Resolution of Singularities of Arithmetical Threefolds II.

Vincent Cossart * Olivier Piltant *

December 1, 2014

Dedicated to Shreeram Shankar Abhyankar, in memoriam.

Abstract

We prove Grothendieck's Conjecture on Resolution of Singularities for quasi-excellent schemes $\mathcal{X}$ of dimension three and of arbitrary characteristic. This applies in particular to $\mathcal{X} = \mathrm{Spec}A$, A a reduced complete Noetherian local ring of dimension three and to algebraic or arithmetical varieties of dimension three. Similarly, if F is a number field, a complete discretely valued field or more generally the quotient field of any excellent Dedekind domain $\mathcal{O}$, any regular projective surface X/F has a proper and flat model $\mathcal{X}$ over $\mathcal{O}$ which is everywhere regular.

Contents

[1]Laboratoire de Mathématiques LMV UMR 8100, Université de Versailles, 45, avenue des États-Unis, 78035 VERSAILLES Cedex, France
cossart@math.uvsq.fr, piltant@math.uvsq.fr

1 Introduction.

The Resolution of Singularities conjecture has been, and still is a long standing conjecture since it was formulated by A. Grothendieck in the 1960's [37](7.9.6). Grothendieck emphasized its importance for studying homological and homotopical properties of schemes. Even since H. Hironaka's celebrated theorem [40] proved fifty years ago, some new results have bettered our understanding of the problem in equal characteristic zero [10][66][67]. These results focus on the constructivity and functoriality of their algorithms for Resolution in contrast with Hironaka's.

In arbitrary characteristic, a major advance towards Grothendieck's conjecture is due to A.J. de Jong [48] theorem 4.1 and theorem 6.5. He proved a weaker form of the above conjecture for varieties X over a field or a complete discrete valuation ring. A significant difference with Grothendieck's formulation is that de Jong's alterations allow a finite extension of the function field. Furthermore, de Jong's result does not in general provide a regular compactification $\overline{X}$ of some étale covering U of the regular locus $\mathrm{Reg}X$.

Resolution of Singularities in its full birational form was to this date restricted to surfaces [1][4][41][52][30][33][25], only to mention some contributions. In dimension three, some partial results do exist for algebraic varieties over an algebraically closed field k of positive characteristic $p \geq 7$ [5][32]. These results extend to all characteristics $p > 0$ and fields k with $[k : k^p] < +\infty$ [26][27] theorem on p. 1839. For arithmetical schemes (unequal residue characteristic), birational Resolution of Singularities was sofar restricted to surfaces. The first and main purpose of this article is to prove:

Theorem 1.1. *Let $\mathcal{X}$ be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. There exists a proper birational morphism $\pi : \mathcal{X}' \to \mathcal{X}$ with the following properties:*

(i) $\mathcal{X}'$ is everywhere regular;

(ii) π induces an isomorphism $\pi^{-1}(\mathrm{Reg}\mathcal{X}) \simeq \mathrm{Reg}\mathcal{X}$;

(iii) $\pi^{-1}(\mathrm{Sing}\mathcal{X})$ *is a strict normal crossings divisor on* $\mathcal{X}'$.

If furthermore a finite affine covering $\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_n$ *is specified, one may take* $\pi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i$ *projective,* $1 \leq i \leq n$.

We emphasize that no assumption is made on the characteristic of $\mathcal{X}$ in this theorem. A proper birational morphism π with property (i) was called a resolution of singularities by Grothendieck [37](7.9.1), though more recent terminology (this article included) tends to require property (ii) as well. When property (iii) also holds, one says that π is a good resolution or a log-resolution. In dimension three, the hard part is to prove (i). The following corollary gives a strong basis for the local study of three dimensional singularities via Resolution of Singularities:

Corollary 1.2. *Let* A *be a reduced complete Noetherian local ring of dimension three. Then* $\mathcal{X} := \mathrm{Spec}A$ *has a good resolution of singularities which is projective.*

Since the class of quasi-excellent schemes is stable by morphisms of finite type, theorem 1.1 applies in particular to algebraic varieties and to arithmetical varieties over excellent Dedekind rings. Similarly, theorem 1.1 applies to formal completions of affine Noetherian schemes along quasi-excellent subschemes. An important application of theorem 1.1 is to constructing regular integral models of projective surfaces:

Corollary 1.3. *Let* $\mathcal{O}$ *be an excellent Dedekind domain with quotient field* F *and* Σ/F *be a regular projective surface. There exists a proper and flat* $\mathcal{O}$*-scheme* $\mathcal{X}$ *with generic fiber* $\mathcal{X}_F = \Sigma$ *which is everywhere regular.*

We remark at this point that the morphism π in theorem 1.1 is *not* constructed as a composition of *Hironaka-permissible* blowing ups, i.e. with regular centers along which the successive strict transforms of $\mathcal{X}$ are normally flat (Hironaka Resolution). Similarly, it is not even known if such π can be obtained by blowing up an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ whose zero locus is $\mathrm{Sing}\mathcal{X}$, even when $\mathcal{X}$ is affine.

On the other hand, a certain local version of theorem 1.1 is proved using only local Hironaka-permissible blowing ups in theorem 1.4 below. This fact appears to be a piece of evidence that Hironaka Resolution could be true for threefolds of nonzero residue characteristic, *vid.* also [19][55] in positive characteristic. It is however restricted to certain hypersurface threefolds

of multiplicity not bigger than the residue characteristic and the problem remains widely open even in dimension three.

In higher dimensions $n \geq 4$, the Resolution of Singularities conjecture for algebraic varieties over a field is considered in several recent papers [7][8][11][45][46][49][50][56] but remains open to this date. Its local variant for valuations is also considered in [47][51][57][63][64][65] but remains equally unsolved. The case of arithmetical schemes has apparently attracted less attention.

The second purpose of this article is to explore the Resolution of Singularities Conjecture as formulated by A. Grothendieck [37](7.9.6). The text includes numerous examples and prospective remarks aimed at preparing the ground for further research in higher dimension. For this purpose, we consider finite morphisms $\eta : \mathcal{X} \to \mathrm{Spec} S$, where S is an arbitrary excellent regular local ring. A test case for Resolution if S has positive characteristic $p > 0$ is when η is purely inseparable; this was already recognized by O.Zariski [70] p.88 and S. Abhyankar [5] and recently confirmed by M. Temkin's purely inseparable Local Uniformization Theorem [65] theorem 1.3.2, *vid.* remark 1.3.5 (iii). In residue characteristic $p > 0$, we also include Galois coverings of degree p to this test case, *vid.* assumption (ii) below. The main step in proving theorem 1.1 consists in proving:

Theorem 1.4. *Let (S, m_S, k) be an excellent regular local ring of dimension $n = 3$, quotient field $K := QF(S)$ and residue characteristic* $\mathrm{char} k = p > 0$. *Let*

$$h := X^p + f_1 X^{p-1} + \cdots + f_p \in S[X], \ f_1, \ldots, f_p \in S \tag{1.1}$$

be a reduced polynomial, $\mathcal{X} := \mathrm{Spec}(S[X]/(h))$ and $L := \mathrm{Tot}(S[X]/(h))$ be its total quotient ring. Assume that h satisfies one of the following assumptions:

(i) $\mathrm{char} K = p$ *and* $f_1 = \cdots = f_{p-1} = 0$, *or*

(ii) $\mathcal{X}$ is G-invariant, where $G := \mathrm{Aut}_K(L) = \mathbb{Z}/p$.

Let μ be a valuation of L which is centered in m_S. There exists a composition of local Hironaka-permissible blowing ups:

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{1.2}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that $(\mathcal{X}_r, x_r)$ is regular.

We develop an approach to the Resolution of Singularities Conjecture for hypersurface singularities defined by an equation (1.1) such that (i) or (ii) holds (condition **(G)** in the text) *in any dimension* $n := \dim S \geq 1$. No other assumption on S is required here than excellence of S; we do not even assume that $[k : k^p] < +\infty$ as suggested by A. Grothendieck *loc.cit.* An extra condition **(E)** on η (definition 2.11) is also assumed: (i) the image in $\mathrm{Spec}S$ of the locus $\mathrm{Sing}_p\mathcal{X}$ of multiplicity p, or (ii) the discriminant locus of $\mathcal{X} \to \mathrm{Spec}S$ is contained in a normal crossings divisor E; when S has characteristic zero (so (ii) holds), E has characteristic p. This condition **(E)** can be achieved by preparatory blowing ups in dimension three (corollary 4.13), applying known Resolution theorems for two-dimensional schemes.

The basic structure we work with is the triple (S, h, E) thus defined. The main combinatorial data attached with the singularity $\mathcal{X}$ is a *characteristic polyhedron* [42][28]:

$$\Delta_S(h; u_1, \ldots, u_n; Z) \subseteq \mathbb{R}^n_{\geq 0}, \tag{1.3}$$

where $Z := X - \phi$, $\phi \in S$, is a linear coordinate change minimizing this polyhedron (beginning of chapter 2).

Resolution for hypersurface singularities in residue characteristic zero uses two primary invariants: the multiplicity function $x \mapsto m(x)$ and the (normalized) slope function $x \mapsto \epsilon(x)$. The latter is not well-behaved in residue characteristic $p > 0$: it is in general not a constructible function on $\mathcal{X}$; the pair $(m(x), \epsilon(x))$ in general increases after performing Hironaka-permissible blowing ups. This pair is denoted $(\nu, \tilde{\epsilon})$ for surfaces in [42] p.253.

In contrast, we construct a numerical function (definition 2.16)

$$\iota : \mathcal{X} \to \{1, \ldots, p\} \times \mathbb{N} \times \{1, \geq 2\} : \; x \mapsto (m(x), \omega(x), \kappa(x)) \tag{1.4}$$

which refines the multiplicity function at those points $x \in \mathcal{X}$ such that $m(x) = p$. This function is differential in nature and has "expected" properties: ι is invariant by regular base change $S \subset \tilde{S}$, $\tilde{S}$ excellent (theorem 2.20) and is constructible on $\mathcal{X}$ (corollary 3.12).

Remark 1.1. The differential multiplicity $\omega(x)$ sprouts from Hironaka's $\epsilon(x)$ if one requires invariance by smooth base change, *vid.* theorem 2.20. A difference takes place between (i) the purely inseparable case, and (ii) the Galois case considered in theorem 1.4: eventually ι is uppersemicontinuous in case (i) but only constructible in general in case (ii), *vid.* corollary 3.12 and following example 3.3.

We develop a notion of permissible blowing up for ι refining that of H. Hironaka. Permissible centers $\mathcal{Y} \subset \mathcal{X}$ are of two different kinds (definitions 3.1 and 3.2), first kind being "ϵ-constant". They also extend to permissible centers under regular base change (theorem 3.4). The function ι is nonincreasing with respect to permissible blowing ups (theorem 3.6). Differential multiplicities and permissible centers have a similar behavior to adapted multiplicities and permissible blowing ups considered in Resolution of Singularities for differential forms and vector fields [61][12][13][14][53][58] and for toroidalization of morphisms [34][31].

Remark 1.2. Our notion of permissible blowing up also sprouts from Hironaka's ϵ-constant blowing ups if one requires invariance by smooth base change, *vid.* theorem 3.4. Permissibility at a point $y \in \mathcal{X}$ implies permissibility on a nonempty Zariski open subset $\mathcal{U} \subseteq \mathcal{Y} := \overline{\{y\}}$ (theorem 3.11). Example 3.1 shows the relevance of permissible blowing ups of the second kind whenever $\mathcal{X}$ has dimension $n \geq 3$. Section 3.3 includes further results intended to serve as a guideline for $n \geq 4$.

Beginning from chapter 4, dimension $n = 3$ is assumed and we focus on the proof of theorem 1.1. Chapter 4 reduces the proof of theorem 1.1 to that of theorem 1.4 and is adapted from [26] to our arbitrary characteristic context.

The last four chapters contain the technical bulk of this article. In chapter 5, the function κ in (1.4) is refined with values in $\{1, 2, 3, 4\}$. For fixed $\iota(x)$, we attach a generic projection from Spec$S$ to dimension two. In contrast with residue characteristic zero, there is no obvious way to attach a projected two-dimensional structure similar to (S, h, E). This difficulty (no reasonable notion of "maximal contact") seems to be inherent to residue characteristic $p > 0$ and has proved to be quite a match. Our method consists in projecting only the combinatorial structure provided by the characteristic polyhedron given in (1.3), say:

$$\mathbf{p}_2 : \ [\Delta_S(h; u_1, u_2, v; Z) \subseteq \mathbb{R}^3_{\geq 0}] \mapsto [\Delta_2(h; u_1, u_2; v; Z) \subseteq \mathbb{R}^2_{\geq 0}]. \tag{1.5}$$

Here, $\mathbf{p}_2$ is a linear projection and $v := u_3 - \phi_2$, $\phi_2 \in S$, is a linear coordinate change minimizing the image polygon. New combinatorial invariants are associated to the right-hand side polygon; their control under permissible blowing ups eventually leads to a smaller value $\iota(x') < \iota(x)$. This is the content of the projection theorem 5.1 from which theorem 1.4 follows easily

by induction on $\iota(x)$ (corollary 5.2). The strategy follows that of [27] but also contains very substantial improvements:

- the sequence (1.2) which is constructed involves Hironaka-permissible blowing ups only, in contrast with [27]. It does *not* depend on the given valuation μ and can be considered as a version of Hironaka's Local Control (Hironaka's A/B Game, in residue characteristic zero) for equations (1.1). Precise statements use the notion of independent sequence (definition 2.18) and theorem 5.1 is stated in these terms. The authors hope that theorem 1.4 could be extended to a Resolution of Singularities $\pi : \mathcal{X}' \to \mathcal{X}$, π a composition of Hironaka-permissible (global) blowing ups (and with G-invariant centers under assumption (ii)).

- all resolution invariants used in this text are defined in terms of initial form polynomials $\mathrm{in}_\sigma h$ w.r.t. certain faces σ of the characteristic polyhedron attached to h. Furthermore, these initial form polynomials provide control for the invariants under blowing up. These facts are the main reason why our proof is characteristic free: $\mathrm{in}_\sigma h$ is a polynomial with coefficients in the residue field $k(x)$. They are also the reason why the extra assumption $[k(x) : k(x)^p] < +\infty$ is not required in the proof.

- the role played by small residue characteristics is very minor (essentially the extra twist in lemma 7.27 for $p = 2$). Difficulties caused by nonperfect residue fields $k(x)$ appear mostly technical in nature, because one is led to carry along (absolute) p-bases $(\lambda_l)_{l\in\Lambda_0}$ in the construction (section 2.4). Nontrivial issues are related to regular base change (proposition 2.5, theorem 2.20 and theorem 3.4), the Hilbert-Samuel stratum (proposition 2.15) and Zariski closure of formal centers (proposition 3.8) in arbitrary dimension $n \geq 1$. For $n = 3$, *vid.* remark 2.4, proposition 5.3 and section 7.5; real difficulties come from lemma 7.15(3)(3') for inseparable extensions of degree $d = p = 2$.

