof. Let A_E be the subalgebra spanned by E. We prove by induction the inclusion $A_n \subset A_E$ (this is true for $n = 0$). Let $a \in A_n$, then there exists $b \in E$ such that $a - b \in \mathfrak{M}^2$ and we may assume that $a \in \mathfrak{M}^2$. Let us write $a = \sum_i a_i b_i$ with $a_i, b_i \in \mathfrak{M}$. Considering the graded decomposition of a_i and b_i are kepping in a only the degree n part we get an equality $a = \sum_i \alpha_i\beta_i$ with $\deg \alpha_i + \deg \beta_i = n$ and the degrees of α_i and β_i are positive. This implies that the degrees of α_i and β_i are strictly less than n and proves the claim. □

Applying this result we get that the morphism $S(E) \to A$ is surjective. But $E \simeq \mathfrak{M}/\mathfrak{M}^2$ and $S(E) \simeq k[T_xX]$ thus we have a surjective graded morphism

$$k[T_xX] \to A$$

and X is a $\mathbb{G}_m$-stable closed subvariety in T_xX. furthermore x is mapped to $0 \in T_xX$. The unique fixed point in T_xX is 0 thus x is the unique fixed point in X. If X is smmoth then $\dim X = \dim T_xX$ thus the closed embedding $X \subset T_xX$ is an isomorphism.

Note that because the weights of $\mathbb{G}_m$ on T_xX are negative, then the limit of $t \cdot z$ when t goes to 0 is 0 thus any T-stable non empty subvariety of T_xX (and thus of X) contains x.

Assume now that X is only connected. Note that $\mathbb{G}_m$ being connected, any irreducible component of X is $\mathbb{G}_m$-stable. Let X_1 be such a component containing x. Then by the previous argument X is closed $\mathbb{G}_m$-stable subvariety in T_xX_1 and x is its only fixed point. If X is not irreducible, then exists X_2 another irreducible component meeting X_1. Then $X_1 \cap X_2$ is non empty and T-stable thus contains x. Thus X_2 is also a closed subvariety of T_xX with the same properties as X_1. Going on with this process, any irreducible component of X contains x.

Let $X_1, \cdots, X_n$ the irreducible components of X and P_i the corresponding minimal prime ideals of A. Then we have $\cap_i P_i = 0$ (because X is reduced) and $P_i \subset \mathfrak{M}$ for all i. Applying Krull's intersection Theorem in the domain A/P_i we get that the intersection $\cap_n \mathfrak{M}^n$ is contained in each P_i and thus vanishes. We conclude as in the irreducible case. $\square$

($\imath\imath$) Assume first that U' is connected. Then because U' is normalised by T, we know that $\mathfrak{g}_\alpha \subset L(U') \Leftrightarrow U_\alpha \subset U'$ and the same proof as for ($\imath$) gives the result.

Let us prove that U' is connected and let V be the connected component containing the identity. Then by ($\imath$) and ($\imath\imath$) we have that if $E = \{\alpha \in R^+ \ / \ U_\alpha \not\in V\}$ for $W = \prod_{\alpha \in E} U_\alpha$ we have that the multiplication map

$$V \times W \to U$$

is an isomorphism. Restriction to U' gives an isomorphism $V \times (U' \cap W) \simeq U'$ and quotienting by V gives $U' \cap W \simeq U'/V$ which is finite. But this group is also T-stable since U' and W are. Therefore because T is connected, all the points of $U' \cap W$ are centralised by T thus in $C_G(T) = T$ thus in $U \cap T = \{e\}$ and $U' \cap W = \{e\}$ and U' is connected.

($\imath\imath\imath$) Follows from what we already proved. $\square$

9.5.4 Bialynicki-Birula decomposition and Bruhat decomposition

Using Proposition 9.5.3 we prove a special case of Bialynicki-Birula decomposition. Let us first prove the following easy Lemma.

Lemma 9.5.23 *Let T be a torus and V be a linear representation. Then $\mathbb{P}(V)$ is covered by affine T-invariant open subsets.*

Proof. We already used this implicitely. Let (e_i) be an basis of eigenvectors for the action of T. Let $(e_i^\vee)$ the dual basis which is again composed of eigenvectors for the dual action. Then $\mathbb{P}(V)$ is covered by the affine subsets $D(e_i^\vee) = \{[v] \in \mathbb{P}(V) \ / \ e_i^\vee(v) \neq 0\}$. $\square$

Theorem 9.5.24 (Białynicki-Birula decomposition) *Let V be a $\mathbb{G}_m$-representation of finite dimension and let X be a closed irreducible subvariety in $\mathbb{P}(V)$ stable under $\mathbb{G}_m$. Let $X^{\mathbb{G}_m}$ the vqriety of fixed points, then for each element x in $X^{\mathbb{G}_m}$ we define the set*

$$X(x) = \{y \in X \ / \ \lim_{t \to 0} t \cdot y = x\}.$$

($\imath$) Then all the varieties $X(x)$ are locally closed subvarieties of X isomorphic to an affine space of dimension $n(x)$ and we have the cellular decomposition

$$X = \coprod_{x \in X^{\mathbb{G}_m}} X(x).$$

($\imath\imath$) If moreover $X^{\mathbb{G}_m}$ is finite then there exists an unique point $x \in X^{\mathbb{G}_m}$ such that $X(x)$ is open (and dense) in X. This point is called the attractive point and denoted by x_-. There also exists a unique point x_+ such that $X(x^+) = \{x^+\}$.

Example 9.5.25 Take $\mathbb{G}_m$ acting on k^{n+1} with weights $0, 1, 2, \cdots, n$. Then the Białynicki-Birula decomposition is given by $\{[x_0 : \cdots : x_n] \ / \ x_1 = \cdots = x_{i-1}, \ x_i = 1\} \simeq k^{n-i}$.

Proof. (i) Let $x \in X^{\mathbb{G}_m}$. By the previous Lemma there exists an open affine neigbourhood U of x in $\mathbb{P}(V)$ and thus in X by intersection with X which is T-stable. Let $A = k[U]$ and $\mathfrak{M} = \mathfrak{M}_x, U$ the maximal ideal corresponding to x. There is a decomposition

$$T_xX = T_x^+X \oplus T_x^{\leq 0}X,$$

where T_x^+X and $T_x^{\leq 0}X$ are respectively the direct sums of weight spaces with positive and non positive weights. Let E be a $\mathbb{G}_m$-stable supplementary of $\mathfrak{M}^2$ in $\mathfrak{M}$. Let E^- and $E^{\geq 0}$ be respectively the direct sums of weight spaces with negative and non negative weights. We have an isomorphism of $\mathbb{G}_m$-representations:

$$E \simeq \mathfrak{M}/\mathfrak{M}^2 = (T_xX)^\vee \simeq (T_x^+X)^\vee \oplus (T_x^{\leq 0}X)^\vee$$

mapping E^- to $(T_x^+X)^\vee$ and $E^{\geq 0}$ to $(T_x^{\leq 0}X)^\vee$. The inclusion $E \subset \mathfrak{M}$ induces a $\mathbb{G}_m$-equivariant morphism

$$\phi^\sharp : k[T_xX] = S(E) \to A = k[U]$$

corresponding to a $\mathbb{G}_m$-equivariant morphism $\phi : U \to T_xX$. Let us set $Y = \phi^{-1}(T_x^+X)$ which is a closed subset in U defined by the ideal J spanned by $\phi^\sharp(I)$ in A with I the defining ideal of T_x^+X in T_xX *i.e.* the ideal spanned by the linear forms vanishing on T_x^+X. The space of these forms is $E^{\geq 0}$ thus $J = Rad(AE^{\geq 0})$.