The proof of theorem 5.1 is spread along chapters 6 ($\kappa(x) = 1$), 7 ($\kappa(x) = 2$), 8 and 9 ($\kappa(x) = 3, 4$). Chapter 9 uses blowing ups along Hironaka-permissible curves which are not necessarily of the first or second kind. The authors do not know if such blowing ups are required in general in order to achieve Resolution (in contrast with permissible blowing ups of the second kind, *vid.* example 3.1). They do not appear in [19].

Quoting H. Hironaka's euphemism from [42] p.254: "in the case of dimension 3 or more, the behavior of [the characteristic polyhedron] appears to be far more complicated and has not yet been fully investigated [...] a little experiments lead us to an aphorism: Reduction of singularities is sharpening of polyhedra."

When the hypersurface singularity $\mathcal{X}$ has dimension 3 and satisfies the assumptions of theorem 1.4, our results give a precise content to this aphorism:

(1) the numerical character $\iota(x) = (m(x), \omega(x), \kappa(x))$ is attached to the initial form polynomial $\mathrm{in}_{m_S} h$ w.r.t. the initial face of the characteristic polyhedron;

(2) permissible blowing ups produce a smaller value $\iota(x')$, or a monic form for the new initial $\mathrm{in}_{m_{S'}} h'$, with $(m(x'), \omega(x')) = (m(x), \omega(x))$. This monic form corresponds to a certain vertex $\mathbf{v}'$ of the characteristic polyhedron;

(3) projecting from $\mathbf{v}'$ produces a characteristic *polygon* with numerical character $\gamma(x') \in \mathbb{N}$;

(4) further Hironaka-permissible blowing ups either produce a smaller value $\iota(x'') < \iota(x)$, or achieve

$$\iota(x'') = \iota(x'),\ \mathrm{in}_{m_{S''}} h'' \text{ in monic form with } \gamma(x'') < \gamma(x').$$

Acknowledgement: the authors acknowledge many stimulating discussions held during the "Fall School on Resolution of Threefolds in Positive Characteristic", University of Regensburg, October 1-11/2013. They hereby thank H. Kawanoue, S. Perlega, S. Saito, M. Spivakovsky, A. Voitovitch, A. Weber and J. Włodarczyk for numerous questions and suggestions, with very special thanks to the organizers U. Jannsen and B. Schober.

1.1 Overview of the content and proof of theorem 1.1.

This article is organized as follows: in chapter 2, we introduce our main tool which is the Hironaka Characteristic Polyhedron [42] (definition 2.1). This is performed for any polynomial equation

$$h := X^m + f_{1,X} X^{m-1} + \cdots + f_{m,X} \in S[X],\ f_{1,X}, \ldots, f_{m,X} \in S$$

where S is an excellent regular local ring of dimension $n \geq 1$.

Our notation $\Delta_S(h; \{u_j\}_{j\in J}; X)$ for polyhedra (definition 2.1) slightly differs from Hironaka's because we focus our attention on the *variation* of the characteristic polyhedron along regular subschemes

$$W := (\{u_j\}_{j\in J}) \subseteq \mathrm{Spec} S,\ J \subseteq \{1, \ldots, n\}.$$

To a given face $\sigma = \sigma_\alpha$ defined by a weight vector $\alpha \in \mathbb{R}^n_{\geq 0}$, an initial form polynomial $\mathrm{in}_\alpha h$ is attached (definition 2.2). Proposition 2.4 is imported from [28] and is an essential tool for studying these variations along W. It states that $\Delta_S(h; u_1, \ldots, u_n; X) \subseteq \mathbb{R}^n_{\geq 0}$ can be made minimal by a suitable linear coordinate change $Z := X - \phi$, $\phi \in S$. Denote

$$\mathcal{X} := \mathrm{Spec}(S[Z]/(h)),\ \eta : \mathcal{X} \longrightarrow \mathrm{Spec} S.$$

If $x \in \eta^{-1}(m_S)$ is a point of multiplicity $m(x) = m$, then

$$\eta^{-1}(m_S) = \{x\},\ k(x) = S/m_S.$$

Hironaka's slope for $\Delta_S(h; u_1, \ldots, u_n; Z)$ is denoted by $\delta(x) \geq 1$ when this polyhedron is minimal (proposition 2.3 and definition 2.5).

Assume that a reduced normal crossings divisor

$$E = \mathrm{div}(u_1 \cdots u_e) \subseteq \mathrm{Spec} S \tag{1.6}$$

is specified. Well adapted coordinates $(u_1, \ldots, u_n; Z)$ are coordinates such that (1.6) holds and $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal (definition 2.8). Relevant numerical data are defined for well adapted coordinates only. For such coordinates, h has weights

$$d_j := \min\{x_j : (x_1, \ldots, x_n) \in \Delta_S(h; u_1, \ldots, u_n; Z)\},\ 1 \leq j \leq e.$$

When $m = p$, assumptions (i) or (ii) of theorem 1.4 (condition **(G)** in the text) and **(E)** (definition 2.11) imply that

$$p\delta(x),\ H_j := pd_j \in \mathbb{N} \text{ (corollary 2.12)} \tag{1.7}$$

and provide the structure theorem 2.14 for the initial form polynomials $\mathrm{in}_\alpha h$ with respect to its compact faces (definition 2.2). This fact allows us to reproduce part of the equicharacteristic $p > 0$ constructions used in [27].

Note that E is always assumed to be equicharacteristic $p > 0$ (definition 2.11).

For example when $\alpha = \mathbf{1} := (1, \ldots, 1)$, $\sigma_{\mathbf{1}}$ is the *initial face* of the polyhedron $\Delta_S(h; u_1, \ldots, u_n; Z)$; the corresponding homogeneous polynomial

$$\mathrm{in}_{\mathbf{1}} h \in G(m_S)[Z],\ G(m_S) := \mathrm{gr}_{m_S} S \simeq k(x)[U_1, \ldots, U_n]$$

(denoted by $\mathrm{in}_{m_S} h$ in the text) has degree $p\delta(x)$, setting $\deg Z := \delta(x)$. Theorem 2.14 can be stated as follows: assume that $\Delta_S(h; u_1, \ldots, u_n; Z)$ is *not* an orthant with vertex in $\mathbb{R}^e$ ($\epsilon(x) \neq 0$ in the text); then

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z]. \tag{1.8}$$

Let $H := \prod_{j=1}^{e} U_j^{H_j} \in G(m_S)$ with notations as in (1.7). We denote (definition 2.9):

$$\epsilon(x) := \deg(\mathrm{in}_{m_S} h) - \deg H = p\delta(x) - \sum_{j=1}^{e} H_j \in \mathbb{N}.$$

This leads us to define the function ι in (1.4) (definition 2.16). The function ω is a differential version of Hironaka's ϵ-function [42] and requires introducing a differential structure (S, h, E) adapted to the normal crossings divisor $E \subset \mathrm{Spec} S$ (section 2.4). This is done by considering the $G(m_S)$-module $\Omega^1_{G(m_S)/\mathbb{F}_p}(\log U_1 \cdots U_e)$ of absolute logarithmic differentials and its dual space of derivatives $\mathcal{D}(m_S)$. The derivatives

$$H^{-1} \frac{\partial}{\partial Z},\ \{H^{-1} D\}_{D \in \mathcal{D}(m_S)} \tag{1.9}$$

act on $\mathrm{in}_{m_S} h$. If $G = 0$, we simply let $\kappa(x) \geq 2$, *vid.* (1.4), and

$$\omega(x) := \begin{cases} \epsilon(x) & \text{if } \frac{\partial F_{p,Z}}{\partial U_j} = 0,\ e+1 \leq j \leq n \\ \\ \epsilon(x) - 1 & \text{otherwise} \end{cases}. \tag{1.10}$$

If $G \neq 0$, the definition is more delicate but only relies on elementary linear algebra. In any case, we have

$$(\omega(x) = \epsilon(x),\ \kappa(x) = 1) \text{ or } (\omega(x) = \epsilon(x) - 1,\ \kappa(x) \geq 2). \tag{1.11}$$

Another important notion is that of the affine cone $\mathrm{Max}(x)$ and affine space $\mathrm{Dir}(x)$ (definition 2.17). These are respectively the stratum and directrix of the space of forms of degree $\omega(x)$ obtained by applying those derivatives in (1.9). Once again, the definition is more delicate when $G \neq 0$ but elementary in nature. For applications to dimension three, we always have $\mathrm{Max}(x) = \mathrm{Dir}(x)$, *vid.* remark 2.4.

When $\omega(x) = 0$ in (1.4), a simple combinatorial blowing up algorithm (similar to residue characteristic zero) makes the value of the multiplicity function smaller than p at all points of the blown up space mapping to x (theorem 2.23). There remains to deal with points $x \in \mathcal{X}$ such that $m(x) = p$, $\omega(x) > 0$.

Chapter 3 develops a notion of permissible blowing up $\pi : \mathcal{X}' \to \mathcal{X}$ which refines that of H. Hironaka. Roughly speaking, a Hironaka permissible center $\mathcal{Y} \subset \mathcal{X}$ is permissible in our sense if $\mathcal{X}$ is "differentially equimultiple" along $\mathcal{Y}$ (definition 3.1 and definition 3.2). The notion is somewhat subtle but has good properties, the main result being theorem 3.6: ι is nonincreasing along permissible blowing ups. Furthermore, ι decreases except possibly at exceptional points $x' \in \pi^{-1}(x)$ belonging to some embedded projective cone

$$PC(x, \mathcal{Y}) \subset \pi^{-1}(x)$$

given in definition 3.3. The cone $PC(x, \mathcal{Y})$ is the projectivization of a certain cone containing $\mathrm{Max}(x)$ and coincides with it when $\omega(x) = \epsilon(x)$. We also mention:

- persistence of permissibility under regular base change (theorem 3.4);
- the strict transform $\mathcal{Z}' \subset \mathcal{X}'$ of a permissible center $\mathcal{Z} \subset \mathcal{X}$ under a permissible blowing up π with center $\mathcal{Y} \subset \mathcal{Z}$ is permissible (theorem 3.7);
- the support of a formal arc can be made permissible at its special point by performing permissible blowing ups (proposition 3.8);
- Hironaka permissible centers are permissible in a dense open subset of their support (theorem 3.11).

Remark 1.3. Example 3.2 points out a substantial difference between permissibility for ι and Hironaka-permissibility when $n \geq 4$. It states that the support $\mathcal{Z} \subseteq \mathcal{X}$ of a formal arc cannot in general be made permissible for

ι at its special point x by iterated quadratic transforms. This phenomenon also occurs for $n = 3$ but only for $\omega(x) = 1$; it is then easily dealt with.

The section concludes with the constructibility on $\mathcal{X}$ of the function ι (corollary 3.12). Dimension $n = 3$ is assumed in the next chapters.

Chapter 4 contains what can be deduced from known Embedded Resolution results in excellent regular threefolds. We also adapt some of the equal characteristic $p > 0$ material from [26] to our arbitrary characteristic context and prove:

(4.1) reduction of theorem 1.1 to its Local Uniformization form along valuations;

(4.2) reduction of Local Uniformization to theorem 1.4;

(4.3) the normal crossings condition **(E)** can be achieved (corollary 4.13).

Chapter 5 collects together all previous results. A projection number $\kappa(x) \in \{1, 2, 3, 4\}$ (definition 5.1) is associated to a singular point $x \in \mathcal{X}$ such that $m(x) = p, \omega(x) > 0$. This function basically expresses the transverseness or tangency of the initial form (1.8) of the characteristic polyhedron with respect to the initial face. For convenience of the reader, we give *a sample* of the main types of initial form polynomials occurring when $E = \mathrm{div}(u_1)$; we take $\omega(x) > 0$, $\lambda \in k(x)$ and all exponents are integers in these formulæ. Furthermore, we have $\lambda \neq 0$, $\lambda \notin k(x)^p$ if

$$(d_1, \omega(x)/p) \in \mathbb{N}^2 \text{ (resp. if } d_1 + \omega(x)/p \in \mathbb{N})$$

in the second (resp. fifth) formula:

$$\mathrm{in}_{m_S} h = \begin{cases} Z^p - \left(\lambda U_1^{d_1 + \frac{\omega(x)}{p}}\right)^{p-1} Z & & \kappa(x) = 1 \\ Z^p + \lambda U_1^{pd_1} U_3^{\omega(x)} & \omega(x) \equiv 0 \mathrm{mod} p & \kappa(x) = 2 \\ Z^p + \lambda U_1^{pd_1} U_2 U_3^{\omega(x)} & \omega(x) \equiv 0 \mathrm{mod} p & \kappa(x) = 2 \\ Z^p + \lambda U_1^{pd_1} U_3^{1+\omega(x)} & 1 + \omega(x) \not\equiv 0 \mathrm{mod} p & \kappa(x) = 3 \\ Z^p + \lambda U_1^{pd_1 + \omega(x)} & & \kappa(x) = 4 \\ Z^p + \lambda U_1^{pd_1 + \omega(x)} U_2 & & \kappa(x) = 4 \end{cases}$$

The complete definition of $\kappa(x)$ takes into account all possible $\mathrm{in}_{m_S}h$ and E which may occur. The simpler forms listed above are "monic forms" in the sense that a certain monomial computing $\omega(x)$ occurs in $\mathrm{in}_{m_S}h$. We now explain these definitions and the hierarchy between them: for fixed $\omega(x)$, the singularity is considered as milder as $\kappa(x)$ decreases. To begin with, $\omega(x)$ is computed from $\mathrm{in}_{m_S}h$ by applying certain derivatives (1.9)-(1.11).

• when this derivative is transverse to the base Spec$S$, i.e. applying $H^{-1}\frac{\partial}{\partial Z}$ in (1.9), we set $\kappa(x) = 1$; otherwise $\kappa(x) \geq 2$.

• when $\kappa(x) \geq 2$, we set $\kappa(x) = 4$ if the *directrix* affine space $\mathrm{Dir}(x)$ has equations in $U_1, \ldots, U_e$, i.e. in those coordinates corresponding to E. Otherwise, $\mathrm{Dir}(x)$ has an equation which is transverse to E, say $U_3 = 0$ with $e = 1$ or $e = 2$. The very transverse case $\kappa(x) = 2$ means that a derivative transverse to U_3 is involved in (1.9), i.e. a derivative w.r.t. another variable U_1, U_2 or to a constant in $k(x)$:

$$D = H^{-1}U_1\frac{\partial}{\partial U_1},\ D = H^{-1}\frac{\partial}{\partial U_2}\ (e = 1),\ \text{or}\ D = H^{-1}\frac{\partial}{\partial \lambda}.$$

Theorem 5.1 states that $\iota(x)$ can be made smaller by performing local Hironaka permissible blowing ups. Theorem 1.4 then follows easily by descending induction on $\iota(x)$.

The proof of theorem 5.1 is very long and intricate. For $\kappa(x) = 1$ (resp. $2, 3, 4$), the proof is given in corollary 6.2 (resp. theorem 7.18, theorem 9.6, *ibid.*). Three main phenomena are responsible for these intricacies:

(i) no obvious way shows up for reducing theorem 5.1 for (S, h, E) to some statement on the *coefficients* of the polynomial h. When this is possible (for $\kappa(x) = 1$ and in part for $\kappa(x) = 3, 4$), the proofs are notably simplified. This is done in section 6 where some weak form of maximal contact with a component of E is assumed for ι.