Lemma 9.5.26 *The morphism $\phi|_Y : Y \to T_x^+X$ is an isomorphism $\mathbb{G}_m$-equivariant.*

Proof. We have the commutative diagrams

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\phi|_Y"'] & X \arrow[d, "\phi"] \\ T_x^+X \arrow[r] & T_xX \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A/J & A \arrow[l] \\ k[T_x^+X] \arrow[u, "\phi|_Y^\sharp"] & k[T_xX] \arrow[l] \arrow[u, "\phi^\sharp"'] \end{tikzcd}$$

We are therefore left to prove that $\phi|_Y^\sharp$ is an isomorphism. Let $r = \dim E^{\geq 0} = \dim T_xX - \dim T_x^+X$. On the one hand, since $AE^{\geq 0}$ is spanned by r elements we have the inequality $\dim Y \geq \dim X - r = \dim T_x^+X$ because X is smooth. On the other hand the cotangent space to Y at x is $(\mathfrak{M}/J)/(\mathfrak{M}^2 + J/J) \simeq \mathfrak{M}/\mathfrak{M}^2 + J$ which is a quotient of $\mathfrak{M}/\mathfrak{M}^2 + E^{\geq 0} \simeq E^+$ and thus of dimension at most $\dim X - r$. Therefore Y is of dimension $\dim X - r$ and we have the equality $\mathfrak{M}^2 + J = \mathfrak{M}^2 + E^{\geq 0}$ *i.e.* $T_xY \simeq T_x^+X$ and Y is smooth at x. By Proposition 9.5.3 we get that $\phi|_Y$ is an isomorphism. □

We want to prove that $Y = X(x)$. For this we now prove the Lemma.

Lemma 9.5.27 *Let $y \in U$, the following propositions are equivalent.*
(i) $\lim_{t\to 0} t \cdot y = x$;
(ii) $y \in Y = \phi^{-1}(T_x^+X)$.

Proof. (ii)⇒(i) If $y \in Y$, then $\phi(y) \in T_x^+(Y)$ thus $\lim_{t\to 0} t \cdot \phi(y) = 0$ and since ϕ is equivariant with $\phi(x) = 0$ we get the result (recall that $\phi|_Y$ is an isomorphism).

(i)⇒(ii) Assume that $\lim_{t\to 0} t \cdot y = x$, then applying ϕ, we get $\lim_{t\to 0} t \cdot \phi(y) = \phi(x) = 0$. Let us write $\phi(y) = \sum_i v_i$ with v_i of weight i with respect to $\mathbb{G}_m$. The limit has to be equal to 0 therefore all v_i with $i < 0$ vanish and the result follows. □

From the former Lemma we get that $Y \subset X(x)$. Conversely, for $y \in X(x)$, the orbit $\mathbb{G}_m \cdot y$ contains x in its closure thus has to meet U. But U being $\mathbb{G}_m$-invariant, the orbit is contained in U, thus $y \in U$. Again by the previous Lemma we get that $y \in Y$. This proves the isomorphisms

$X(x) = Y \simeq T_x^+X$ thus $X(x)$ is locally closed (since Y is closed in U) and isomorphic to the affine space T_x^+X. Furthermore, since any orbit $\mathbb{G}_m \cdot y$ for $y \in X$ contains a fixed point in its closure (recall the description of the orbits of $\mathbb{G}_m$ on the projective space) we get the partition

$$X = \coprod_{x \in X^{\mathbb{G}_m}} X(x).$$

(ıı) Assume that $X^{\mathbb{G}_m}$ is finite. The varieties $X(x)$ are locally closed therefore open in their closure. Therefore there can only be one $X(x)$ dense in X. There must be one which we call x_- which is the unique fixed point x such that al the weights of $\mathbb{G}_m$ on T_xX are positive: we have the equivalences

$$T_x^+X = T_xX \Leftrightarrow \dim X(x) = \dim X \Leftrightarrow X(x) \text{ is open.}$$

We also have the equivalences

$$T_x^-X = T_xX \Leftrightarrow \dim X(x) = 0 \Leftrightarrow X(x) = \{x\}.$$

Considering these equivalences for the opposite action of $\mathbb{G}_m$ (by composing with $t \mapsto t^{-1}$) we get that there exists a unique x_+ such that $X(x_+) = \{x_+\}$. □

Bruhat decomposition. We are now in position to prove a first version of Bruhat decomposition. Let G be a reductive connected algebraic group. Let T be a maximal torus and B be a Borel subgroup containing T. We denote by W the Weyl group $W(G,T)$ by $R^+ = R^+(B)$ the set of positive roots defined by B and by U the unipotent radical of B. We also denote by $\mathfrak{b}$ and $\mathfrak{n}$ the Lie algebras of B and U. For $w \in W$, we denote by e_w the T-fixed point wB/B in G/B.

Definition 9.5.28 *Let $C^+ = \{\phi \in X_*(T) \otimes_Z R \ / \ \langle \alpha, \phi \rangle > 0$ for all $\alpha \in R^+\}$. The Weyl chamber is called the dominant Weyl chamber for B.*

The Weyl group acts simply transitively on the Weyl chambers therefore there exists an unique element $w_0 \in W$ such that $w_0(C^+) = -C^+$. Furthermore w_0 is an involution since $w_0^2(C^+) = C^+$. Let us denote by n_0 a representative of w_0 in $N_G(T)$.

Let us fix $\phi \in C^+$ and consider the corresponding Białynicki-Birula decomposition:

$$G/B = \coprod_{w \in W} C(w) \quad \text{with} \quad C(w) = \{x \in G/B \ / \ \lim_{t \to 0} \phi(t) \cdot x = e_w\}.$$

Theorem 9.5.29 (Bruhat decomposition) *The cell $C(w)$ is the U-orbit Ue_w. We thus have the (cellular) Bruhat decompositions:*

$$G/B = \coprod_{w \in W} Un_wB/B \quad \text{et} \quad G = \coprod_{w \in W} Un_wB.$$

Furthermore the open orbit under U in G/B (resp. of $U \times B$ in G) is Un_0B/B (resp. Un_0B).

Proof. We let $\mathbb{G}_m$ act on G and G/B via a cocharacter $\phi \in C^+$. Let $x \in C(w)$, since the weights of $\mathbb{G}_m$ on $\mathfrak{n} = L(U)$ are positive, then for all $u \in U$ and $t \in \mathbb{G}_m$, we have $\lim_{t \to 0} \phi(t)u\phi(t)^{-1} = 1$. This implies that following equality:

$$\lim_{t \to 0} \phi(t)ux = \lim_{t \to 0} \phi(t)u\phi(t)^{-1}\phi(t)x = (\lim_{t \to 0} \phi(t)u\phi(t)^{-1}) \cdot (\lim_{t \to 0} \phi(t)x) = e_w.$$

This proves that $C(w)$ is stable under the action of U. Therefore $Ue_w \subset C(w)$.

Conversely, if Ux is a non empty U-orbit in $C(w)$, by Kostant-Rosenlicht Theorem, this orbit is closed therefore $e_w \in Ux$ and $Ux = Ue_w$. This proves that $C(w) = Ue_w$. This proves the decomposition results.

For the last assertion, recall that the tangent space $T_{e_{w_0}}(G/B)$ can be identified with

$$\mathfrak{g}/n_0(\mathfrak{b}) \simeq \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$$

therefore all the weights are positive of the tangent space and by the Białynicki-Birula decomposition Theorem this is the dense orbit. $\square$

The subgroups U_w and U^w. Let us set $R^- = -R^+$ and $U^- = n_0(U)$. We have the easy fact.

Fact 9.5.30 *For any $\alpha \in R$ we have the equivalences*

$$U_\alpha \subset U \Leftrightarrow \alpha \in R^+ \quad \text{and} \quad U_\alpha \subset U^- \Leftrightarrow \alpha \in R^-.$$

Definition 9.5.31 *For $w \in W$ we define U_w and U^w by*

$$U_w = U \cap n_w(U) \quad \text{et} \quad U^w = U \cap n_w(U^-).$$

The groups U_w and U^w are closed subgroup of U and are normalised by T. Therefore these subgroups are the products of the U_α that they contain.

Fact 9.5.32 *(ı) We have the equality $U_w \cap U^w = \{e\}$.*
(ıı) The multiplication induces an isomorphism $U_w \times U^w \to U$.

Proof. (ı) This intersection is normalised by T thus it is equal to the product of the U_α that it contains. But we have the equivalences

$$U_\alpha \subset U_w \Leftrightarrow \alpha \in R^+ \cap w(R^+)$$
$$U_\alpha \subset U^w \Leftrightarrow \alpha \in R^+ \cap w(R^-).$$

These conditions are exclusive proving the result.

(ıı) Any root in R^+ satisfy one of the above conditions proving (ıı). $\square$

Proposition 9.5.33 *(ı) For $w \in W$, the stabiliser of e_w in G (resp. $\mathfrak{g}$, resp. U, resp. $\mathfrak{n}$) is $n_w(B)$ (resp. $n_w(\mathfrak{b})$, resp. U_w, resp. $\mathfrak{n} \cap n_w(\mathfrak{b}) = L(U_w)$).*

(ıı) We have the equality $Ue_w = U^w e_w$ and the orbit morphism $U^w \to U^w e_w = Ue_w$ is an isomorphism. In particular, $\dim Ue_w = n(w)$ with $n_w = |R^+ \cap w(R^-)|$.