(ii) reducing theorem 5.1 to the "monic forms" corresponding to $\kappa(x)$ is achieved by a casuistic analysis which seems for the moment out of reach in higher dimensions. Sections 7.2, 8.3 and part of 8.1, 8.2 are concerned with this problem.

(iii) blowing up a monic form along a permissible center (e.g. a closed point) may lead to a bigger value $\iota(x') = (p, \omega(x), 4) > \iota(x)$ when

$\kappa(x) = 2, 3$. These situations are also dealt with by a casuistic analysis whose extension to higher dimensions seems out of reach. Section 7.1 and part of 8.1, 8.2 are concerned with this problem.

Chapter 6 proves theorem 5.1 for sequences of permissible blowing ups with centers lying inside a fixed irreducible component of E. This proves theorem 5.1 in the case $\kappa(x) = 1$ and prepares the ground in the cases $\kappa(x) = 3, 4$. The proof is similar to that of Resolution for excellent surfaces [42][16][17], but does not follow from it.

Chapter 7 proves theorem 5.1 when $\kappa(x) = 2$. The above phenomenon (iii) is studied in section 7.1. The proofs are essentially the same as in [27] chapter 2.**II** except that all statements and proofs are phrased only in terms of initial form polynomials $\mathrm{in}_\alpha h$ w.r.t. certain faces σ_α of $\Delta_S(h; u_1, u_2, u_3; Z)$. Section 7.2 defines the "monic forms" (definition 7.1) and deals with the above phenomenon (ii) in proposition 7.8.

No obvious reduction to Resolution for surfaces is available (phenomenon (i)). The proof then follows our strategy as indicated at the end of the previous section (3) and (4). Section 7.3 builds up the projected polygon $\Delta_2(h; u_1, u_2; v; Z)$ of (1.5) (theorem 7.12) and defines secondary numerical invariants (definition 7.4). The main invariant is denoted by $\gamma(x) \in \mathbb{N}$. Two main difficulties arise here: rationality over S (i.e. v can be chosen in S and not only in $\hat{S}$), and independence of choices of coordinates. Section 7.4 studies the behavior of the invariants under blowing up a closed point. Finally, section 7.5 proves that permissible blowing ups produce some point x' with $\iota(x') \leq (p, \omega(x), 1)$ (theorem 7.18). The algorithm blows up permissible curves only when $\gamma(x) = 0, 1$.

Chapters 8 and 9 prove theorem 5.1 for $\kappa(x) = 3, 4$. Since only Hironaka-permissible centers are used, this chapter contains many new features in comparison with the corresponding [27] chapter 3.**II**. Definition 8.1 states what is required of the "monic forms", called respectively (**) ($\kappa(x) = 3, 4$) and (T**) ($\kappa(x) = 4$). Phenomenon (iii) seems to be untractable here and is the reason for these stronger conditions imposed on h. Reduction to these monic forms is harder than in chapter 7 and is spread along sections 8.1, 8.2 and 8.3 (propositions 8.6 and 8.8).

Section 9.2 reduces a monic form (T**) to (**) or to $\kappa(x) \leq 2$ (proposition 9.1). The proof is an application of theorem 6.1 since a weak form of maximal contact with a component of E holds for this reduction. Section 9.3 finally proves that monic forms (**) can be reduced to $\kappa(x) \leq 2$ (proposition 9.5).

When $\omega(x) \geq p$, this reduction is achieved by blowing up along Hironaka-permissible curves, not necessarily permissible of the first or second kind, but contained in the locus

$$\Omega_+(\mathcal{X}) := \{y \in \mathcal{X} : \omega(y) > 0\}.$$

In order to ensure Hironaka-permissibility, the condition $E = \eta(\mathrm{Sing}_p\mathcal{X})$ is required (section 9.3.1, condition (**E'**) in the text). Section 9.3.2 builds up the projected polygon $\Delta_2(h; u_1, u_2; v; Z)$ (definition 9.3 and proposition 9.11) and defines secondary numerical invariants (definition 9.4). Said blowing ups along Hironaka-permissible curves are performed mostly in propositions 9.14 and 9.16.

2 Adapted structure and primary invariants.

All along this article, we will denote by S a regular local ring of arbitrary dimension $n \geq 1$, and by $(u_1, \ldots, u_n)$ a regular system of parameters (r.s.p. for short) of S. Its maximal ideal is denoted by $m_S := (u_1, \ldots, u_n)$ and its formal completion w.r.t. m_S by $\hat{S}$. The order function ord_{m_S} on S is defined by:

$$\mathrm{ord}_{m_S} f := \sup\{n \in \mathbb{N} : f \in m_S^n\} \in \mathbb{N} \cup \{+\infty\},\ f \in S.$$

This order function extends to a discrete valuation on the quotient field $K := QF(S)$ of S.

We will assume that $\mathrm{char}(S/m_S) > 0$ except for the next three sections. We also assume that S is *excellent* beginning from proposition 2.4 on. The basic reference for excellent rings is [37] 7.8 and 7.9. A useful *compendium* is [54] pp. 255-260; some extensions and examples of non excellent regular local rings can be found in [47] pp. 7-22. Let

$$h := X^m + f_{1,X}X^{m-1} + \cdots + f_{m,X} \in S[X],\ f_{1,X}, \ldots, f_{m,X} \in S \tag{2.1}$$

be a unitary polynomial of degree $m \geq 2$. We denote by

$$\mathcal{X} := \mathrm{Spec}(S[X]/(h)) \text{ and } \eta :\ \mathcal{X} \longrightarrow \mathrm{Spec}S \tag{2.2}$$

respectively the corresponding hypersurface and induced projection.

The total ring of fractions $\mathcal{X}$ is denoted by $L := \mathrm{Tot}(S[X]/(h))$. Given a point $y \in \mathcal{X}$, its residue field is denoted by $k(y)$ and its multiplicity by $m(y)$. Explicitly, we have:

$$m(y) = \mathrm{ord}_{m_{S[X]_y}} h.$$

The singular locus of $\mathcal{X}$ is denoted by :

$$\mathrm{Sing}\mathcal{X} = \{y \in \mathcal{X} : m(y) \geq 2\}.$$

The *locus of multiplicity* m of $\mathcal{X}$ is viewed as an embedded reduced subscheme of $\mathcal{X}$:

$$\mathrm{Sing}_m\mathcal{X} := \{y \in \mathrm{Spec}(S[X]) : \mathrm{ord}_{m_{S[X]_y}} h = m\} \subseteq \mathrm{Sing}\mathcal{X}.$$

Both of $\mathrm{Sing}\mathcal{X}$ and $\mathrm{Sing}_m\mathcal{X}$ are proper closed subsets of $\mathcal{X}$ if S is excellent.

Given a "linear change of" (one also says "translation on") the X-coordinate, say $X' := X - \phi$, $\phi \in \hat{S}$, we still denote by

$$h = X'^m + f_{1,X'}X'^{m-1} + \cdots + f_{m,X'} \in \hat{S}[X']$$

the corresponding expansion of $h(X' + \phi)$, $f_{1,X'}, \ldots, f_{m,X'} \in \hat{S}$. The explicit formula for this change of coordinate is :

$$f_{i,X'} = \binom{m}{i}\phi^i + \sum_{j=1}^{i}\binom{m-j}{i-j} f_{j,X}\phi^{i-j},\ 1 \leq i \leq m. \tag{2.3}$$

Given $\phi \in S$ and a rational number $d \leq \mathrm{ord}_{m_S}\phi$, we denote by $\mathrm{cl}_d\phi$ the *initial form* of ϕ in $\mathrm{gr}_{m_S}S \simeq S/m_S[U_1, \ldots, U_n]$ (resp. the null form) if $d = \mathrm{ord}_{m_S}\phi$ (resp. otherwise). Similarly, if $I \subseteq S$ and $d \leq \mathrm{ord}_{m_S}I$, we denote

$$\mathrm{cl}_d I := \mathrm{Vect}(\{\mathrm{cl}_d\phi\}_{\phi \in I}) \subseteq S/m_S[U_1, \ldots, U_n]_d.$$

Suppose that a weight vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0}$ is given. Let $\Gamma_\alpha := \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n \subset \mathbb{R}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$, denote

$$\mid \mathbf{x} \mid_\alpha := \alpha_1 x_1 + \cdots + \alpha_n x_n \in (\Gamma_\alpha)_{\geq 0}.$$

An associated valuation μ_α of K is defined by setting for $f \in S$, $f \neq 0$:

$$\mu_\alpha(f) := \max\{a \in \Gamma_\alpha : f \in I_\alpha(a) := (\{u_1^{x_1} \cdots u_n^{x_n} :\mid \mathbf{x} \mid_\alpha \geq a\})\}.$$

It easily follows from the Noetherianity of S that $\mu_\alpha(f)$ is well defined. One sets

$$\mu_\alpha(f/g) := \mu_\alpha(f) - \mu_\alpha(g) \text{ for } f, g \in S, fg \neq 0.$$

Note that $\mathrm{ord}_{m_S} = \mu_{\mathbf{1}}$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n_{>0}$. We will systematically use the graded ring $\mathrm{gr}_\alpha S$ of S w.r.t. μ_α:

$$\mathrm{gr}_\alpha S \simeq S/(\{u_i : \alpha_i > 0\})[\{U_i : \alpha_i > 0\}].$$

If $a \in \Gamma_\alpha$ and $\phi \in S$ is given with $a \leq \mu_\alpha(\phi)$, its initial form $\mathrm{cl}_{\alpha,a}\phi \in \mathrm{gr}_\alpha S$ is defined as before. Similarly, if $I \subset S$ and $a \leq \mu_\alpha(I)$, we associate a $(\mathrm{gr}_\alpha S)_0$-module denoted by

$$\mathrm{cl}_{\alpha,a} I := \mathrm{Span}(\{\mathrm{cl}_{\alpha,a}\phi\}_{\phi \in I}) \subseteq (\mathrm{gr}_\alpha S)_a.$$

2.1 Characteristic polyhedron and first invariants.

Given an equation $h \in S[X]$ (2.1) and a r.s.p. $(u_1, \ldots, u_n)$ of S, there is an associated Newton polyhedron w.r.t.the variables $(u_1, \ldots, u_n, X)$:

$$NP(h; u_1, \ldots, u_n; X) \subseteq \mathbb{R}^{n+1}_{\geq 0}.$$

Let $P := (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}_{\geq 0}$, so $P \in \frac{1}{m} NP(h; u_1, \ldots, u_n; X)$, and

$$\mathbf{p} : \mathbb{R}^{n+1} \setminus \{P\} \longrightarrow \mathbb{R}^n$$

be the projection on the $(u_1, \ldots, u_n)$-space. We define a polyhedron by:

$$\Delta_S(h; u_1, \ldots, u_n; X) := \mathbf{p}\left(\frac{1}{m} NP(h; u_1, \ldots, u_n; X) \cap \{x_{n+1} < 1\}\right) \subseteq \mathbb{R}^n_{\geq 0}.$$

The *characteristic polyhedron* is introduced in a more general context in [42]. In our setting, it consists in minimizing $\Delta_S(h; u_1, \ldots, u_n; X')$ over all linear changes of coordinates $X' = X - \phi$, $\phi \in \hat{S}$ (2.3).

In this section, we review and adapt notations to fit our purposes. A fundamental algebraicity result is borrowed from [28] in proposition 2.4 below. Then some of the invariance properties of the characteristic polyhedron under base change are listed.

Let S and $(u_1, \ldots, u_n)$ be fixed as above. Given a subset $J \subseteq \{1, \ldots, n\}$, we denote by

$$I_J := (\{u_j\}_{j \in J}) \subset S \text{ and } \overline{S}^J := S/I_J.$$

We also use the notation $s^J \in \mathrm{Spec} S$ to denote the point $s^J = I_J$, reserving the idealistic notation I_J to commutative algebraic formulæ.

Proposition 2.1. *Let $f \in S$. There exists a unique finite set $\mathbf{S}^J(f) \subset \mathbb{N}^J$ such that the following holds:*

(i) the set of monomials $\{\prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \in \mathbf{S}^J(f)\}$ forms a minimal system of generators of the ideal

$$I(f) := \left(\left\{\prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \in \mathbf{S}^J(f)\right\}\right);$$

(ii) there is an expansion

$$f = \sum_{\mathbf{a}\in\mathbf{S}^J(f)} \gamma(f,\mathbf{a}) \prod_{j\in J} u_j^{a_j} \in S,\ \gamma(f,\mathbf{a}) \in S \tag{2.4}$$

such that $\gamma(f,\mathbf{a}) \notin I_J$ for every $\mathbf{a} \in \mathbf{S}^J(f)$.

Proof. Let $\widehat{S}^J$ be the formal completion of S *along* I_J. Since $I_J \subseteq m_S$, $\widehat{S}^J$ is faithfully flat over S [54] theorem 8.14(3). Thus $I\widehat{S}^J \cap S = I$ for any ideal $I \subseteq S$, in particular for any monomial ideal in $\{u_j\}_{j\in J}$. One deduces that property (i) and existence of an expansion (2.4) descend from $\widehat{S}^J$ to S.

Suppose that an expansion (2.4) exists for a given $\mathbf{S}^J(f)$ satisfying (i). Each S/I_J^{n+1}, $n \geq 0$ has a structure of free $\overline{S}^J$-module with basis

$$\left\{\prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \text{ and } \sum_{j\in J} a_j \leq n\right\}.$$

Therefore the class $\gamma(f,\mathbf{a}) + I_J$ is independent of the chosen expansion (2.4) by the minimality property in (i). This proves that the property $\gamma(f,\mathbf{a}) \notin I_J$ in (ii) also descends from $\widehat{S}^J$ to S. In other terms, we may assume that S is I_J-adically complete.

Independent monomial generators in S/I_J^n lift to independent monomial generators in S/I_J^{n+1} for every $n \geq 1$. One easily deduces the existence of an expansion (ii) satisfying (i) for some finite subset $\mathbf{S}^J(f) \subset \mathbb{N}^J$, since S is I_J-adically complete and Noetherian.