Proof. The morphism $\pi : G \to G/B$ is separable and $\ker d_e\pi = \mathfrak{b}$. The stabiliser of $e_w = n_wB/B$ is obviously $n_w(B)$ since the stabiliser of $eB/B = e$ is B. Translating π by n_w we get $\pi_w : G \to G/B$ defined by $\pi_w(g) = gn_wB/B$. This morphism is again separable. The stabiliser of eB/B is now $n_w(B)$ and $\ker d_e\pi_w$ is $n_w(\mathfrak{b})$ proving the first two results.

The stabiliser of e_w in U is $U \cap n_w(B) = U \cap n_w(U) = U_w$ and the the kernel of the restriction of $d_e\pi_w$ to U is $\ker d_e\pi_w \cap \mathfrak{n} = n_w(\mathfrak{b}) \cap \mathfrak{n} = L(U_w)$. Since $U_w \times U^w \to U$ is an isomorphism, the morphism $U^w \to U^w e_w$ is bijective and the kernel of the differential is $L(U^w) \cap L(U_w) = 0$ therefore it is separable and an isomorphism. $\square$

Theorem 9.5.34 (Bruhat decomposition) *(ı) We have the decompositions*

$$G = \coprod_{w\in W} U^w n_w B, \quad \text{and} \quad G/B = \coprod_{w\in W} U^w n_w B/B,$$

and for any $w \in W$, the morphism $U^w \times B \to U^w n_w B$ defined by $(u, b) \mapsto u n_w b$ is an isomorphism. In particular any element $g \in G$ can be written uniquely as

$$g = u n_w t u', \quad \text{with } u \in U^w,\ t \in T \text{ and } u' \in U.$$

(ıı) Furthermore, we have the open coverings

$$G = \bigcup_{w\in W} n_w U^- B \quad \text{and} \quad G/B = \bigcup_{w\in W} n_w U^- B/B.$$

Proof. (ı) We already proved everything.

(ıı) We know that U^-B and U^-B/B are open subsets containing e. Therefore their translate by n_w are open subsets containing n_w. The fact that the union is the all of G or G/B comes from the inclusion $n_w U^- B = (n_w U^- n_w^{-1}) n_w B \supset U^w n_w B$ and the decomposition in (ı). □

Definition 9.5.35 *Let us define $N(w) = \{\alpha \in R^+ \ / \ w^{-1}(\alpha) \in R^-\} = R^+ \cap w(R^-)$ and define $n(w) = |N(w)|$.*

Fact 9.5.36 *For $w \in W$.*

(ı) We have $\dim C(w) = \dim U^w = n(w)$.

(ıı) We have $n(w) = n(w^{-1})$ and $n(w_0 w) = n(w w_0) = |R^+| - n(w)$.

9.6 Structure of semisimple groups

Theorem 9.6.1 *Let G be a semisimple connected algebraic group and let T be a maximal torus. Let R be the corresponding root system.*

(ı) We have the equality

$$Z(G) = \bigcap_{\alpha\in R} \ker \alpha$$

and this group is finite.

(ıı) The root system R spans $X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the dual root system $R^\vee$ spans $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(ııı) The group G is spanned by the subgroups $(U_\alpha)_{\alpha\in R}$. We have the equality $G = D(G)$.

Proof. (ı) We already know the inclusion $Z(G) \subset T$ (indeed $Z(G) \subset C_G(T) = T$). Let $t \in Z(G)$. For $\alpha \in R$ and $u_\alpha(x) \in U_\alpha$ we have $u_\alpha(x) = t u_\alpha(x) t^{-1} = u_\alpha(\alpha(t)x)$. Therefore, since u_α is an isomorphism we have $\alpha(t) = 1$ for all $\alpha \in R$ proving the inclusion $Z(G) = \bigcap_{\alpha\in R} \ker \alpha$.

Conversely, for t an element in this intersection, then t commutes with any element in T and in U_α for all $\alpha \in R$. Since G is generated by T and the $(U_\alpha)_{\alpha\in R}$ the element t lies in $Z(G)$.

Finaly we know that $Z(G)^0 = R(G)$ is trivial (since G is semisimple) therefore the center is finite.

(ıı) Consider the algebra $k[T]$. This is the algebra of the character group $X^*(T)$. In $k[T]$ there is a subalgebra A spanned by the roots $A = k[\alpha, \alpha \in R]$. Let T' be the quotient of T such that $T' = \operatorname{Spec} A$. This is a torus and since the map $A \to k[T]$ is an inclusion, the map $T \to T'$ is dominant. The kernel of this map is the center $Z(G)$ thanks to (ı). Therefore the dimension of T and T' are the same and these dimension are respectively the dimension of the spaces $X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ and the subspace generated by R. The result follows.

(ııı) Let S be the subgroup of T spanned by the coroots $\alpha^\vee(\mathbb{G}_m)$ for $\alpha\in R$. We start to prove that $T=S$ *i.e.* that T is spanned by the $\alpha^\vee(G_m)$. Indeed, we have a restriction map $\pi: X^*(T)\to X^*(S)$ which is surjective (because S is a closed subgroup of T) and by (ıı) has a finite kernel. Because T is connected $X^*(T)$ is torsion free thus the kernel is trivial and $S=T$.

Lemma 9.6.2 *For any $\alpha\in R$, the group $\alpha^\vee(\mathbb{G}_m)$ is contained in the subgroup spanned by U_α and $U_{-\alpha}$.*

Proof. It is enough to prove this result in the subgroup $Z_\alpha=C_G(S_\alpha)$ with $S_\alpha=(\ker\alpha)^0$. But by definition of $\alpha^\vee$, the group $\alpha^\vee(\mathbb{G}_m)$ is contained in $D(Z_\alpha)$. Let H_α be the subgroup of $D(Z_\alpha)$ generated by U_α and $U_{-\alpha}$. This subgroup closed, connected and is normalised by T. It contains U_α and $U_{-\alpha}$. Therefore it is normalised by T, U_a and $U_{-\alpha}$. Therefore it is normal in Z_α. Therefore this group is not unipotent otherwise we would have an exact sequence $1\to H_\alpha\to Z_\alpha\to Z_\alpha/H_\alpha\to 1$ with $\dim Z_\alpha/H_\alpha\leq 1$ thus solvable and Z_α would be solvable. A contradiction. Therefore H_α is of dimension 3 thus $H_\alpha=Z_\alpha$. □

This proves that G is spanned by the subgroups $(U_\alpha)_{\alpha\in R}$. Furthermore the above lemma proves that $T\subset D(G)$. The last assertion follows from the fact that $U_\alpha\subset D(G)$. Indeed, for $x\in\mathbb{G}_m$ and $t\in T$, we have

$$u_\alpha(x)tu_\alpha(-x)t^{-1}=u_\alpha((1-\alpha(t))x)$$

proving the desired inclusion. □

Corollary 9.6.3 *The multiplication induces a surjection $D(G)\times Z^0(G)\to G$ whose kernel is finite.*

Proof. The kernel of this map is the intersection $D(G)\cap Z^0(G)$ which is finite. Furthermore, $Z^0(G)=R(G)$ thus $G/Z^0(G)$ is semisimple thus $D(G/Z^0(G))=G/Z^0(G)$. But for the surjective map $G\to G/Z^0(G)$, this implies that the map $D(G)\to D(G/Z^0(G))=G/Z^0(G)$ is also surjective. Therefore the map is surjective (by dimension count). □

Remark 9.6.4 (ı) Note that $D(G)$ is semisimple.

(ıı) On the Lie algebra level, this results corresponds to the decomposition $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus\mathfrak{z}(\mathfrak{g})$ for $\mathfrak{g}$ reductive.

Let me state without proof the following result (a proof along the same lines as the previous proofs can be given by I will skip it by lack of time.

Theorem 9.6.5 *Let G be a semisimple connected algebraic group, let T be a maximal torus and let R be the corresponding root system. Decompose R into orthogonal root systems*

$$R=\coprod_{i=1}^r R_i$$

and let G_i be the subgroup of G spanned by the U_a for $ainR_i$.

(ı) For $i\neq j$, the subgroups G_i and G_j commute. In particular these subgroups are normal in G and therefore semisimple.