Uniqueness of $\mathbf{S}^J(f)$ is also checked by taking images in S/I_J^{n+1} for some $n >> 0$. $\square$

Definition 2.1. (Associated Polyhedron). Given an equation $h \in S[X]$ (2.1) and $J \subseteq \{1, \ldots, n\}$, we define a rational polyhedron:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) := \mathrm{Conv}\left(\bigcup_{i=1}^{m} \bigcup_{\mathbf{a}\in \mathbf{S}^J(f_{i,X})} \left\{\frac{\mathbf{a}}{i} + \mathbb{R}^J_{\geq 0}\right\}\right) \subseteq \mathbb{R}^J_{\geq 0}.$$

Definition 2.2. (Initial forms). Let $\alpha = (\{\alpha_j\}_{j\in J}) \in \mathbb{R}^J_{>0}$ be a weight vector. We define

$$\delta_\alpha(h; \{u_j\}_{j\in J}; X) := \min\{|\ \mathbf{x}\ |_\alpha \colon \mathbf{x} \in \Delta_S(h; \{u_j\}_{j\in J}; X)\}.$$

The weight vector defines a *compact face* σ_α of $\Delta_S(h; \{u_j\}_{j\in J}; X)$ by:

$$\sigma_\alpha := \{\mathbf{x} \in \Delta_S(h; \{u_j\}_{j\in J}; X) : |\ \mathbf{x}\ |_\alpha = \delta_\alpha(h; \{u_j\}_{j\in J}; X)\}.$$

The *initial form* $\mathrm{in}_\alpha h$ of h w.r.t. α is the polynomial

$$\mathrm{in}_\alpha h := X^m + \sum_{i=1}^{m} F_{i,X,\alpha} X^{m-i} \in (\mathrm{gr}_\alpha S)[X], \tag{2.5}$$

where

$$F_{i,X,\alpha} := \sum_{\mathbf{x}\in\sigma_\alpha} \overline{\gamma}(f_{i,X}, i\mathbf{x}) U^{i\mathbf{x}},$$

and bars denotes images in $(\mathrm{gr}_\alpha S)_0 = \overline{S}^J$, i.e.

$$\overline{\gamma}(f_{i,X}, i\mathbf{x}) := \mathrm{cl}_{\alpha,0}\gamma(f_{i,X}, i\mathbf{x}) \in (\mathrm{gr}_\alpha S)_0 = \overline{S}^J.$$

By convention, we take $\overline{\gamma}(f_{i,X}, i\mathbf{x}) = 0$ in these formulæ whenever $i\mathbf{x} \notin \mathbf{S}^J(f_{i,X})$.

Remark 2.1. Any vertex of $\Delta_S(h; \{u_j\}_{j\in J}; X)$ has coordinates in $\frac{1}{m!}\mathbb{N}$. We have:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \emptyset \Leftrightarrow h = X^m.$$

It is worth emphasizing that the polynomial $\mathrm{in}_\alpha h$ only depends on the face σ_α and not on the specific weight vector α defining it. Given h and α, the grading of $\mathrm{gr}_\alpha S$ can be extended to $(\mathrm{gr}_\alpha S)[X]$ by setting:

$$\deg X := \delta_\alpha(h; \{u_j\}_{j\in J}; X).$$

Then $\mathrm{in}_\alpha h$ is a *homogeneous* polynomial of degree $m\delta_\alpha(h; \{u_j\}_{j\in J}; X)$ for this grading.

We now briefly review the behaviour of polyhedra and initial forms under basic operations such as formal completion, localization and projection onto a regular subscheme. The case of regular local morphisms $S \subset \tilde{S}$, $\tilde{S}$ excellent will be considered further on.

With notations as above, let $\alpha \in \mathbb{R}^J_{>0}$ be a weight vector and

$$\sigma_\alpha \subset \Delta_S(h; \{u_j\}_{j\in J}; X), \ \mathrm{in}_\alpha h \in (\mathrm{gr}_\alpha S)[X].$$

Formal Completion: $\hat{S}$ is excellent [37] theorem 7.8.3(iii). Proposition 2.1 and definition 2.1 give an identification

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X). \tag{2.6}$$

This identification preserves the initial form $\mathrm{in}_\alpha h$ for each weight vector α via the inclusion $\mathrm{gr}_\alpha S \subseteq \mathrm{gr}_\alpha \hat{S} \simeq \mathrm{gr}_\alpha S \otimes_S \hat{S}$.
Localization: the regular local ring S_{s^J} is excellent if S is excellent [37] theorem 7.4.4. Similarly, the identifications

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \Delta_{S_{s^J}}(h; \{u_j\}_{j\in J}; X) \tag{2.7}$$

also preserve the initial form $\mathrm{in}_\alpha h$ (2.5) via the inclusion

$$\mathrm{gr}_\alpha S \subseteq \mathrm{gr}_\alpha S_{s^J} \simeq (\mathrm{gr}_\alpha S) \otimes_S QF(\overline{S}^J).$$

Projection: let $J \subseteq \{1, \ldots, n\}$ and denote by $J' := \{1, \ldots, n\} \backslash J$ its complement. The regular local ring $\overline{S}^J$ is excellent if S is excellent. A r.s.p. of $\overline{S}^J$ is $(\{\overline{u}_{j'}\}_{j'\in J'})$, where bars denote images in $\overline{S}^J$. With notations as above, we have:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \mathrm{pr}^J \Delta_S(h; u_1, \ldots, u_n; X), \tag{2.8}$$

where $\mathrm{pr}^J : \mathbb{R}^n \to \mathbb{R}^J, \ \mathbf{x} \mapsto \mathbf{y} = (\{x_j\}_{j\in J})$ denotes the projection. Let

$$f_{i,X} = \sum_{\mathbf{a}\in \mathbf{S}(f_{i,X})} \gamma(f_{i,X}, \mathbf{a}) u_1^{a_1} \cdots u_n^{a_n} \in S,$$

be an expansion (2.4) (for the subset $\{1, \ldots, n\}$, where $\mathbf{S}(f_{i,X})$ here stands for $\mathbf{S}^{\{1,\ldots,n\}}(f_{i,X})$), $1 \le i \le m$. Then (2.5) is given by

$$F_{i,X,\alpha} := \sum_{\mathbf{y}\in\sigma_\alpha} \left(\sum_{\mathrm{pr}^J(\mathbf{x})=\mathbf{y}} \overline{\gamma}(f_{i,X}, i\mathbf{x}) \prod_{j'\in J'} \overline{u}_{j'}^{ix_{j'}} \right) \prod_{j\in J} U_j^{iy_j}, \tag{2.9}$$

where bars denotes images in $(\mathrm{gr}_\alpha S)_0 = \overline{S}^J$ as before (recall that by convention, we take $\overline{\gamma}(f_{i,X}, i\mathbf{x}) := 0$ in this formula if $i\mathbf{x} \notin \mathbf{S}(f_{i,X})$).

Definition 2.3. (Solvable vertices). Let $\mathbf{x} \in \mathbb{R}^J$ be a vertex of the polyhedron $\Delta_S(h; \{u_j\}_{j\in J}; X)$, that is, a 0-dimensional face $\sigma = \{\mathbf{x}\}$. Denote by

$$\mathrm{in}_{\mathbf{x}} h = X^m + \sum_{i=1}^{m} F_{i,X,\mathbf{x}} X^{m-i} \in (\mathrm{gr}_\alpha S)[X]$$

the initial form polynomial (2.5) w.r.t. any defining weight vector α. We will say that $\mathbf{x}$ is solvable if $\mathbf{x} \in \mathbb{N}^J$ and there exists $\overline{\lambda} \in \overline{S}^J$ such that

$$\mathrm{in}_{\mathbf{x}} h = (X - \overline{\lambda} U^{\mathbf{x}})^m.$$

Explicitly, with notations as in (2.5) *sqq.*, the latter equality means that

$$\overline{\gamma}(f_{i,X}, i\mathbf{x}) = (-1)^i \binom{m}{i} \overline{\lambda}^i \in \overline{S}^J, \ 1 \leq i \leq m.$$

Note that $\binom{m}{i} \in \overline{S}^J$ is not a unit in general when $\mathrm{char}(S/m_S) > 0$.

The following result is a rewriting of [42] in this hypersurface situation.

Proposition 2.2. *(Hironaka).* *There exists a linear change of the X-coordinate $Z := X - \theta$, with $\theta \in \hat{S}$, such that*

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \min_{X'} \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X'), \tag{2.10}$$

where the minimum is taken w.r.t. inclusions and over all possible linear changes of coordinates $X' := X - \phi$, $\phi \in \hat{S}$.

Given $X' := X - \phi$, $\phi \in \hat{S}$, $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X')$ achieves equality in (2.10) if and only if it has no solvable vertex.

If S is excellent, there is an equivalence

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \emptyset \Leftrightarrow \exists g \in S : h = (X - g)^m.$$

Proof. This is respectively [42] Hironaka's vertex preparation lemma (3.10) and theorem (4.8), and [28] lemma II.1. □

Definition 2.4. (Characteristic Polyhedron). For $X' := X - \phi$, $\phi \in \hat{S}$, we will say that the polyhedron $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X')$ is minimal if it has no solvable vertex.

Example 2.1. Let p be a prime number and $n \in \mathbb{Z}$ not divisible by p. We take:

$$S := \mathbb{Z}_{(p)} \text{ and } h := X^p - np^a \in S[X],\ a \geq 0.$$

The following holds:

(1) if $a \notin p\mathbb{Z}$, then $\Delta_{\mathbb{Z}_p}(h; p; X) = [a/p, +\infty[$ is minimal;

(2) if $a \in p\mathbb{Z}$, then $\Delta_{\mathbb{Z}_p}(h; p; Z)$ is minimal, where $Z := X - np^{a/p}$ and we have:

$$\Delta_{\mathbb{Z}_p}(h; p; Z) = \begin{cases} [\frac{a+1}{p}, +\infty[& \text{if} \quad n^p - n \notin p^2\mathbb{Z} \\ \\ [\frac{a}{p} + \frac{1}{p-1}, +\infty[& \text{if} \quad n^p - n \in p^2\mathbb{Z} \end{cases}.$$

With notations and conventions as in (2.1) and (2.2), we have the following result in the case $J = \{1, \ldots, n\}$ and $\alpha = \mathbf{1}$ (so $\mu_{\mathbf{1}} = \mathrm{ord}_{m_S}$) [42] [20]:

Proposition 2.3. *The rational number $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z)$ is independent of the r.s.p. $(u_1, \ldots, u_n)$ and $Z = X - \theta$, $\theta \in \hat{S}$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal.*

If $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, the following characterizations hold:

(i) $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z) > 0 \Leftrightarrow (\eta^{-1}(m_S) = \{x\}$ and $k(x) = S/m_S)$;

(ii) $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z) \geq 1 \Leftrightarrow \eta^{-1}(m_S) \cap \mathrm{Sing}_m \mathcal{X} \neq \emptyset$.

Proof. Let $(Z', u'_1, \ldots, u'_n)$ and $(Z, u_1, \ldots, u_n)$ be two systems of coordinates such that both polyhedra $\Delta_{\hat{S}}(h; u'_1, \ldots, u'_n; Z')$ and $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ are minimal. Suppose that $\delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z') > \delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z)$. Then

$$f_{i,Z'}^{m!} \in m_S^{\frac{m!}{i}\delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z')}$$

for each i, $1 \leq i \leq m$, hence

$$\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z') \geq \delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z') > \delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z).$$

This contradicts the assumption $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ minimal. The first assertion follows by symmetry.

Let $\overline{h} \in S/m_S[Z]$ be the reduction of h modulo m_S. Since

$$\eta^{-1}(m_S) = \mathrm{Spec}(S/m_S[Z]/(\overline{h})),$$

(i) and the "only if" part in (ii) are immediate from the definitions. We have

$$\mathrm{ord}_x h(Z) \leq \mathrm{ord}_x \overline{h}(Z) \leq m,$$

hence $x \in \mathrm{Sing}_m \mathcal{X}$ implies $\overline{h}(Z) = (Z - \lambda)^m$ for some $\lambda \in S/m_S$. Since $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, $\mathbf{0} \in \mathbb{R}^n$ is not a solvable vertex and therefore we have $\lambda = 0$. This proves that (i) holds, the "if" part in (ii) being then obvious. $\square$

Definition 2.5. Let $s \in \mathrm{Spec} S$, $(v_1, \ldots, v_{n(s)})$ be a r.s.p. of S_s and $y \in \eta^{-1}(s)$. Let $Z := X - \theta$, $\theta \in \widehat{S}_s$ be such that $\Delta_{\widehat{S}_s}(h; v_1, \ldots, v_{n(s)}; Z)$ is minimal, where $\widehat{S}_s$ denotes the formal completion of S_s w.r.t. its maximal ideal. We let:

$$\delta(y) := \delta_{\mathbf{1}}(h; v_1, \ldots, v_{n(s)}; Z) = \min_{1 \leq i \leq m} \left\{ \frac{\mathrm{ord}_{m_{\widehat{S}_s}} f_{i,Z}}{i} \right\} \in \frac{1}{m!}\mathbb{N}.$$

This invariant is classical and appears in e.g. [15], [16] and [7] definition 4.2 and proposition 4.8 in an equal characteristic context. Our main resolution invariants will be defined in terms of coordinates $(u_1, \ldots, u_n)$ and $Z = X - \theta$, $\theta \in \hat{S}$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. Since minimizing polyhedra involves in principle choosing *formal* coordinates, an *algebraic* version will be useful for proving the constructibility of our invariants. The following proposition is fundamental for this purpose. When $\mathrm{char} S/m_S = 0$, the first statement in the proposition easily follows from proposition 2.2 by applying the Tschirnhausen transformation (take $\theta = -\frac{1}{m} f_{1,X}$ below).

We assume from this point on that S is excellent.

Proposition 2.4. *[28] Given $h \in S[X]$ (2.1) and a r.s.p. $(u_1, \ldots, u_n)$ of S, there exists $Z := X - \theta$, $\theta \in S$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal.*

For any such Z, the following holds: for every subset $J \subseteq \{1, \dots, n\}$, the polyhedron $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ is also minimal and is computed by:

$$\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}^J \Delta_{\hat{S}}(h; u_1, \dots, u_n; Z), \tag{2.11}$$

where $\mathrm{pr}^J : \mathbb{R}^n \to \mathbb{R}^J$, $\mathbf{x} \mapsto \mathbf{y} = (\{x_j\}_{j\in J})$ *denotes the projection. In particular, we have*

$$\delta(y) = \min \left\{ \frac{1}{i} \sum_{j\in J} a_j,\ \mathbf{a} \in \mathbf{S}^{\{1,\dots,n\}}(f_{i,Z}),\ 1 \le i \le m \right\},\ y \in \eta^{-1}(s^J).$$

Proof. The proposition is trivial if $\mathbf{0} \in \mathbb{R}^n$ is a nonsolvable vertex of the polyhedron $\Delta_{\hat{S}}(h; u_1, \dots, u_n; Z)$, taking $Z := X$. Otherwise it can be assumed that $f_{i,X} \in m_S$, $1 \le i \le m$. The first statement is [28] corollary II.4.