(ıı) We have the equality $G=G_1\cdots G_r$ and the product is almost direct: for all i, the intersection $G_i\cap(\prod_{j\neq i}G_j$ is finite.

(ııı) Any normal connected subgroup of G is the product of the subgroups G_i contained in it.

Chapter 10

Representations of semisimple algebraic groups

10.1 Basics on representations

Let G be a semisimple algebraic group. We fix T a maximal torus of G and B a Borel subgroup containing T. We denote by W the Weyl group and by C the dominant Weyl chamber. We also denote by R^+ the set of positive roots defined by B.

Definition 10.1.1 *Let V be a finite dimensional representation of G. Then V can be writen as a direct sum*

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

with $V_\chi = \{v \in V \ / \ t \cdot v = \chi(t)v\}$. The characters χ such that the space V_χ is not trivial are called the weights of V. The dimension $\dim V_\chi$ is the multiplicity of the weight.

Lemma 10.1.2 *Let $n \in N_G(T)$ and $w \in W$ such that $w = \bar{n}$. Let χ be a character of T. Then we have*

$$n \cdot V_\chi = V_{w(\chi)}.$$

Proof. Let $t \in T$ and $v \in V_\chi$. We have

$$t \cdot (n \cdot v) = n \cdot (n^{-1}tn \cdot v) = n \cdot (\chi(n^{-1}tn)v) = n \cdot (w(\chi)(t)v) = w(\chi)(t)n \cdot v$$

and the result follows. □

Corollary 10.1.3 *For V a finite dimensional repesentation, all the weights of V in the same W-orbit have the same multiplicity.*

Proposition 10.1.4 *Let V be a representation of G, then for $\alpha \in R$, the group U_α maps V_χ to*

$$\sum_{k \in \mathbb{Z}_{\geq 0}} V_{\chi + k\alpha}.$$

Proof. Replacing G by its image in $GL(V)$, we may assume that the representation is faithful. By Lie-Kolchin Theorem, we may assume that the matrices representing elements of U_α are upper trianglar (pick successive fixed subspaces in V) and we may also assume that Furthermore, since T normalises U_α that the matrices representing T are diagonal.

For $t \in T$, we write $t = diag(t_1, \cdots, t_n)$ and for $u = (u_{i,j}) \in U_\alpha$, we have $(tut^{-1})_{i,j} = t_i t_j^{-1} u_{i,j}$.

Consider the composition $\mathbb{G}_a \to U_\alpha \to \mathbb{A}^1_k$ defined by $x \mapsto u_\alpha(x) \mapsto u_\alpha(x)_{i,j}$. Then $u_\alpha(x)_{i,j} = \sum_k a_k x^k$ is a polynomial in x with $a_k \in k$. We thus have $u_\alpha(\alpha(t)x)_{i,j} = \sum_k a_k(\alpha(t)x)^k$. On the other hand, we also have $u_\alpha(\alpha(t)x) = tu_\alpha(x)t^{-1}$ thus $u_\alpha(\alpha(t)x)_{i,j} = t_i t_j^{-1} u_\alpha(x)_{i,j} = t_i t_j^{-1} \sum_k a_k x^k$. We thus have the equality

$$\sum_k a_k(\alpha(t)^k - t_i t_j^{-1})x^k.$$

This is true for all x thus for all k we have $a_k(\alpha(t)^k - t_i t_j^{-1}) = 0$. Denote by χ the character of T such that $\chi(t) = t_i t_j^{-1}$. We thus have for all k the equality $a_k(\alpha^k - \chi) = 0$. This implies that $a_k = 0$ for all k except maybe one for which $\alpha^k = \chi$ (if there are 2 such k then α would be of torsion in $X^*(T)$). We thus have for some $k_{i,j} \geq 0$ the equalities

$$u_\alpha(x)_{i,j} = a_{k_{i,j}} x^{k_{i,j}} \quad \text{and} \quad \alpha(t)^{k_{i,j}} t_j = t_i.$$

Let us now consider the eigenvector e_j of the eigenbasis (e_i) of V. The weight of e_j is λ such that $\lambda(t) = t_j$. Then we have

$$u_\alpha(x)e_j = \sum_i u_\alpha(x)_{i,j} e_i = \sum_i x^{k_{i,j}} e_i.$$

Applying $t \in T$ to e_i we get

$$t \cdot e_i = t_i e_i = \alpha(t)^{k_{i,j}} t_j e_i = \alpha(t)^{k_{i,j}} \lambda(t) e_i$$

therefore the weight of e_i is $\lambda + k_{i,j}\alpha$ and the result follows. □

Definition 10.1.5 *Let V be a finite dimensional representation of G, a maximal vector (or highest weight vector) of V is an eigenvector v for T such that $U_\alpha \cdot v = v$ for all $\alpha \in R^+$.*

Fact 10.1.6 *There always exist maximal vectors.*

Proof. Indeed, the class of a maximal vector is a fixed point of B in $\mathbb{P}(V)$ and thus exists. □

Proposition 10.1.7 *Let V be a finite dimensional representation of G and let v be a maximal vector of weight λ. Then all the weights of the G-subrepresentation V' of V spanned by v are of the form $\lambda - \sum_{\alpha \in R^+} c_\alpha \alpha$ with $c_\alpha \in \mathbb{Z}_{\geq 0}$. The weight λ has multiplicity one in V' and V' has a unique maximal submodule.*

Proof. The weight description follows from the previous proposition.

Let us prove that λ has multiplicity one. If we consider $U^- B \cdot v$, then since $B \cdot v = kv$ and $U^- \cdot v$ lives in $kv + \sum_{\alpha \in R^+} V_{\lambda - c_\alpha \alpha}$ with $c_\alpha \in Z_{\geq 0}$. But $U^- B$ is dense in G (see the Bruhat decomposition) thus $V' = \langle G \cdot v \rangle = \langle (U^- B) \cdot v \rangle$ is contained in that space and the multiplicity result follows.

Now a proper subrepresentation W of V' does not contain v (otherwise it is not proper). It is therefore a sum of eigenspaces of weight different from λ. The sum of all proper subrepresentations of V' is again a sum of eigenspaces of weight different from λ thus again proper. This sum is the maximal proper submodule. □

Corollary 10.1.8 *The weight λ of a maximal vector is a dominant weight* i.e $\langle \lambda, \alpha^\vee \rangle \geq 0$ *for all* $\alpha \in R^+$.

Proof. Let $\alpha \in R^+$, then $s_\alpha(\lambda)$ is again a weight of the representation V' spanned by the maximal vector. Thus $s_\alpha(\lambda) = \lambda - \sum_{\beta \in R^+} c_\beta \beta$ with $c_\beta \geq 0$. Thus $\lambda - \langle \lambda, \alpha^\vee \rangle \alpha = \lambda - \sum_{\beta \in R^+} c_\beta \beta$ and $\langle \lambda, \alpha^\vee \rangle = c_\alpha \geq 0$. □

Definition 10.1.9 *A representation V is simple if it has no non trivial submodule.*

Example 10.1.10 Let SL_2 act on $S^p(k^2)$ the p-th graded part of the symmetric algebra $S^(k^2)$. Then this representation os irreducible if char$k \neq p$. Otherwise, the subset of p-th powers is a subspace and therefore a subrepresentation.

Theorem 10.1.11 *Let V be a simple representation.*

($\imath$) There exists a unique B-stable 1-dimensional subspace. It is spanned by a maximal vector of some dominant weight λ with multiplicity 1. This weight is called the highest weight of V.

($\imath\imath$) All the weights of V are of the form $\lambda - \sum_{\alpha \in R^+} c_\alpha \alpha$ with $c_\alpha \in \mathbb{Z}_{\geq 0}$.

($\imath\imath\imath$) If V' is another simple module of highest weight λ', then V' is isomorphic to V is and only if $\lambda' = \lambda$.

Proof. We know that there is at least one B-stable subspace. Pick a vector v in it then the subrepresentation spanned by v has to be V proving ($\imath$) and ($\imath\imath$).

($\imath\imath\imath$) If V and V' are isomorphic then obviously $\lambda = \lambda'$. Conversely consider $W = V \oplus V'$, then if v and v' maximal in V and V', then $v + v'$ has weight λ and we may consider the submodule V'' of W spanned by $v + v'$. Note that $v + v'$ is also maximal thus λ has multiplicity 1 in V'' thus v and v' are not in V''. We have morphisms $V'' \to V$ and $V'' \to V'$ given by projection on the first and second factors of W. The image is a submodule and contains v resp. v' thus the map is surjective. Furthermore, the kernel is a (proper since v and v' are not in V'') submodule of V' resp. V. It is thus trivial and the maps are isomorphisms. The result follows. □

10.2 Parabolic subgroups of G

Let G be a semisimple algebraic group, let T be a maximal torus B be a Borel subgroup, W be the Weyl group, R^+ the set of positive roots and S the corresponding basis (cf. the lecture on Lie algebras). The basis associated to R^+ is the subset of R^+ of indecomposable roots α in R^+ *i.e.* α can not be written as a sum $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$.

10.2.1 Existence of maximal parabolic subgroups

In this subsection we prove that for each simple root $\alpha \in S$, there exists a unique maximal parabolic subgroup associated to this root.

Let us recall the following fact proved in Exercise sheet 5 Exercise 4.

Fact 10.2.1 *Let ϕ be a cocharacter of an algebraic group G and define*

$$P(\phi) = \{x \in G \ / \ \lim_{t \to 0} \phi(t) x \phi(t)^{-1} \ \textit{exists}\}.$$

Then $P(\phi)$ is a closed subgroup of G.

We are going to study such a subgroup for special cocharacters. Indeed, let $(\alpha)_{\alpha\in S}$ be the simple basis of the root system. We define the fundamental cocharacters as the dual basis $(\varpi_\alpha^\vee)_{\alpha\in S}$ in $X_*(T)\otimes_{\mathbb{Z}}\mathbb{R}$ and set $P_\alpha = P(\varpi_\alpha^\vee)$.

Lemma 10.2.2 *The subgroup $P(\alpha)$ is a proper parabolic subgroup of G containing B. Furthermore, for $\beta \in S\setminus\{\alpha\}$ then any element $n_{s_\beta}\in N_G(T)$ representing $s_\beta\in W$ is in P_α.*

Proof. Obviously the torus T is contained in P_α since for $x,t\in T$ we have $\varpi_\alpha^\vee(t)x\varpi_\alpha^\vee(t)^{-1} = x$. Furthermore, for any root γ, we have $\varpi_\alpha^\vee(t)u_\gamma(x)\varpi_\alpha^\vee(t)^{-1} = u_\gamma(t^{\langle\gamma,\varpi_\alpha^\vee\rangle}x)$ and if γ is in R^+, then $scal\gamma,\varpi_\alpha^\vee \geq 0$ and the above expression has a limit when t goes to 0. Therefore U is contained in P_α thus $B\subset P_\alpha$ and this is a parabolic subgroup.

Let $\beta\in S\setminus\{\alpha\}$. Then we have

$$\begin{aligned}\varpi_\alpha^\vee(t)n_{s_\beta}\varpi_\alpha^\vee(t)^{-1}n_{s_\beta}^{-1} &= \varpi_\alpha^\vee(t)s_\beta(\varpi_\alpha^\vee)(t)^{-1}\\ &= \varpi_\alpha^\vee(t)(\varpi_\alpha^\vee - \langle\beta,\varpi_\alpha^\vee\rangle\beta)(t)^{-1}\\ &= \varpi_\alpha^\vee(t)\varpi_\alpha^\vee(t)^{-1}\\ &= e.\end{aligned}$$

We deduce that $\varpi_\alpha^\vee(t)n_{s_\beta}\varpi_\alpha^\vee(t)^{-1} = n_{s_\beta}$ and therefore has a limit. ☐

10.2.2 Description of all parabolic subgroups

Definition 10.2.3 *Let I be any subset of S.*

(ı) We define W_I as the subgroup of W spanned by the elements $(s_\alpha)_{\alpha\in I}$.

(ıı) We define R_I as the subset of R of roots which are linear combinations of roots in I.

(ııı) We define $S_i = \left(\bigcap_{\alpha\in I}\ker\alpha\right)^0$ and $L_I = C_G(S_I)$.

Using classical results we have the following fact.

Fact 10.2.4 *The group L_I is a connected reductive subgroup of G containing T and $B_I = B\cap L_I$ is a Borel subgroup of L_I.*

Lemma 10.2.5 *(ı) The root system of L_I with respect to T is R_I and its Weyl group is W_I.*

(ıı) The system of positive roots in R_I is $R_I^+ = R_I\cap R^+$ and the corresponding basis of R_I is I.

Proof. (ı) The subgroups U_α contained in L_I are the subgroup centralising S_I. Let $t\in T$ and $x\in\mathbb{G}_a$, we have $tu_\alpha(x)t^{-1}u_\alpha(-x) = u_\alpha(\alpha(t)x)u_\alpha(-x) = u_\alpha((\alpha(t)-1)x)$ thus

$$u_\alpha(x)tu_\alpha(-x) = tu_\alpha((\alpha(t^{-1})-1)x).$$

If U_α is contained in L_I, then for all $t\in S_I$, the right hand term has to be t thus $\alpha(t^{-1}) = 1$ for all $t\in S_I$ *i.e.* α is in $(R_I^\perp)^\perp = R_I$. Conversely for α in this set we have $\alpha(t^{-1}) = 1$ for all $t\in S_I$ thus $U_a\subset L_I$. The result follows.

(ıı) The statement on R_I^+ is obvious as well as the one on basis (by dimension count: I is obviously in the basis and is the all basis by dimension counts). ☐

Theorem 10.2.6

(i) Let $P_I = \coprod_{w\in W_I} C(w)$, this is a parabolic subgroup of G containing B and L_I.

(ii) The unipotent radical $R_U(P_I)$ is generated by the U_α for $\alpha \in R^+ \setminus R_I$.

(iii) The product $L_I \times R_u(P_I) \to P_I$ is an isomorphism of varieties.

(iv) Any parabolic subgroup P containing B is of the form P_I for some subset I of S.

Proof.

(i) To prove that P_I is a group, we only need to prove that it is stable under left multiplication by s_α for $\alpha \in R_I$ since it it stable under taking the inverse and left multiplication by B. Now we only need to prove axiom (T_1) of Tits system (see Exercise sheet 13). We are in fact proving that $s_\alpha Bw \subset BwB \cup Bs_\alpha wB$. This will prove the statement. Write $B = T\prod_{\beta\in R^+} U_\beta$. We have $s_\alpha Bw = T\prod_{\beta\in R^+,\beta\neq\alpha} U_\beta U_{-\alpha}s_\alpha w$. In particular we only need to prove the inclusion $U_{-\alpha}s_\alpha w \subset BwB\cup Bs_\alpha wB$. If $w^{-1}(\alpha)\in R^+$, then $U_{-\alpha}s_\alpha w = s_\alpha wU_{w^{-1}(\alpha)} \subset Bs_\alpha wB$. If not, then using Bruhat decomposition in Z_α we have $U_{-\alpha} \subset B\cup Bs_\alpha B$. We deduce the inclusions

$$U_{-\alpha}s_\alpha w \subset Bs_\alpha w \cup Bs_\alpha Bs_\alpha w.$$

But we have $Bs_\alpha Bs_\alpha w = BU_{-\alpha}w = BU_{-w^{-1}(\alpha)} \subset B$.

The set P_I contains $C(e)$ and thus $C(e) = B$. Let us also remark that Bruhat decomposition in $Z_\alpha = C_G(S_\alpha)$ with $S_\alpha = (\ker\alpha)^0$ implies the equality

$$Z_\alpha = C(e)\cup C(s_\alpha)$$

therefore U_α and $U_{-\alpha}$ are contained in the union $C(e)\cup C(s_\alpha)$. This implies that P_I contains L_I. Futhermore since the $C(w)$ are locally closed, P_I contains a dense subset of its closure thus it is a closed subgroup.

(ii) Note that U is a maximal unipotent subgroup of P_I thus $R_u(P_I)$ is the identity component of the intersection

$$\bigcap_{w\in W_I} w(U).$$

But we can write $U = U_I U^I$ with $U_I = \prod_{\alpha\in R_I^+} U_\alpha$ and $U^I = \prod_{\alpha\in R^+\setminus R_I} U_\alpha$. We see that $w\in W_I$ map $R^+\setminus R_I$ onto itself thus the above intersection is

$$\left(\bigcap_{w\in W_I} w(U_I)\right) U^I.$$

But L_I is reductive and U_I in a maximal unipotent of L_I and W_I the Weyl group of L_I the left hand term is $\{e\}$ proving the result.