Formula (2.11) follows from (2.6) (2.7) (2.8). To prove minimality, suppose that $\mathbf{y} \in \mathbb{N}^J$ is a solvable vertex of $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ defined by some $\alpha \in \mathbb{R}^J_{>0}$. By definition,

$$\exists \overline{\lambda} \in QF(\overline{S}^J) : \mathrm{in}_{\mathbf{y}} h = (Z - \overline{\lambda} U^{\mathbf{y}})^m. \tag{2.12}$$

By (2.9), we have $\overline{\lambda}^m = (-1)^m U^{-m\mathbf{y}} F_{m,Z,\alpha} \in \overline{S}^J$. Hence $\overline{\lambda} \in \overline{S}^J$, since the regular ring $\overline{S}^J$ is integrally closed. By (2.11), there exists a *vertex* $\mathbf{x} \in \Delta_{\hat{S}}(h; u_1, \dots, u_n; Z)$ such that $\mathbf{y} = \mathrm{pr}^J(\mathbf{x})$. Lifting up, there exists $\beta \in \mathbb{R}^n_{>0}$, $\alpha = \mathrm{pr}^J(\beta)$ defining $\mathbf{x}$, and we let $\alpha' := \mathrm{pr}^{J'}(\beta)$. There is an induced valuation $\mu_{\alpha'}$ on $\overline{S}^J$. The initial form of $\overline{\lambda}$ in $\mathrm{gr}_{\alpha'} \overline{S}^J$ has the form

$$\Lambda = \lambda \prod_{j'\in J'} \overline{U}_{j'}^{x_{j'}},\ \lambda \in S/m_S,\ \lambda \neq 0,\ \{x_{j'}\}_{j'\in J'} \in \mathbb{N}^{J'}.$$

Collecting together (2.9) and (2.12), we get $\mathrm{in}_{\mathbf{x}} h = (Z - \lambda U^{\mathbf{x}})^m$, i.e. $\mathbf{x}$ is a solvable vertex: a contradiction. Therefore $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ has no solvable vertex, hence is minimal by the second statement in proposition 2.2. The last statement is a rewriting of definition 2.5. □

Remark 2.2. This proposition allows us to skip the reference to formal completion when stating that a certain polyhedron is minimal, i.e. given $Z := X - \phi$, $\phi \in S$, the statement "$\Delta_S(h; u_1, \dots, u_n; Z)$ is minimal" stands for "$\Delta_{\hat{S}}(h; u_1, \dots, u_n; Z)$ is minimal". On the other hand, we will keep the reference to the regular local ring S since we are also interested in base change.

Let $S \subseteq \tilde{S}$ be a *local* base change which is *regular*, i.e. flat with geometrically regular fibers [37] definition 6.8.1(iv). In particular $\tilde{S}$ is regular [37] proposition 6.5.1(ii) and faithfully flat. The ring $\tilde{S}$ is not excellent in general, but this certainly holds in the following cases:

(i) $\tilde{S} = \hat{S}$ [37] 7.8.3(iii);

(ii) $\tilde{S}$ is ind-étale over S [47] theorem I.8.1(iv), or

(iii) $\tilde{S}$ is essentially of finite type over S, i.e. smooth over S [37] proposition 7.8.6(i).

An important special case of (ii) is when $\tilde{S}$ is the Henselization or strict Henselization of S. When regular base changes are concerned, we always assume that $\tilde{S}$ is excellent. These conditions are preserved by localizing, i.e. replacing $S \subseteq \tilde{S}$ by $S_s \subseteq \tilde{S}_{\tilde{s}}$, $\tilde{s} \in \mathrm{Spec}\tilde{S}$ and $s \in \mathrm{Spec}S$ its image.

Notation 2.1. Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent, $\tilde{s} \in \mathrm{Spec}\tilde{S}$ with image $m_S \in \mathrm{Spec}S$. Any r.s.p. $(u_1, \ldots, u_n)$ of S can be extended to a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$. We let $\tilde{h} \in \tilde{S}[X]$ be the image of h and

$$\tilde{\eta}: \ \tilde{\mathcal{X}} = \mathcal{X} \times_S \mathrm{Spec}\tilde{S} \to \mathrm{Spec}\tilde{S}.$$

It follows from definition 2.3 that, if $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ is a nonsolvable vertex of $\Delta_S(h; u_1, \ldots, u_n; Z)$, the vertex

$$(\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(h; u_1, \ldots, u_{\tilde{n}}; Z) \subseteq \mathbb{R}^{\tilde{n}}_{\geq 0}$$

is nonsolvable provided that $S/m_S \subseteq \tilde{S}/m_{\tilde{S}}$ is inseparably closed. This is of course always satisfied when S/m_S is perfect (e.g. $\mathrm{char} S/m_S = 0$). An obvious consequence of the second statement in proposition 2.2 is:

Proposition 2.5. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Assume that $S/m_S \subseteq \tilde{S}/m_{\tilde{S}}$ is inseparably closed. Let $Z = X - \theta$, $\theta \in S$, be such that $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal. Then*

$$\Delta_{\tilde{S}}(h; u_1, \ldots, u_{\tilde{n}}; Z) = \Delta_S(h; u_1, \ldots, u_n; Z) \times \mathbb{R}^{\tilde{n}-n}_{\geq 0} \subseteq \mathbb{R}^{\tilde{n}}_{\geq 0}$$

and this polyhedron is minimal.

Note that the assumptions of the proposition are satisfied in the above situation (ii): $\tilde{S}$ is ind-étale over S. In situation (iii), i.e. $\tilde{S}$ smooth over S, the following example will make the situation clear:

Example 2.2. Let (S, m_S, k) be an excellent DVR, char$k = p > 0$, and $\gamma \in S$ be a unit. Let $\lambda \in k$ be the residue of γ and assume furthermore that

$$h := X^p - \gamma u_1^{pa} \in S[X],\ a \geq 1,\ \lambda \in k \backslash k^p.$$

Then $\Delta_S(h; u_1; X) = [a, +\infty[$ and is minimal. Take $\tilde{S} = S[t]_{(u_1, P(t))}$, where P is a monic polynomial with irreducible residue $\overline{P}(t) \in k[t]$ (resp. $P = 0$). Let $u_2 := P(t)$, so (u_1, u_2) (resp. (u_1)) is a r.s.p. of $\tilde{S}$. Let

$$\tilde{k} := \tilde{S}/m_{\tilde{S}} = k[t]/(\overline{P}(t)) \text{ (resp. } \tilde{k} = k(t))$$

be the residue field of $\tilde{S}$. Setting $\{\tilde{x}\} = \tilde{\eta}^{-1}(m_{\tilde{S}})$, we have

$$\begin{cases} \delta(\tilde{x}) = a & \text{if} \quad \lambda \notin \tilde{k}^p \\ \delta(\tilde{x}) = a + \frac{1}{p} & \text{if} \quad \lambda \in \tilde{k}^p \end{cases}.$$

This is obvious if $\lambda \notin \tilde{k}^p$; if $\lambda \in \tilde{k}^p$, take

$$Z := X - \tilde{\gamma} u_1^a, \text{ where } \tilde{v} := \tilde{\gamma}^p - \gamma \in m_{\tilde{S}}.$$

Then $(u_1, \tilde{v})$ is a r.s.p. of $\tilde{S}$ (S excellent) and we have:

$$\Delta_{\tilde{S}}(\tilde{h}; u_1, \tilde{v}; Z) = (a, 1/p) + \mathbb{R}^2_{\geq 0}.$$

In particular, the function

$$\mathbb{A}^1_k = \{x\} \times \mathbb{A}^1_k \subset \mathcal{X} \times_k \mathbb{A}^1_k \to \frac{1}{p}\mathbb{N},\ \tilde{x} \mapsto \delta(\tilde{x})$$

is not a constructible function.

Proposition 2.4 and proposition 2.5 suggest the following question. An affirmative answer would be very useful in order to build geometrical invariants from characteristic polyhedra. Proposition 2.5 answers in the affirmative when S/m_S is perfect, with $\tilde{S} := S$.

Question 2.1. Let S be an excellent regular local ring with r.s.p. $(u_1, \ldots, u_n)$ and $h \in S[X]$ (2.1). Does there exist a smooth local base change $S \subseteq \tilde{S}$, a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$ extending $(u_1, \ldots, u_n)$ and $Z = X - \tilde{\phi}$, $\tilde{\phi} \in \tilde{S}$, such that the following holds:

"for every smooth local base change $\tilde{S} \subseteq S'$ and r.s.p. $(u_1, \ldots, u_{n'})$ of S' extending $(u_1, \ldots, u_{\tilde{n}})$, the polyhedron $\Delta_{S'}(h; u_1, \ldots, u_{n'}; Z)$ is minimal"?

Uncovering transformation rules for the characteristic polyhedron under blowing up is a major problem, *vid.* [42] p.254. A good behaviour is known in the special case of a blowing up along a Hironaka permissible subscheme and an exceptional point at the origin of some standard chart.

Proposition 2.6. *With notations as before, let $J \subseteq \{1, \ldots, n\}$, $y \in \eta^{-1}(s^J)$ and assume that $\delta(y) \geq 1$. Fix $j_0 \in J$ and let $S' := S[\{u'_j\}_{j \in J}]_{(u'_1, \ldots, u'_n)}$, where*

$$\begin{cases} u'_j & := & u_j/u_{j_0} & \text{if} & j \in J \backslash \{j_0\}; \\ u'_j & := & u_j & \text{if} & j \in J' \cup \{j_0\}. \end{cases}$$

Let $Z = X - \theta$, $\theta \in S$, with $\Delta_S(h; u_1, \ldots, u_n; Z)$ minimal and define:

$$h'(Z') := u_{j_0}^{-m} h(Z) = Z'^m + u_{j_0}^{-1} f_{1,Z} Z'^{m-1} + \cdots + u_{j_0}^{-m} f_{m,Z} \in S'[Z'], \quad (2.13)$$

where $Z' := Z/u_{j_0}$. Define a map $l: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

$$\mathbf{x} = (x_1, \ldots, x_n) \mapsto \mathbf{x}' = (x_1, \ldots, x_{j_0 - 1}, \sum_{j \in J} x_j - 1, x_{j_0+1}, \ldots, x_n). \quad (2.14)$$

Then $l(\Delta_S(h; u_1, \ldots, u_n; Z)) = \Delta_{S'}(h'; u'_1, \ldots, u'_n; Z')$ and this polyhedron is minimal.

Proof. The assumption $\delta(y) \geq 1$ forces $f_{i,Z} \in I_J^i$ by the last statement in proposition 2.4. Therefore (2.13) makes sense, i.e. $h'(Z') \in S'[Z']$. Since l is one-to-one, we have

$$\frac{1}{i} \mathbf{S}^{\{1, \ldots, n\}}(f_{i,Z'}) \subseteq l\left(\frac{1}{i} \mathbf{S}^{\{1, \ldots, n\}}(f_{i,Z})\right), \ 1 \leq i \leq m,$$

with notations as in proposition 2.1. By definition 2.1, we get:

$$l(\Delta_S(h; u_1, \ldots, u_n; Z)) = \Delta_{S'}(h'; u'_1, \ldots, u'_n; Z').$$

Let $\mathbf{x}' = l(\mathbf{x})$ be a vertex of $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z')$. Denote

$$\mathrm{in}_{\mathbf{x}} h = Z^m + \lambda_1 U^{\mathbf{x}} Z^{m-1} + \cdots + \lambda_m U^{m\mathbf{x}}, \ \lambda_1, \ldots, \lambda_m \in S/m_S,$$

with the convention as before that $\lambda_i = 0$ if $i\mathbf{x} \notin \mathbb{N}^n$, $1 \leq i \leq m$. Applying l (2.14), we get

$$\mathrm{in}_{\mathbf{x}'} h = Z'^m + \lambda_1 U'^{\mathbf{x}'} Z'^{m-1} + \cdots + \lambda_m U'^{m\mathbf{x}'}.$$

Since $S'/m_{S'} = S/m_S$, definition 2.3 then shows that $\mathbf{x}'$ is solvable if and only if $\mathbf{x}$ is solvable. Since $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal, the polyhedron $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z')$ is also minimal by proposition 2.2. $\square$

2.2 Normal crossings divisors.

We now introduce a normal crossings divisor $E \subseteq \mathrm{Spec}S$. This section fixes the terminology and notations for blowing ups and base changes with respect to E, then introduces the Hironaka ϵ function on $\mathcal{X}$.

Definition 2.6. A r.s.p. $(u_1, \ldots, u_n)$ of S is said to be adapted to E if $E = \mathrm{div}(u_1 \cdots u_e)$ for some e, $0 \leq e \leq n$.

We emphasize that we allow $e = 0$, i.e. $E = \emptyset$ in this definition. In this context, we use the following notion of Hironaka permissible center:

Definition 2.7. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is Hironaka-permissible (resp. Hironaka-permissible with respect to E) at $x \in \mathcal{Y}$ if condition (i) (resp. condition (ii)) below is satisfied:

(i) $m(y) = m(x)$ and $\mathcal{Y}$ regular at x;

(ii) $\mathcal{Y} \subseteq \mathrm{Sing}_m\mathcal{X}$ and $W := \eta(\mathcal{Y})$ has normal crossings with E at $s := \eta(x)$.

We remind the reader that an integral closed subscheme $W \subseteq \mathrm{Spec}S$ has normal crossings with $E = \mathrm{div}(u_1 \cdots u_e)$ if the family $(u_1, \ldots, u_e)$ can be extended to a r.s.p. $(u_1, \ldots, u_n)$ of S such that the ideal $I(W)$ of W is of the form $I_J = (\{u_j\}_{j \in J}) \subseteq S$, for some $J \subseteq \{1, \ldots, n\}$.

Note that a Hironaka-permissible center w.r.t. any E (e.g. $E = \emptyset$) is Hironaka-permissible: since $\mathcal{Y} \subseteq \mathrm{Sing}_m\mathcal{X}$, we have $m(y) = m(x) = m$ and $y \in \eta^{-1}(w) \cap \mathrm{Sing}_m\mathcal{X}$, where w is the generic point of W; by proposition 2.3 applied to S_w, the map $\mathcal{Y} \to W$ is birational, hence an isomorphism since W is regular.

Since the notion is local on $\mathcal{X}$, a Hironaka-permissible blowing up (w.r.t. E) is simply the blowing up along a center $\mathcal{Y} \subset \mathcal{X}$ which is Hironaka-permissible (w.r.t. E) at each point of its support. By a *local* Hironaka-permissible blowing up, we simply mean the localization at some point of the exceptional divisor $\pi^{-1}(\mathcal{Y})$ of the blowing up π along a Hironaka-permissible center. The important fact is that Hironaka-permissible blowing ups w.r.t. E preserve our structure:

Proposition 2.7. *Let S, $h \in S[X]$ (2.1), $\mathcal{X}$ and $E = \mathrm{div}(u_1 \cdots u_e)$ be as above. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a Hironaka-permissible blowing up w.r.t. E at*

$x \in \mathcal{X}$. There exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}S & \xleftarrow{\sigma} & \mathcal{S}' \end{array} \tag{2.15}$$

where $\sigma : \mathcal{S}' \to \mathrm{Spec}S$ is the blowing up along W.

For every $s' \in \sigma^{-1}(s)$, $S' := \mathcal{O}_{\mathcal{S}',s'}$, there exists $h' \in S'[X']$ unitary of degree m such that $\mathcal{X}'_{s'} = \mathrm{Spec}(S'[X']/(h'))$.

Furthermore, there exists a r.s.p. $(u'_1, \ldots, u'_n)$ of S' adapted to the stalk $E'_{s'}$, $E' := \sigma^{-1}(E \cup W)_{\mathrm{red}}$.