(iii) We have for $w\in W_I$ the equality $C(w) = U^w wB = U^w wTU_IU^I$ and since for $w\in W_I$ we have $U^w \subset L_I$ we get the inclusion $C(w)\subset L_IR_u(P_I)$. The map is thus surjective an easily seen to be injective. By inspection on Lie algebras we get that it is an isomorphism.

(iv) Let P be a subgroup containing B and let R_P be the set of roots of $P/R_u(P)$ for the action of T. Define $I = R_P\cap S$. We see that for $\alpha\in I$, the image in $P/R_u(P)$ of the intersection of Z_α with P has the same Lie algebra as Z_α thus is of the same dimension and by connectedness of Z_α we get that $Z_\alpha\subset P$.

In particular $U_{\pm\alpha}\subset P$ thus $L_I\subset P$ thus since $R_u(P_I)\subset B\subset P$ we have $P_I\subset P$.

Conversely, the root systems of L_I and P are the same thus $\dim L_I = \dim P/R_U(P)$ and thus $P\subset L_IR_u(P)\subset L_IB\subset L_IR_u(P_I)\subset P_I$ (the first inclusion holds because since L_I is reductive the intersection $L_I\cap R_u(P)$ is trivial. □

Corollary 10.2.7 *Any parabolic subgroup P is of the form $P(\phi)$ for some cocharacter ϕ.*

Proof. We only need to take ϕ such that if $P = P_I$, then $I = \{\alpha \in S \ / \ \langle \alpha, \phi \rangle = 0\}$. $\square$

10.3 Existence of representations

Theorem 10.3.1 *Let χ be a dominant weight, then the exists an irreducible representation of highest weight χ.*

Proof. First of all, note that we only need to prove that there exists a representation with a maximal vector of weight χ. Furthermore, there are some easy constructions producing representations from other representations. In particular we have the lemma.

Lemma 10.3.2 *If V and V' are representations with maximal weights v and v' of weights χ and χ', then $V \otimes V'$ is a representation with $v \otimes v'$ a maximal vector of weight $\chi + \chi'$.*

Proof. Indeed, the action of $b = tu \in B$ with $t \in T$ and $u \in U$ on $v \otimes v'$ is given by $b \cdot (v \otimes v') = bv \otimes bv' = \chi(t)\chi(t')v \otimes v'$. $\square$

Now we need to construct representations. For this we use Chevalley's Theorem. Consider for $\alpha \in S$ the maximal parabolic subgroup P_α and choose a representation V_α of G such that P_α stabilises a line kv_α for some $v_\alpha \in V_\alpha$. The elements n_β for $\beta \in S \setminus \{\alpha\}$ are in P_α thus $n_\beta \cdot kv_\alpha = kv_\alpha$. By construction v_α is a maximal vector of some weight χ. Let us write $\chi = \sum_{\beta \in S} a_\beta \varpi_\beta$ (with $(\varpi_\beta)_{\beta \in S}$ the dual basis of $(\beta^\vee)_{\beta \in S}$). For $\beta \in S \setminus \{\alpha\}$, we have $s_\beta(\chi) = \chi$ thus $\langle \chi, \beta^\vee \rangle = 0$ therefore $\chi_\beta = 0$ Thus $\chi = a_\alpha \alpha$. This already proves that big enough multiple of any dominant weight is the highest weight of an irreducible representation.

Let us now prove the existence in general. For this we only need to ove that the result is true for the fundamental weights $(\varpi_\alpha)_{\alpha \in S}$. We look for special functions in $k[G]$. Let ϖ_α be a fundamental weight define a function c_α on $U^- B = U^- TU$ by $c_\alpha(utu') = \varpi_\alpha(t)$. This is well defined since the multiplication $U^- \times T \times U \to U^- B$ is an isomorphism. This is a rational function on G and we want to extend it to a function on G *i.e.* an element of $k[G]$.

But a rational function $f \in k(G)$ is defined in $x \in G$ if and only if one of its power is defined. In the above representation V_α, let V'_α be the span of all weight spaces different from kv_α. We have a direct sum $V_\alpha = kv_\alpha \oplus V'_\alpha$ and we map define the linear map r_α by $r_\alpha(v_\alpha) = 1$ and $r_\alpha(V'_\alpha) = 0$. We may then define $d_\alpha(g) = r_\alpha(g \cdot v_\alpha)$. We may then compute for $utu' \in U^- B$ the value of d_α:

$$d_\alpha(utu') = r_\alpha(utu' \cdot v_\alpha) = r_\alpha(ut \cdot v_\alpha) = r_\alpha(u\varpi_\alpha(t)^{a_\alpha} v_\alpha) = \varpi_\alpha^{a_\alpha}(t) = c_\alpha(utu')^{a_\alpha}.$$

Therefore the function $c_\alpha^{a_\alpha}$ is d_α and defined on G thus c_α is defined on G.

We know that there is a finite dimensional subrepresentation of $k[G]$ containing c_α and the weight of T on c_α is ϖ_α bacause of the equaliy $t' \cdot c_\alpha(utu') = c_\alpha(utu't') = \varpi_\alpha(t)\varpi_\alpha(t')$. This finishes the proof. $\square$

More generaly, let us introduce for χ a character the set of functions

$$F_\chi = \{f \in k[G] \ / \ f(xy) = \chi(x) f(y) \text{ for } x \in B^- \text{ and } y \in G\}.$$

Fact 10.3.3 *the set F_χ is a subspace of $k[G]$ stable under right translations.*

Furthermore the function c_α above is in F_{ϖ_α} or more generaly for χ dominant the corresponding function c_χ defined from the representation of highest weight χ is in F_χ.

Lemma 10.3.4 *Let F be a simple subrepresentation of F_χ with highest weight χ', then $\chi' = \chi$.*

Proof. Indeed, let f be a mxaimal vector in f of weight χ', then we have $f(xy) = \chi'(y)f(x)$ for $x \in G$ and $y \in B$. Applying this with $t \in T$, we get

$$f(e) = f(tet^{-1}) = \chi'(t)^{-1}f(te) = \chi'(t^{-1})\chi(t)f(e).$$

So we only need to prove that $f(e) \neq 0$ but in that case $f(utu') = \chi(u)\chi'(t)f(e) = 0$ thus f is trivial on the dense open U^-B and thus trivial a contradiction.

Note that we even have $f = f(e)c_\chi$. □

Note that the above space F_χ is the space of section of a globally generated line bundle. In characteristic 0 this representation is simple.

Chapter 11

Uniqueness and existence Theorems, a review

In this chapter. We want to give a quick review of the classification of connected semisimple algebraic groups over an algebraically closed field k. For more details we refer to [Spr98].

11.1 Uniqueness Theorem

11.1.1 Structure constants

Recall that we proved that, for α a root of a connected semisimple group G, if we fix $u_\alpha : \mathbb{G}_a \to U_\alpha$ an isomorphism then we have

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x).$$

Lemma 11.1.1 *We may choose the maps u_α such that if we define*

$$n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1),$$

this element lies in $N_G(T)$ and its image in W the Weyl group is s_α. For such a choice we have the following properties.

- *For $x \in \mathbb{G}_m \subset \mathbb{G}_a$, we have the equality $u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) = \alpha^\vee(x)n_\alpha$.*
- *We have the equalities $n_\alpha^2 = \alpha^\vee(-1)$ and $n_{-\alpha} = n_\alpha^{-1}$.*

Furthermore, if $(u'_\alpha)_{\alpha\in R}$ is another family of maps satisfying the above condition, then there exists $c_\alpha \in \mathbb{G}_m$ for all α such that $u'_\alpha(x) = u_\alpha(c_\alpha x)$ and $c_\alpha c_{-\alpha} = 1$.

Proof. These conditions essentially deal with semisimple groups of rank one. We can therefore restrict ourselves to the group $D(Z_\alpha)$ with Z_α as usual. Then we have to use the (unproved) fact that this group is isomorphic to SL_2 or PGL_2. Because of the surjective map from the first one to the other we only need to deal with SL_2. Then we compute all these properties explicitely on the matrices.

If we have another morphism u'_α, then because the only group morphisms $\mathbb{G}_a \to \mathbb{G}_a$ are $x \mapsto ax$ we get that there exists $c_\alpha \in \mathbb{G}_m$ with $u'_\alpha(x) = u_\alpha(c_\alpha x)$. $\square$

Definition 11.1.2 *A realisation of the root system is a family of map $(u_\alpha)_{\alpha\in R}$ such that $u_\alpha : \mathbb{G}_a \to U_\alpha$ is a group isomophism and these maps satisfy the above lemma.*

Remark 11.1.3 A realisation determines the coroots.

Let us fix once and for all a total ordering of the set of roots.

Proposition 11.1.4 (Structure constants) *Let α and β be roots with $\alpha \neq \pm\beta$. Then there exists constants $c_{\alpha,\beta,i,j} \in k$ such that for all $x, y \in \mathbb{G}_a$ we have the equality:*

$$u_\alpha(x)u_\beta(y)u_\alpha(x)^{-1}u_\beta(x)^{-1} = \prod_{i\alpha+j\beta\in R; i,j>0} u_{i\alpha+j\beta}(c_{\alpha,\beta;i,j}x^iy^j)$$

the order of the multiplication being given by the ordering on the roots.

Proof. It is easy to prove that one can assume α and β to be positive. We then know because of the structure of U that we have polynomials P_γ for any positive root γ such that

$$u_\alpha(x)u_\beta(y)u_\alpha(x)^{-1}u_\beta(x)^{-1} = \prod_\gamma u_\gamma(P_\gamma(x, y)).$$

Conjugating by $t \in T$, we get the equality

$$P_\gamma(\alpha(t)x, \beta(t)y) = \gamma(t)P_\gamma(x, y).$$

But by linear invariance of the characters, we get that there is a unique pair (i, j) such that $\gamma = i\alpha+j\beta$ and $P_\gamma(x, y) = c_{\alpha,\beta;i,j}x^iy^j$ proving the result. $\square$

11.1.2 The elements n_α

In this subsection we want to explicit some useful properties of the elements n_α defined by

$$n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1).$$

Proposition 11.1.5 *Assume that α and β are simple roots and let $m(\alpha, \beta)$ be the order of $s_\alpha s_\beta$ in the Weyl group W. Then we have*

$$n_\alpha n_\beta n_\alpha n_\beta \cdots = n_\beta n_\alpha n_\beta n_\alpha \cdots$$

with $m(\alpha, \beta)$ factors on both sides. We also have

$$\chi(n_\alpha^2) = (-1)^{\langle\chi,\alpha^\vee\rangle}.$$

Proof. This follows from a careful (and non obvious) study of rank 2 roots systems. $\square$

Corollary 11.1.6 *There is a well defined morphism $\phi : W \to N_G(T)$ with $\phi(s_\alpha) = n_\alpha$ for $\alpha \in S$ and if $w = s_{\alpha_1} \cdots s_{\alpha_n}$ then $\phi(w) = n_{\alpha_1} \cdots n_{\alpha_n}$.*

Proposition 11.1.7 *For α and β simple roots and $w \in W$ with $w(\alpha) = \beta$ then*

$$\phi(w)u_\alpha(x)\phi(w)^{-1} = u_\beta(x).$$

Proof. Again this follows by inspection on the rank two cases. $\square$

11.1.3 Presentation of G

In this subsection we describe G as an abstract group. For this let us fix the root datum and a total order on the positive roots as well as structure constants $(c_{\alpha,\beta;i,j})$ for a realisation (u_α) of G.

For α and β simple, we define by $R_{\alpha,\beta}$ the rank 2 root system spanned by α and β and let

$$R_1 = \bigcup_{\alpha,\beta\in S,\alpha\neq\beta} R_{\alpha,\beta}.$$

We give a presentation of G with generators and relations. Let us first define

$$\mathbb{T} = \mathrm{Hom}(X^*(T), \mathbb{G}_m)$$

which is the group of morphisms of abelian group.

Fact 11.1.8 *These is a natural isomorphism $\pi : \mathbb{T} \to T$ defined by $\chi(\pi(\mathbf{t})) = \mathbf{t}(\chi)$. The inverse image of π is given by $\pi^{-1}(t)(\chi) = \chi(t)$.*

Definition 11.1.9 *($\imath$) For $\chi \in X^*(T)$ we define a character $\overline{\chi} : \mathbb{T} \to \mathbb{G}_m$ by $\overline{\chi}(\mathbf{t}) = \mathbf{t}(\chi)$.*
($\imath\imath$) For ϕ a cocharacter of T, we define $\overline{\phi} : \mathbb{G}_m \to \mathbb{T}$ by $\overline{\phi}(x)(\chi) = x^{\langle\chi,\phi\rangle}$.
($\imath\imath\imath$) We define an action of the Weyl group on $\mathbb{T}$ by $w(\mathbf{t})(\chi) = \mathbf{t}(w^{-1}\cdot\chi)$.

Now we define for $\gamma \in R_1$ and $k \in \mathbb{G}_a$ a generator $\mathbf{u}_\gamma(x)$ and we impose the relations for $\gamma, \delta \in R_{\alpha,\beta} \subset R_1$ for some simple roots α, β.

$$\mathbf{u}_\gamma(x+y) = \mathbf{u}_\gamma(x)\mathbf{u}_\gamma(y)$$

$$\mathbf{u}_\gamma(x)\mathbf{u}_\delta(y)\mathbf{u}_\gamma(x)^{-1}\mathbf{u}_\delta(x)^{-1} = \prod_{i\gamma+j\delta\in R;i,j>0} \mathbf{u}_{i\gamma+j\delta}(c_{\gamma,\delta;i,j}x^iy^j),$$

$$\mathbf{t}\mathbf{u}_\gamma(x)\mathbf{t}^{-1} = \mathbf{u}_\gamma(\overline{\gamma}(\mathbf{t})x).$$

For $\gamma \in R_1$ we define the elements

$$\mathbf{n}_\gamma = \mathbf{u}_\gamma(1)\mathbf{u}_{-\gamma}(-1)\mathbf{u}_\gamma(1)$$

and we impose the following relations.

$$\mathbf{n}_\gamma\mathbf{u}_\gamma(x)\mathbf{n}_\gamma^{-1} = \mathbf{u}_{-\gamma}(-x)$$

$$\mathbf{n}_\gamma^2(\chi) = (-1)^{\langle\chi,\gamma^\vee\rangle}$$

$$\mathbf{n}_\gamma\mathbf{n}_\delta\cdots = n_\delta\mathbf{n}_\gamma\cdots$$

such that in the last relation there are $m(\gamma,\delta)$ factors with δ and γ simple roots. Finnaly we impose the relation:

$$\mathbf{u}_\gamma(x)\mathbf{u}_{-\gamma}(-x^{-1})\mathbf{u}_\gamma(x) = \overline{\gamma^\vee}(x)\mathbf{n}_\gamma.$$

Fact 11.1.10 *The elements $\mathbf{n}_\gamma$ normalise $\mathbb{T}$, we have the formula*

$$\mathbf{n}_\gamma\mathbf{t}\mathbf{n}_\gamma^{-1}(\chi) = \mathbf{t}(s_\gamma(\chi)).$$

Proof. We compute

$$\begin{aligned}\mathbf{n}_\gamma \mathbf{t} \mathbf{n}_\gamma^{-1} \mathbf{t}^{-1} &= \mathbf{n}_\gamma \mathbf{t} \mathbf{u}_\gamma(-1)\mathbf{u}_{-\gamma}(1)\mathbf{u}_\gamma(-1)\mathbf{t}^{-1} \\ &= \mathbf{n}_\gamma \mathbf{u}_\gamma(-\overline{\gamma}(\mathbf{t}))\mathbf{u}_{-\gamma}(\overline{\gamma}(\mathbf{t})^{-1})\mathbf{u}_\gamma(-\overline{\gamma}(\mathbf{t})) \\ &= \mathbf{n}_\gamma \overline{\gamma^\vee}(-\overline{\gamma}(\mathbf{t}))\mathbf{n}_\gamma = \overline{\gamma^\vee}(-\overline{\gamma}(\mathbf{t}))\mathbf{n}_\gamma^2.\end{aligned}$$

Evaluation in χ we get the answer. $\square$

Let $\mathbb{G}$ be the group generated by $\mathbb{T}$ and the $u_\gamma(x)$ satisfying the above relations.