Proof. By the above remarks, there exists $J \subseteq \{1, \ldots, n\}$ such that $I(W) = I_J = (\{u_j\}_{j \in J})$. By proposition 2.4, there exists $Z := X - \theta$, $\theta \in S$, such that $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal. Since $x, y \in \mathrm{Sing}_m \mathcal{X}$, we have

$$\eta^{-1}(s) = \{x\},\ \eta^{-1}(W) = \mathcal{Y} \text{ and } \delta(x) \geq 1,\ \delta(y) \geq 1$$

by proposition 2.3. In particular, the ideal of $\mathcal{Y}$ at x is

$$I(\mathcal{Y}) = (Z, \{u_j\}_{j \in J}).$$

Since $\delta(y) \geq 1$, the point at infinity $(1 : 0 : \cdots : 0)$ does not belong to $\mathcal{X}'$ so $(\{u_j\}_{j \in J})\mathcal{O}_{\mathcal{X}'}$ is invertible. By the universal property of blowing up, there is a commutative diagram (2.15).

Let $s' \in \sigma^{-1}(s)$ and $j_0 \in J$ be such that u_{j_0} is a local equation of $\pi_0^{-1}(W)$. We take $X' := Z/u_{j_0}$ and

$$h' := u_{j_0}^{-m} h(Z) = X'^m + u_{j_0}^{-1} f_{1,Z} X'^{m-1} + \cdots + u_{j_0}^{-m} f_{m,Z}. \tag{2.16}$$

Note that $h' \in S'[X']$ follows from the last statement in proposition 2.4. The last statement is obvious because $E' = \sigma^{-1}(E \cup W)_{\mathrm{red}}$ is a normal crossings divisor on $\mathcal{S}'$. □

We will stick to these notations when local Hironaka-permissible blowing ups are concerned, or compositions of such local blowing ups. We always refer to the reduced total transform of E on the blown up base $\mathrm{Spec}S$.

Suppose a base change is given as considered in the previous section, i.e. formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent.

Notation 2.2. Given $S \subseteq S'$ such a base change, we denote

$$E' := E \times_S \mathrm{Spec} S',\ \eta' :\ \mathcal{X}' = \mathcal{X} \times_S \mathrm{Spec} S' \to \mathrm{Spec} S'.$$

The image of h in $S'[X]$ is denoted $h' \in S'[X]$. This notation is used consistently with notation 2.1.

For instance if $s \in \mathrm{Spec} S$, there exists a r.s.p. $(v_1, \ldots, v_{n(s)})$ of S_s which is adapted to E_s, where E_s is the stalk of E at s. We then have $E_s = \mathrm{div}(v_1 \cdots v_{e(s)})$ and may choose $v_j = u_{\varphi(j)}$ for some injective map $\varphi :\ \{1, \ldots, e(s)\} \to \{1, \ldots, e\}$. It is of course not possible in general to extend a given $(v_1, \ldots, v_{n(s)})$ to a r.s.p. $(u_1, \ldots, u_n)$ of S. We let $h_s \in S_s[X]$ be the image of h.

Definition 2.8. Let $s \in \mathrm{Spec} S$ and $(v_1, \ldots, v_{n(s)})$ be an r.s.p. of S_s which is adapted to E_s, $E_s = \mathrm{div}(v_1 \cdots v_{e(s)})$. We say that coordinates

$$(v_1, \ldots, v_{n(s)}; Z_s),\ Z_s := X - \phi_s,\ \phi_s \in S_s,$$

are well adapted at $y \in \eta^{-1}(s)$ if $\Delta_{S_s}(h; v_1, \ldots, v_{n(s)}; Z_s)$ is minimal.

Definition 2.9. Let $(u_1, \ldots, u_n)$ be a r.s.p. of S which is adapted to E. Let j, $1 \leq j \leq e$, and let $\mathcal{Y}_j \subset \mathcal{X}$ be an irreducible component of $\eta^{-1}(\mathrm{div}(u_j))$ with generic point $y_j \in \mathcal{X}$. We let

$$d_j := \delta(y_j) \in \frac{1}{m!}\mathbb{N}.$$

For any $s \in \mathrm{Spec} S$ and $y \in \eta^{-1}(s)$, we let

$$\epsilon(y) := m \left(\delta(y) - \sum_{\mathrm{div}(u_j) \subseteq E_s} d_j \right) \in \frac{1}{(m-1)!}\mathbb{Z}.$$

Summing up results from the previous section, we have:

Proposition 2.8. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. With notations as above, we have*

$$d_j = \min \left\{ \frac{a_j}{i},\ \mathbf{a} \in \mathbf{S}^{\{1,\ldots,n\}}(f_{i,Z}),\ 1 \leq i \leq m \right\},\ 1 \leq j \leq e.$$

For $s \in \mathrm{Spec} S$ and $y \in \eta^{-1}(s)$, we have $\epsilon(y) \geq 0$.

Proof. The first (resp. second) statement follows from the last one in proposition 2.4 applied to S and $J := \{j\}$ (resp. to S_s and each $J := \{j\}$ with $\mathrm{div}(u_j) \subseteq E_s$). □

2.3 The Galois or purely inseparable assumption.

In this section, we introduce the assumptions of theorem 1.4. The main result is proposition 2.11 which analyzes the consequence w.r.t. the slopes $\delta_\alpha(h; u_1, \ldots, u_n; Z)$ and initial form polynomials $\mathrm{in}_\alpha h$ from definition 2.2. We assume furthermore that the following property holds:

(G) $m = p$ is a prime number, h is reduced, the ring extension $L|K$ is normal and $\mathcal{X}$ is G-invariant, where $G := \mathrm{Aut}_K(L)$.

*Assumption **(G)** is maintained up to the end of this chapter.*

Since $[L : K] = p$ is a prime number, we have either $G = \mathbb{Z}/p$ ($L|K$ separable, cases (a) and (b) below) or $G = (1)$ ($L|K$ inseparable, case (c) below). Case (a) is included here for the sake of completeness and because residual actions in case (b) may lead to case (a). The three cases to be considered are:

(a) h is totally split (product of p pairwise distinct linear factors) over K;

(b) h is irreducible and Galois over K with group $G = \mathbb{Z}/p$;

(c) h is irreducible, $\mathrm{char} S = p$, $f_{i,X} = 0$, $1 \leq i \leq p-1$.

Assumption **(G)** is also preserved by those base changes considered in the previous sections, i.e. formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent. Note that in any case, h reduced implies respectively h_s, $\hat{h}$ (since S is excellent) and $\tilde{h}$ reduced (notation 2.2). Recall notations and definitions of initial forms from definition 2.2.

Proposition-Definition 2.9. *Assume that* $\mathrm{char} S/m_S = p$. *Let* $(u_1, \ldots, u_n)$ *be a given r.s.p. of* S *and* $\alpha \in \mathbb{R}^n_{>0}$ *be a weight vector. The integer*

$$i_0(\alpha) := \min\{i \in \{1, \ldots p\} : F_{i,Z,\alpha} \neq 0\}$$

does not depend on $Z = X - \theta$, $\theta \in \hat{S}$ *such that* $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ *is minimal. If* $i_0(\alpha) < p$, *the form* $F_{i_0(\alpha),Z,\alpha}$ *is also independent of the choice of* $Z = X - \theta$ *as above.*

In case $\alpha = \mathbf{1}$, *the integer* $i_0(\mathbf{1})$ *(also denoted by* $i_0(x)$ *for* $x \in \eta^{-1}(m_S)$*) and form* $F_{i_0(\mathbf{1}),Z} = F_{i_0(\mathbf{1}),Z,\mathbf{1}}$ *(if* $i_0(\mathbf{1}) < p$*) are also independent of the*

choice of the r.s.p. $(u_1, \ldots, u_n)$ *of* S *and* $Z = X - \theta$, $\theta \in \hat{S}$ *such that* $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ *is minimal.*

Proof. Take $Z' = Z - \phi$ such that both polyhedra $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ and $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z')$ are minimal. By minimality, we have

$$\mu_\alpha(\phi) \geq a := \delta_\alpha(h; u_1, \ldots, u_n; Z).$$

The initial forms $\mathrm{in}_\alpha h(Z) \in (\mathrm{gr}_\alpha S)[Z]$ and $\mathrm{in}_\alpha h(Z') \in (\mathrm{gr}_\alpha S)[Z']$ are related by

$$\mathrm{in}_\alpha h(Z') = \mathrm{in}_\alpha h(Z - \mathrm{cl}_{\alpha,a}\phi).$$

The first statement follows from the elementary fact that $\mu_\alpha \begin{pmatrix} p \\ i \end{pmatrix} > 0$ for $1 \leq i \leq p-1$, since $p \in m_S$. The second statement then follows from proposition 2.3. □

Proposition 2.10. *For* $x \in \mathrm{Sing}\mathcal{X}$, $s := \eta(x)$, *we have:*

$$\eta^{-1}(s) = \{x\},\ k(x) = k(s) \text{ and } \delta(x) > 0. \tag{2.17}$$

Assume that a normal crossings divisor $E = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec}S$ *is specified and let* $\pi : \mathcal{X}' \to \mathcal{X}$ *be a Hironaka-permissible blowing up w.r.t.* E *at* x. *Then, with notations as in proposition 2.7, for every* $s' \in \sigma^{-1}(s)$, $\mathcal{X}'_{s'}$ *satisfies again* ***(G)****.*

Proof. It can be assumed that $s = m_S$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $\overline{h}(Z) \in S/m_S[Z]$ be the reduction of h modulo m_S. By **(G)**, G acts transitively on the fiber $\eta^{-1}(s)$. Then $\overline{h}(Z)$ is either a pth-power or satisfies again **(G)** w.r.t. the zero-dimensional regular local ring S/m_S.

If $\overline{h}(Z)$ satisfies **(G)**, then $(h(Z), u_1, \ldots, u_n)$ is a r.s.p. of the local ring $S[Z]_{m_x}$, so x is a regular point of $\mathcal{X}$.

Assume now that $\overline{h}(Z) = (Z - \overline{\lambda})^p$ for some $\overline{\lambda} \in S/m_S$. Now $(0, \ldots, 0)$ is a solvable vertex of $\Delta_S(h; u_1, \ldots, u_n; Z)$ unless $\overline{\lambda} = 0$. Since $(u_1, \ldots, u_n; Z)$ are well adapted coordinates at x, we have $\overline{\lambda} = 0$.

The last statement follows from proposition 2.7 and the fact that x is G-invariant by (2.17). □

Proposition 2.11. *Let* $x \in \eta^{-1}(m_S)$ *and* $(u_1, \ldots, u_n; Z)$ *be well adapted coordinates at* x. *For* $\alpha \in \mathbb{R}^n_{>0}$ *a weight vector, the following holds:*

(i) the polynomial $\mathrm{in}_\alpha h \in (\mathrm{gr}_\alpha S)[Z]$ *satisfies again* ***(G)*** *w.r.t. the local ring* $(\mathrm{gr}_\alpha S)_{(U_1,\dots,U_n)}$*;*

(ii) if ($\mathrm{char} S/m_S = p$ *and* $i_0(\alpha) < p$*), then*

$$\delta_\alpha(h; u_1, \dots, u_n; Z) \in \Gamma_\alpha = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n;$$

(iii) if $\mathrm{char} S/m_S = 0$ *or if (*$\mathrm{char} S/m_S = p$ *and* $i_0(\alpha) = p$*), then*

$$\delta_\alpha(h; u_1, \dots, u_n; Z) \in \frac{1}{p}\Gamma_\alpha.$$

Proof. If $\delta(x) = 0$, we have $\delta_\alpha(h; u_1, \dots, u_n; Z) = 0$ and $\mathrm{in}_\alpha h = \overline{h}(Z)$ with notations as in the previous proof, so the proposition is trivial. Assume that $\delta(x) > 0$.

By proposition 2.2, we have $\Delta_S(h; u_1, \dots, u_n; Z) \neq \emptyset$ and this polyhedron has no solvable vertex. Therefore $\mathrm{in}_\alpha h$ is not a pth-power. Let $z \in L$ be the image of Z and ν_α be any extension of μ_α to L. Then ν_α is centered at x, since $\mathcal{X}$ is G-invariant and $\eta^{-1}(m_S) = \{x\}$ by proposition 2.3(i). We have:

$$\nu_\alpha(z) = \mu_\alpha(f_{i,Z})/i = \delta_\alpha(h; u_1, \dots, u_n; Z) \in \Gamma_\alpha \otimes_{\mathbb{Z}} \mathbb{Q} \tag{2.18}$$

for each i, $1 \leq i \leq p$ such that $F_{i,Z,\alpha} \neq 0$. Since $L|K$ is normal of degree p, the reduced ramification index e_0 of $\nu_\alpha|\mu_\alpha$ is $e_0 = 1$ or $e_0 = p$.

Assume that ($\mathrm{char} S/m_S = p$ and $i_0(\alpha) = p$). Then $\mathrm{in}_\alpha h$ is in case (c) of **(G)** and we get (iii) from (2.18).

Assume that $\mathrm{char} S/m_S = 0$ or ($\mathrm{char} S/m_S = p$ and $i_0(\alpha) < p$). Then h is in case (a) or (b). Since $G = \mathbb{Z}/p$ in these cases and $\mathcal{X}$ is G-invariant, G acts transitively on the roots of $\mathrm{in}_\alpha h$. We have:

$$\begin{cases} \mathrm{Tot}((\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)) & = & \prod_{\nu_\alpha} QF(\mathrm{gr}_\alpha S) & \text{if } \mu_\alpha \text{ splits;} \\ \\ QF((\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)) & = & QF(\mathrm{gr}_{\nu_\alpha} S) & \text{otherwise,} \end{cases}$$

and this proves (i). Statement (iii) follows from (2.18) if $\mathrm{char} S/m_S = 0$.

Assume finally that ($\mathrm{char} S/m_S = p$ and $i_0(\alpha) < p$). By (2.18), we have

$$p\nu_\alpha(z) = p\mu_\alpha(f_{i_0(\alpha),Z})/i_0(\alpha) \in \Gamma_\alpha.$$

Since $\Gamma_\alpha \simeq \mathbb{Z}^r$ for some $r \geq 1$, this implies

$$\delta_\alpha(h; u_1, \dots, u_n; Z) = \mu_\alpha(f_{i,Z})/i_0(\alpha) \in \Gamma_\alpha$$

which completes the proof of (ii). □

Corollary 2.12. *Assume that a normal crossings divisor*

$$E = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec} S$$

is specified. We have $pd_j \in \mathbb{N}$, $1 \leq j \leq e$, *and* $\epsilon(y) \in \mathbb{N}$ *for every* $y \in \mathcal{X}$.

Proof. In view of definition 2.9 and proposition 2.8, this follows from proposition 2.11 (ii)(iii) applied to the local rings $S_{(u_j)}$ and S_s, $s := \eta(y)$. □

This corollary allows us to define the following invariant:

Definition 2.10. Let $(u_1, \ldots, u_n)$ be a r.s.p. of S which is adapted to the normal crossings divisor $E = \mathrm{div}(u_1 \cdots u_e)$. For $y \in \mathcal{X}$, $s := \eta(y)$, we define a principal ideal:

$$H(y) := \left(\prod_{\mathrm{div}(u_j) \subseteq E_s} u_j^{H_j} \right) \subseteq S,$$

where $H_j := pd_j \in \mathbb{N}$.

2.4 The discriminant assumption.

We now introduce the critical locus of the map $\eta : \mathcal{X} \to \mathrm{Spec} S$ together with its scheme structure given by the discriminant $D := \mathrm{Disc}_X h \in S$. We are interested in the case where D is a normal crossings divisor. Theorem 2.14 below is basically a reduction to characteristic $p > 0$ as dealt with in [26] [27].

Note that D is by definition independent of the choice of regular parameters of S and invariant by those translations $X' := X - \phi$, $\phi \in \hat{S}$ used in minimizing polyhedra. If (S, h, E) is in case (c) of **(G)**, then $D = 0$.

Definition 2.11. Let S, $h \in S[X]$ (2.1), $\mathcal{X}$ and $E = \mathrm{div}(u_1 \cdots u_e)$ be specified. We say that (S, h, E) satisfies assumption **(E)** if $\mathrm{char}(S/m_S) = p > 0$ and one of the following properties hold:

$$\begin{cases} (i) \quad D = 0 \quad \text{and} \quad \eta(\mathrm{Sing}_p \mathcal{X}) \subseteq E, \\ \\ (ii) \quad D \neq 0 \quad \text{and} \quad \mathrm{div}(D)_{\mathrm{red}} \subseteq E \subseteq \mathrm{div}(p)_{\mathrm{red}}. \end{cases} \tag{2.19}$$

*Assumption **(E)** is maintained up to the end of this chapter.*

This assumption implies that $\mathrm{Sing}_p\mathcal{X} \subseteq \eta^{-1}(E) \subset \mathcal{X}$, by definition (i) or because $\eta^{-1}(\mathrm{Spec}S\backslash E)$ is regular since $\mathrm{Spec}S\backslash E$ is (ii). In particular $E \neq \emptyset$ if $\mathrm{Sing}_p\mathcal{X} \neq \emptyset$.

Assumption **(E)** is also preserved by those base changes considered in the previous section: formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent. For Hironaka-permissible blowing ups, we have:

Proposition 2.13. *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a Hironaka-permissible blowing up w.r.t. E at $x \in \mathcal{X}$. Then, with notations as in proposition 2.7, for every $s' \in \sigma^{-1}(s)$, (S', h', E') satisfies again **(E)**.*

Proof. Any Hironaka-permissible center $\mathcal{Y} \subset \mathcal{X}$ w.r.t. E at x is contained in E by the above remarks. Therefore the proposition is obvious in case (i) of definition 2.11.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $h(Z) \in S[Z]$ be the corresponding expansion. With notations as in proposition 2.7 and (2.16), we have $h'(X') = u_{j_0}^{-p} h(X' u_{j_0})$ for some $u_{j_0} \in I(W)$. We deduce that

$$D' := \mathrm{Disc}_{X'} h' = u_{j_0}^{-p(p-1)} \mathrm{Disc}_Z h = u_{j_0}^{-p(p-1)} D,$$

hence $\mathrm{div}(D')_{\mathrm{red}} \subseteq E' \subseteq \mathrm{div}(p)_{\mathrm{red}}$ as required. □

Theorem 2.14. ***(Reduction to characteristic** p**).*** *With assumptions as above, let $x \in \eta^{-1}(m_S)$ be such that $\epsilon(x) > 0$. Then $(\mathcal{X}, x)$ is analytically irreducible.*

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $\alpha \in \mathbb{R}^n_{>0}$ be a weight vector. Exactly one of the following properties holds.

(1) $i_0(\alpha) = p$, i.e. $\mathrm{in}_\alpha h = Z^p + F_{p,Z,\alpha}$;

(2) $i_0(\alpha) = p - 1$ i.e. $\mathrm{in}_\alpha h = Z^p + F_{p-1,Z,\alpha} Z + F_{p,Z,\alpha}$, $F_{p-1,Z,\alpha} \neq 0$. Furthermore, we have

$$-f_{p-1,Z} = \gamma_{p-1,Z} \prod_{j=1}^{e} u_j^{A_{p-1,j}} \tag{2.20}$$

with $A_{p-1,j} \in (p-1)\mathbb{N}$, $1 \leq j \leq e$, *and* $\gamma_{p-1,Z} \in S$ *a unit with residue* $\overline{\gamma}_{p-1,Z} \in (S/m_S)^{p-1}$. *In particular,* $-F_{p-1,Z,\alpha} = G^{p-1}$ *for some nonzero* $G \in \mathrm{gr}_\alpha S$, *and we have*

$$\mathrm{cl}_{p(p-1)\delta_\alpha(h;u_1,\ldots,u_n;Z)}(\mathrm{Disc}_Z(h)) =< F^p_{p-1,Z,\alpha} > .$$

Proof. First note that $D = \mathrm{Disc}_Z(h)$ is homogeneous of degree $p(p-1)$ for the grading $\deg f_{i,Z} = i$ on the coefficients of h. In particular, we have

$$\mu_\alpha(D) \geq p(p-1)\delta_\alpha(h; u_1, \ldots, u_n; Z),$$

since $\mu_\alpha(f_{i,Z})/i \geq \delta_\alpha(h; u_1, \ldots, u_n; Z)$ for $1 \leq i \leq p$. We deduce the formula

$$\mathrm{cl}_{\alpha,p(p-1)\delta_\alpha(h;u_1,\ldots,u_n;Z)} D = \mathrm{Disc}_Z(\mathrm{in}_\alpha h). \tag{2.21}$$

On the other hand, $\mathrm{in}_\alpha h$ has a multiple root over an algebraic closure of $QF(\mathrm{gr}_\alpha S)$ if and only if $i_0(\alpha) = p$ by proposition 2.11 (i). When this holds, we are in case (1) of this theorem.

Suppose that h is analytically reducible. By proposition 2.8 and definition 2.5, $\epsilon(x) = \delta(x) - \sum_{i=1}^e d_j$ is determined by $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$, thus invariant by base change $S \subseteq \hat{S}$. Therefore it can be assumed w.l.o.g. that $S = \hat{S}$ in order to prove the first statement, i.e. that h is in case (a) of property **(G)**. Since h splits, there is a factorization

$$h = \prod_{i=1}^p (Z - \varphi_j) \in S[Z],\ \varphi_1, \ldots, \varphi_p \in S.$$

Let $z \in \mathcal{O}_\mathcal{X}$ be the image of Z and $g \in G = \mathbb{Z}/p$, $g \neq 0$. By property **(G)**, we have $g(z) \in \mathcal{O}_\mathcal{X}$ and $g(z)$ is a root of $h(Z)$. Up to reindexing, it can therefore be assumed that

$$g^i(z) = z - \varphi_{i+1} + \varphi_1 \in S,\ 1 \leq i \leq p-1.$$

In particular, we have $g(z) - z = \varphi_1 - \varphi_2 \in S$ and we deduce that

$$g^i(z) - z = \sum_{k=0}^{i-1} g^k(g(z) - z) = i(g(z) - z),\ 1 \leq i \leq p-1.$$

Since $(p-1)!$ is a unit in S, we get a formula

$$D = \mathrm{Disc}_Z(h) = \gamma_0(\varphi_1 - \varphi_2)^{p(p-1)},\ \gamma_0 \in S,\ \gamma_0 \text{ a unit.}$$

By assumption, $(u_1, \ldots, u_n)$ is adapted to E. Then definition 2.11(ii) implies that

$$\varphi_1 - \varphi_2 = \gamma u^{\mathbf{a}},$$

with $\gamma \in S$ a unit, and $a_j = 0$, $e+1 \leq j \leq n$. Take an expansion (2.4):

$$\varphi_1 = \sum_{\mathbf{x} \in \mathbf{S}(\varphi_1)} \gamma_{\mathbf{x}} u^{\mathbf{x}},\ \gamma_{\mathbf{x}} \in S,\ \gamma_{\mathbf{x}} \text{ unit}$$

with $\mathbf{S}(\varphi_1) \subset \mathbb{N}^n$ finite. If $x_j < a_j$ for some $\mathbf{x} \in \mathbf{S}(\varphi_1)$ and some j, $1 \leq j \leq e$, then $\mathbf{x}$ is a vertex of $\Delta_S(h; u_1, \ldots, u_n; Z)$ with initial form

$$\mathrm{in}_{\mathbf{x}} h = (Z - \lambda U^{\mathbf{x}})^p,\ \lambda \in S/m_S,\ \lambda \neq 0.$$

This is a solvable vertex: a contradiction, since $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal. Therefore $\varphi_1 \in (u^{\mathbf{a}})$ and we get $\epsilon(x) = 0$: a contradiction. Hence $(\mathcal{X}, x)$ is analytically irreducible as stated. It can be assumed that h is in case (b) of property **(G)** from now on.

Assume now that $\mathrm{in}_\alpha h$ is in cases (a) or (b) of property **(G)**, i.e. $i_0(\alpha) < p$ and

$$\mathrm{Disc}_Z(\mathrm{in}_\alpha h) \neq 0. \tag{2.22}$$

We now compute $\mathrm{ord}_{(u_j)} D$ for $1 \leq j \leq e$. Let

$$s_j := (u_j) \in \mathrm{Spec} S,\ S_j := S_{s_j} \text{ and } y_j \in \eta^{-1}(s_j).$$

To begin with, $\Delta_{S_j}(h; u_j, Z)$ is minimal by proposition 2.4. We denote by $G(s_j) = k(s_j)[U_j]$ the graded ring of S_j w.r.t. its valuation $\mu_j := \mathrm{ord}_{(u_j)}$ and by in_j the initial form map w.r.t. μ_j. Let:

$$\gamma_{i,j} U_j^{A_{i,j}} := \mathrm{in}_j f_{i,Z} \in G(s_j),\ 1 \leq i \leq p. \tag{2.23}$$

By definition 2.11(ii), we have $\mathrm{char} S/(u_j) = p$. Therefore proposition 2.9 and (2.21) apply to S_j with $\alpha = 1 \in \mathbb{R}$. The corresponding integer $i_0(1)$ is denoted by $i_0(s_j)$ in order to avoid confusion and we have

$$\mu_j(D) \geq p(p-1)\delta(y_j) = (p-1)H_j. \tag{2.24}$$

Case 1: $i_0(s_j) < p$. Then equality holds in the former formula as remarked right after (2.21).

Case 2: $i_0(s_j) = p$. Then inequality is strict in the former formula. Since $\Delta_{S_j}(h; u_j, Z)$ is minimal, we have $\gamma_{p,j} U_j^{A_{p,j}} \notin G(s_j)^p$ and $A_{p,j} = H_j$. Let $z \in L$ be the image of Z. The discrete valuation μ_j of K has a unique extension to L, still denoted by μ_j. There is an embedding $G(s_j) \subset G_j$, where G_j is the graded ring of the valuation ring $\mathcal{O}_j := \{f \in L : \mu_j(f) \geq 0\}$.

Case 2a: $H_j \in p\mathbb{N}$. We have

$$G_j = k(s_j)(\gamma_{p,j}^{\frac{1}{p}})[U_j],\ \mathrm{in}_j z = -\gamma_{p,j}^{\frac{1}{p}} U_j^{\frac{H_j}{p}}; \tag{2.25}$$

Case 2b: $H_j \notin p\mathbb{N}$. We have

$$G_j = k(s_j)[\gamma_{p,j}^{\frac{l_j}{p}} U_j^{\frac{1}{p}}],\ \mathrm{in}_j z = -\gamma_{p,j}^{\frac{1}{p}} U_j^{\frac{H_j}{p}}, \tag{2.26}$$

where l_j satisfies $l_j H_j \equiv 1 \bmod p$, since the element $t := z^{l_j} u_j^{-\frac{l_j H_j - 1}{p}}$ is a regular parameter of $\mathcal{O}_j$ with $(\mathrm{in}_j t)^p = -\gamma_{p,j}^{l_j} U_j$.

Let $g \in G = \mathrm{Gal}(L|K)$ be nontrivial. We have

$$g(z)^p - z^p + \sum_{i=1}^{p-1} f_{i,Z}(g(z)^{p-i} - z^{p-i}) = 0. \tag{2.27}$$

Since $\mu_j(g(z) - z) > \mu_j(z)$ and $\mu_j((p-1)!) = 0$, we deduce from (2.23) and (2.25)-(2.26) that

$$\mathrm{in}_j(f_{i,Z}(g(z)^{p-i} - z^{p-i})) = (-1)^{p-i} i T_j \gamma_{i,j} \gamma_{p,j}^{(p-i-1)/p} U_j^{(p-i-1)\frac{H_j}{p} + A_{i,j}} \tag{2.28}$$

for $1 \leq i \leq p-1$, where $T_j := \mathrm{in}_j(g(z) - z)$. On the other hand, we have

$$g(z)^p - z^p = (g(z) - z)^p + \sum_{i=1}^{p-1} \binom{p}{i} (g(z) - z)^{p-i} z^i. \tag{2.29}$$

Computing $\mu_j(D)$ by the Hilbert formula [71] V.11.(8) gives

$$\mu_j(D) = p(p-1)\mu_j(g(z) - z). \tag{2.30}$$

Since equality is strict in (2.24), we have $\mu_j(H(x)^{-(p-1)}D) > 0$ and we deduce that $\mu_j(g(z)-z) > H_j/p$. Computing initial forms for each term on the right hand side of (2.29), we get for $1 \leq i \leq p-1$:

$$\mathrm{in}_j((g(z)-z)^{p-i}z^i) = (-1)^i T_j^{p-i} \gamma_{p,j}^{\frac{i}{p}} U_j^{i\frac{H_j}{p}}.$$

Since $\mu_j(g(z)-z) > H_j/p$ and $\mu_j(\begin{pmatrix} p \\ i \end{pmatrix}) = \mu_j(p)$, $1 \leq i \leq p-1$, the unique minimal value term in (2.29) inside the summation symbol is obtained with $i = p-1$. This shows

$$\mathrm{in}_j\left(\sum_{i=1}^{p-1} \begin{pmatrix} p \\ i \end{pmatrix} (g(z)-z)^{p-i}z^i\right) = \mathrm{in}_j(p) T_j \gamma_{p,j}^{\frac{p-1}{p}} U_j^{(p-1)\frac{H_j}{p}}. \tag{2.31}$$

Case 2a. By (2.25), all terms $\gamma_{p,j}^{(p-i-1)/p}$ for $1 \leq i \leq p-1$ appearing in (2.28) are linearly independent over $k(s_j)$. Since $p \in S_j$, $pu_j^{-\mu_j(p)}$ is a unit in S_j. Let $\gamma \in k(s_j)$ be its residue, so the family $(\gamma\gamma_{p,j}^{\frac{p-1}{p}}, \{\gamma_{p,j}^{\frac{p-i-1}{p}}\}_{1\leq i\leq p-1})$ is a *basis* of the $k(s_j)$-vector space $k(s_j)(\gamma_{p,j}^{1/p})$. Tracing back to (2.27) an (2.29), the value of $(g(z)-z)^p$ is the value of a sum of terms with linearly independent initial forms in G_j. We deduce the formula

$$\mu_j(g(z)-z)^{p-1} = \min\{\mu_j(p)+(p-1)\frac{H_j}{p}, \min_{1\leq i\leq p-1}\{(p-i-1)\frac{H_j}{p}+A_{i,j}\}\}. \tag{2.32}$$

Case 2b. By (2.26), all values $(p-i-1)H_j/p$ for $1 \leq i \leq p-1$ appearing in (2.28) are pairwise distinct modulo $\mathbb{Z}$. Since $p \in S_j$, the family

$$(\mu_j(p) + (p-1)\frac{H_j}{p}, \{(p-i-1)\frac{H_j}{p} + A_{i,j}\}_{1\leq i\leq p-1})$$

represent all cosets of $(1/p)\mathbb{Z}$ modulo $\mathbb{Z}$. The argument is now similar to case 2a above and (2.32) holds as well. Note that the minimum in the right hand side of (2.32) is achieved exactly once in this case 2b.