Theorem 11.1.11 *The isomorphism $\pi : \mathbb{T} \to T$ extends to an isomorphism of abstract groups $\pi : \mathbb{G} \to G$ with $\pi(\mathbf{u}_\gamma(x)) = u_\gamma(x)$.*

Proof. The relations above are true in G thus the morphism π extends to $\mathbb{G}$. Let us denote by $\mathbb{U}_\gamma$ the image of $\mathbf{u}_\gamma$. We have $\pi(\mathbb{U}_\gamma) = U_\gamma$. Furthermore if we write $U_{\gamma,\delta}$ resp. $\mathbb{U}_{\gamma,\delta}$ the subgroup of G resp. $\mathbb{G}$ generated by the $U_{i\gamma+j\delta}$ with i,j non negative, we have $\pi(\mathbb{U}_{\gamma,\delta}) = U_{\gamma,\delta}$. But the map $\mathbb{G}_a^n \simeq \prod_{i\gamma+i\delta,i,j\geq 0} U_{i\gamma+j\delta} \to U_{\gamma,\delta}$ is an isomorphism thus so is $\pi : \mathbb{U}_{\gamma,\delta} \to U_{\gamma,\delta}$ and thus isomorphisms $\pi : \mathbb{U}_\gamma \to U_\gamma$ for $\gamma \in R_1$.

Note that $\pi(\mathbf{n}_\gamma) = n_\gamma$ and that we deduce a natural map $\phi : W \to \mathbb{G}$ by setting $\phi(w) = \phi(s_{\alpha_1} \cdots s_{\alpha_n}) = n_{\alpha_1} \cdots n_{\alpha_n}$. Note also that $\mathbb{U}_\gamma$ and $\mathbb{U}_{-\gamma}$ normalise $\mathbb{U}_{\gamma,\delta}$ thus so does $\mathbf{n}_\gamma$ and since $n_\gamma U_\delta n_\gamma^{-1} = U_{s_\gamma(\delta)}$ we get $\mathbf{n}_\gamma \mathbb{U}_\delta \mathbf{n}_\gamma^{-1} = \mathbb{U}_{s_\gamma(\delta)}$. We deduce that for $w \in W$ we have $\phi(w)\mathbb{U}_\gamma\phi(w)^{-1} = \mathbb{U}_{w(\gamma)}$ and thus $\pi : \mathbb{U}_\gamma \to U_\gamma$ is bijective for all γ.

Now let $\mathbb{U}$ be the group generated by the $\mathbb{U}_\gamma$ for $\gamma \in R^+$ then by the commuting relations we see that this group is contained in the product of the $\mathbb{U}_\gamma$ for $\gamma \in R^+$ and again by the argument above the restriction of π is bijective.

We deduce that $\mathbb{B}$ the group spanned by $\mathbb{T}$ and $\mathbb{U}$ is in bijection with B and the same it true for the $\mathbb{U}^w$. Thus the $\mathbb{C}(w) = \mathbb{B}\phi(w)\mathbb{B}$ are in bijection with the $C(w) = BwB$.

Now a Tits system argument proves that Bruhat decomposition also holds in $\mathbb{G}$ and the result follows. $\square$

11.1.4 Uniqueness of structure constants

Definition 11.1.12 *Let (u_α) and (u'_α) be two realisations of G and let $(c_{\alpha,\beta;i,j})$ and $(c'_{\alpha,\beta;i,j})$ be the corresponding structure constants. Then we know that there exists constants (c_α) such that $c_\alpha c_{-\alpha} = 1$ and one easily checks that the condition*

$$c'_{\alpha,\beta;i,j} = c_\alpha^{-i} c_\beta^{-j} c_{i\alpha+j\beta} c_{\alpha,\beta;i,j}.$$

Two set of structure constants are called equivalent if there exists constants satisfying the above relations.

Theorem 11.1.13 *The structure constants are uniquely determined by the roots system modulo equivalence.*

Proof. This relies on computations in rank 2 root systems. $\square$

11.1.5 Uniqueness Theorem

For $i \in [1,2]$, let G_i be a connected reductive algebraic group with maximal torus T_i and root datum $(X_i^*, X_{i*}, R_i, R_i^\vee)$.

Theorem 11.1.14 *Let $f : (X_1^*, X_{1*}, R_1, R_1^\vee) \to (X_2^*, X_{2*}, R_2, R_2^\vee)$ be an isomorphism of root datum. Then there exists an isomorphism $\phi : G_1 \to G_2$ with $\phi(T_1) = T_2$ inducing f on the root datum.*

Furthermore there is a unique such isomorphism modulo conjugation by an element in T_1.

Proof. By uniqueness of structure constants and the abstract group description, there exists a bijective map of abstract groups $\phi : G_1 \to G_2$. Furthermore, this map respects the Bruhat decomposition and is therefore a morphism on this open set (by explicit description of the map). Since such open cover G_1 the map is a morphism and the same proof gives that the inverse is also a morphism. □

11.2 Existence Theorem

The main result is as follows.

Theorem 11.2.1 *Given a roots datum, there exists a connected reductive linear algebraic group with this root datum.*

The proof goes mainly in three steps.

Step 1. One proves that if there exists an adjoint group *i.e.* a group such that $X^*(T) = Q(R)$ then there exists an algebraic group with the desired root datum.

I will not discuss on this step.

Step 2. One proves that for simply laced groups, there exists such a group with $Q(R) = X^*(T)$ by realising it as the group of Lie algebra automorphisms of the corresponding Lie algebra.

One can define an explicit presentation of the Lie algbera $\mathfrak{g}$ by

$$[u, u'] = 0,\ [u, e_\alpha] = \langle a, u\rangle e_\alpha,\ [e_\alpha, e_\beta] = c_{\alpha,\beta} e_{\alpha+\beta} \text{ and } [e_\alpha, e_{-\alpha}] = \alpha^\vee.$$

We define T to be a torus with characted group $Q(R)$. Then T acts on $\mathfrak{g}$ by

$$t \cdot u = u \text{ and } t \cdot e_\alpha = \alpha(t) e_\alpha.$$

Then we define $X_\alpha = \operatorname{ad} e_\alpha$ and $X_\alpha^{(2)}$ by mapping any element to 0 except $e_{-\alpha}$ to $-e_\alpha$. We define

$$u_\alpha(x) = 1 + x X_\alpha + x^2 X_\alpha^{(2)}.$$

Theorem 11.2.2 *The group spanned by T and the $u_\alpha(x)$ has the correct root datum.*

Proof. An easy check gives that any element of this group (one only need to check this on the generators) respect the Le bracket in $\mathfrak{g}$ *i.e.* for all $g \in G$ qnd $x, y \in \mathfrak{g}$ we have the equality

$$[gx, gy] = g[x, y].$$

In particular G is a closed subgroup of the algebraic group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of the Lie algebra. By easy derivation computations, we get that $L(G)$ which is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ acts on $\mathfrak{g}$ by derivation thus is contained in $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$. In particular $\dim G \leq \mathfrak{g}$. But obviously the Lie algebra $L(U_\alpha)$ is kX_α and the Lie algebra of T is $\mathfrak{h}$ thus $L(G)$ contains $\mathfrak{h}$ and all the $\mathfrak{g}_\alpha$ thus contains $\mathfrak{g}$. Therefore $L(G) = \mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$.

We also deduce from this that T is a maximal torus of the group (otherwise there would be more trivial weights). By the weight decomposition we get that the root system is R and by dimension counting we get that G has to be reductive. □

Step 3. To get the non simply laced cases, we use automorphisms of the Dynkin diagram: any non simply laced Dynkin diagram can be obtained by folding a simply laced Dynkin diagram. This means that if σ is an automorphism of a Dynkin diagram, then we may extend it to an automorphism of the above group G. Indeed we first extend it to an automorphism of the root system and then of the Lie algebra by simply mapping e_α to $e_{\sigma(\alpha)}$. The element σ lie in $\mathrm{Aut}(\mathfrak{g})$ and we can check that it normalises G as define above thus this induces an automorphism σ of G.

Theorem 11.2.3 *The group G^σ has the correct non simply laced Dynkin diagram.*

Bibliography

[Bou58] Bourbaki, Nicolas. *Algèbre* I-IX, Hermann, 1958.

[Bou60] Bourbaki, Nicolas. *Groupes et algèbres de Lie*, Chapitre I, Hermann, 1960.

[Har77] Hartshorne, Robin. *Algebraic geometry.* GTM 52, Springer-Verlag, NewYork-Heidelberg, 1977.

[Hum72] Humphreys, James E. *Introduction to Lie algebras and representation theory.* GTM 9, Springer-Verlag, Berlin-Heidelberg-NewYork, 1972.

[Spr98] Springer, Tony A. *Linear algebraic groups.* Second edition. Progress in Mathematics, 9. Birkhäuser Boston, Inc., Boston, MA, 1998.