By (2.30) and (2.32), we conclude in all three cases 1, 2a and 2b that

$$\mu_j(H(x)^{-(p-1)}D) = \min\{p\mu_j(p), \min_{1\leq i\leq p-1}\{pA_{i,j} - iH_j\}\}. \tag{2.33}$$

By (2.23) and definition of $i_0(\alpha)$, we have

$$\sum_{j=1}^{e} A_{i_0(\alpha),j}\alpha_j \leq \mu_\alpha(f_{i_0(\alpha),Z}) = i_0(\alpha)\delta_\alpha(h; u_1, \ldots, u_n; Z). \tag{2.34}$$

Collecting together, since it was assumed in (2.22) that $\mathrm{Disc}_Z(\mathrm{in}_\alpha h) \neq 0$, we have

$$\sum_{j=1}^{e} \mu_j(H(x)^{-(p-1)}D)\alpha_j = (p-1)\left(p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j\right)$$

by (2.21). By (2.33)-(2.34), we deduce

$$(p-1-i_0(\alpha))(p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j) \leq 0. \tag{2.35}$$

Suppose that $p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j = 0$. Definition 2.10 implies that $f_{i,Z}^p \in H(x)^i$ for $1 \leq i \leq p$. Definition 2.1 yields the equality

$$\Delta_S(h; u_1, \ldots, u_n; Z) = (\frac{H_1}{p}, \ldots, \frac{H_e}{p}, 0, \ldots, 0) + \mathbb{R}^n_{\geq 0}.$$

This is a contradiction, since it is assumed that $\epsilon(x) > 0$.

We thus have $p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j > 0$. By (2.35), this implies $i_0(\alpha) = p-1$, since $i_0(\alpha) \leq p-1$ was assumed in (2.22).

We may now sharpen (2.35) as follows, since it is an equality: equality holds in (2.34) *and* the minimum on the right hand side of (2.33) is achieved with $i = i_0(\alpha) = p-1$ for each j, $1 \leq j \leq e$. These two properties are equivalent to the existence of an expansion (2.20) with $\gamma_{p-1,Z} \in S$ a unit.

By proposition 2.11(i), $G = \mathbb{Z}/p$ acts on the roots of $\mathrm{in}_\alpha h$. Let

$$z_\alpha \in (\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)$$

be the image of Z. Then $(g(z_\alpha) - z_\alpha)^{p-1} + F_{p-1,Z,\alpha} = 0$ for $g \in G$ nontrivial, so the polynomial $X^{p-1} + F_{p-1,Z,\alpha}$ is totally split over $\mathrm{gr}_\alpha S$, i.e. $-F_{p-1,Z,\alpha}$ is a $(p-1)^{\mathrm{th}}$ in $\mathrm{gr}_\alpha S$ as required. The last formula in the theorem is obvious. $\square$

2.5 Adapted differential structure.

In this section, we introduce the differential structure on the graded algebras $\mathrm{gr}_\alpha S$. We will only consider here the case $\alpha = \mathbf{1} \in \mathbb{R}^J_{>0}$ with notations as in definition 2.2. These algebras appear naturally as blow up algebras of S along regular primes. We will adapt and simplify notations as much as possible in order to fit with the forthcoming computations.

Remark 2.3. This construction uses formal coordinates and Nagata derivatives [54] pp.241-245, and could be considerably simplified when

$$E = \mathrm{Spec}(S/(u_1 \cdots u_e)) \subset \mathrm{Spec} S$$

is essentially of finite type over some field. This extra property is satisfied for example when E is contained in the closed fiber of some previously performed blowing ups. In dimension three, this extra property is easily achieved from embedded resolution theorems in smaller dimensions, *vid.* lemma 4.10.

Notation 2.3. Let $W \subseteq E$ be a regular closed subset of $\mathrm{Spec} S$ having normal crossings with E. We now write

$$I(W) := I_J = (\{u_j\}_{j \in J}) \subset S \text{ for some } J \subseteq \{1, \ldots, n\}.$$

Let $J_E := J \cap \{1, \ldots, e\}$, $J' := \{1, \ldots, n\} \backslash J$, so $(J')_E = \{1, \ldots, e\} \backslash J_E$.

Let $S_W := S/I(W)$ and $\overline{u}_{j'} \in S_W$ be the image of $u_{j'}$, $j' \in J'$, so

$$\overline{m}_S := m_{S_W} = (\overline{u}_{j'})_{j' \in J'}).$$

Since $W \subseteq E$, **(E)** implies that $\mathrm{char} G(W) = \mathrm{char}(S/m_S) = p > 0$. The formal completion $\widehat{S_W}$ of S_W can be written as

$$\widehat{S_W} \simeq S/m_S[[\{\overline{u}_{j'}\}_{j' \in J'}]]. \tag{2.36}$$

The algebra $\mathrm{gr}_{\mathbf{1}} S$ of definition 2.2 is denoted by:

$$G(W) := \mathrm{gr}_{I(W)} S \simeq S_W[\{U_j\}_{j \in J}].$$

We also denote $\widehat{G(W)} := G(W) \otimes_{S_W} \widehat{S_W}$. In the special case $W = \{m_S\}$, we thus have $\widehat{G(m_S)} = G(m_S)$.

The initial form $\mathrm{in}_{\mathbf{1}}h$ w.r.t. the weight vector $\mathbf{1} \in \mathbb{R}^J_{>0}$ is now denoted

$$\mathrm{in}_W h = X^p + \sum_{i=1}^{p} F_{i,X,W} X^{p-i} \in G(W)[X],$$

with $F_{i,X,W} \in G(W)_{i\delta_{\mathbf{1}}(h;u_1,\ldots,u_n;X)}$, $1 \leq i \leq p$.

Any local equation of E has an initial form in $G(W)$, and we denote by $E(W)$ the associated divisor. Explicitly:

$$E(W) := \mathrm{div}\left(\prod_{j\in J_E} U_j \prod_{j'\in(J')_E} \overline{u}_{j'}\right) \subset \mathrm{Spec} G(W).$$

We include in these definitions the case where $W = \mathrm{div}(u_j)$ is an irreducible component of E. This corresponds to $(J')_E = \{1,\ldots,e\}\backslash\{j\}$ and

$$G(W) = S/(u_j)[U_j],\ E(W) = \mathrm{div}\left(U_j \prod_{j'\in(J')_E} \overline{u}_{j'}\right).$$

Let $(\lambda_l)_{l\in\Lambda_0}$ be an absolute p-basis of S/m_S. For this notion and the rest of this section, we refer to [54] pp.201-205 and pp. 235-245. We allow Λ_0 infinite in these constructions. The corresponding derivations $(\frac{\partial}{\partial\lambda_l})_{l\in\Lambda_0}$ of $\mathrm{Der}(S/m_S)$ act on power series in $\widehat{S_W}$ (2.36) coefficientwise. Those derivations $\frac{\partial}{\partial\lambda_l}$, $l \in \Lambda_0$ will be usually called "derivations w.r.t. to constants".

Let $\mathcal{D}(W) \subset \mathrm{Der}(\widehat{G(W)})$ be the submodule generated by the derivations w.r.t. to constants together with

$$\left(\{U_j\frac{\partial}{\partial U_j}\}_{j\in J_E}, \{\frac{\partial}{\partial U_j}\}_{j\in J\backslash J_E}, \{\overline{u}_{j'}\frac{\partial}{\partial \overline{u}_{j'}}\}_{j'\in(J')_E}, \{\frac{\partial}{\partial \overline{u}_{j'}}\}_{j'\in J'\backslash(J')_E}\right). \quad (2.37)$$

Since S_W is excellent and integrally closed, we have $\widehat{S_W}^p \cap S_W = S_W^p$. Therefore for $F \in G(W)$, there is an equivalence:

$$\forall D \in \mathcal{D}(W),\ D\cdot F = 0 \Leftrightarrow F \in G(W)^p. \quad (2.38)$$

If $F \in G(W)_d$ is a *homogeneous* element, $D\cdot F$ is not homogeneous in general for $D \in \mathcal{D}(W)$ because the derivations $(\frac{\partial}{\partial U_j})_{j\in J\backslash J_E}$ lower degrees by one. We

define a homogeneous S_W-submodule of $G(W)_{d-1}$ as follows:

$$\mathcal{V}(F,E,W) :=< \{\mathrm{cl}_{d-1}\frac{\partial F}{\partial U_j}\}_{j\in J\setminus J_E} >\subseteq G(W)_{d-1}. \tag{2.39}$$

Let $\mathcal{D}_W \subseteq \mathcal{D}(W)$ be the submodule defined by

$$\mathcal{D}_W := \{D \in \mathcal{D}(W) : D\cdot(I(W)/I(W)^2) \subseteq (I(W)/I(W)^2)\}.$$

If $D \in \mathcal{D}(W)$, we have

$$D \in \mathcal{D}_W \Leftrightarrow \forall j \in J\setminus J_E,\ < dU_j, D >\in (I(W)/I(W)^2)\widehat{G(W)}, \tag{2.40}$$

and there is an equivalence

$$\mathcal{D}_W = \mathcal{D}(W) \Leftrightarrow W \text{ is an intersection of components of } E. \tag{2.41}$$

If $F \in G(W)_d$ is a *homogeneous* element, we define a homogeneous $\widehat{S_W}$-submodule of $\widehat{G(W)}_d$ as follows:

$$\mathcal{J}(F,E,W) := \mathrm{cl}_d(\mathcal{D}_W\cdot F) \subseteq \widehat{G(W)}_d. \tag{2.42}$$

Let H_W be the initial form in $G(W)$ of the monomial ideal $H(x) \subseteq S$ (definition 2.10), where $x \in \eta^{-1}(m_S)$, i.e.

$$H_W := \left(\prod_{j\in J_E} U_j^{H_j} \prod_{j'\in(J')_E} \overline{u}_{j'}^{H_{j'}}\right) \subseteq G(W)_{d_W}, \tag{2.43}$$

where $d_W := \sum_{j\in J_E} H_j$. If $F \in H_W G(W)_{d-d_W}$, it follows from the above definitions that

$$\mathcal{V}(F,E,W) \subseteq H_W G(W)_{d-d_W-1} \text{ and } \mathcal{J}(F,E,W) \subseteq H_W\widehat{G(W)}_{d-d_W}.$$

For such $F \in H_W G(W)_{d-d_W}$, we denote:

$$\begin{cases} V(F,E,W) &:= & H_W^{-1}\mathcal{V}(F,E,W) & \subseteq & G(W)_{d-d_W-1}, \\ \\ J(F,E,W) &:= & H_W^{-1}\mathcal{J}(F,E,W) & \subseteq & \widehat{G(W)}_{d-d_W}. \end{cases} \tag{2.44}$$

If $F_{p,X,W} \in H_W G(W)_{d-d_W}$, the submodules

$$V(F_{p,X,W},E,W) \subseteq G(W)_{d-d_W-1} \text{ and } J(F_{p,X,W},E,W) \subseteq \widehat{G(W)}_{d-d_W}$$

are well-defined by (2.44). We will continually apply this definition when the following properties (i) and (ii) hold:

(i) $(u_1, \ldots, u_n; X)$ are well adapted coordinates at $x \in \eta^{-1}(m_S)$ (definition 2.8), and

(ii) $d - d_W = \epsilon(y)$ with $\eta^{-1}(s) = \{y\}$, s the generic point of W.

Note that $F_{p,X,W} \in H_W G(W)_{d-d_W}$ is then a consequence of definition 2.9 and proposition 2.8.

Some considerations will require localizing S at some point $s \in W$. We then denote by W_s the stalk of W at s. This notation is used jointly with notation 2.2 *sqq.* about the stalk E_s. The restriction of s is denoted by $\overline{s} \in \mathrm{Spec} S_W = G(W)_0$. We have

$$G(W_s) = \mathrm{gr}_{I(W_s)} S_s \simeq (S_W)_{\overline{s}}[\{U_j\}_{j \in J}].$$

Consistently $\mathrm{in}_{W_s} h \in G(W_s)[X]$ denotes the initial form. The above construction thus allows to associate to any *homogeneous* element $F \in G(W_s)_d$ homogeneous submodules

$$\mathcal{V}(F, E_s, W_s) \subseteq G(W_s)_{d-1}, \ \mathcal{J}(F, E_s, W_s) \subseteq \widehat{G(W_s)}_d.$$

2.6 Cones, ridge and directrix.

In this section, we recollect some facts about the directrix and Hilbert-Samuel stratum of a homogeneous ideal. These facts are then applied to extract numerical invariants from the vector spaces

$$V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1} \text{ and } J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)}$$

defined in the previous section (2.44) when $(u_1, \ldots, u_n; Z)$ are well adapted coordinates at $x \in \eta^{-1}(m_S)$. These considerations are based on elementary linear algebra.

Most difficulties in this section appear only for $n \geq 4$, which will eventually lead us to define our main invariant $\omega(x)$ in a different way than in [27] chapter 1 (for equicharacteristic S of dimension $n = 3$) in the next section.

Let k be a field, R_1 be a k-vector space of finite dimension $n \geq 1$ and $R := k[R_1]$ be the symmetric algebra. Let $\mathbf{V} := \mathrm{Spec} R$ and I be a homogeneous ideal of R which defines a cone $C = C(I) := \mathrm{Spec}(R/I)$. With these notations, we define:

Definition 2.12. The directrix $\mathrm{Vdir}(I)$ of $C = C(I)$ is the smallest k-vector subspace W of R_1 such that $I = (I \cap k[W])R$. We denote

$$\tau(I) := \dim_k \mathrm{Vdir}(I),\ \mathrm{Dir}(I) := \mathrm{Spec}(R/(\mathrm{Vdir}(I))).$$

Definition 2.13. Let $C = C(F)$ be a hypersurface cone, i.e. $I = (F)$ is a nonzero principal ideal. We define a reduced subcone

$$\mathrm{Max}(F) := \{x \in \mathbf{V} : \mathrm{ord}_x F = \mathrm{ord}_0 F\} \subseteq C(F),$$

where 0 is the origin (so $\mathrm{ord}_0 F = \deg F$).

Given a *fixed* degree $d \geq 1$ and an ideal $I = (F_1, \ldots, F_m) \subset R$ defined by homogeneous polynomials $F_1, \ldots, F_m \in R$, $\deg F_i = d$ for $1 \leq i \leq m$, we let

$$\mathrm{Max}(I) := \{x \in \mathbf{V} : \mathrm{ord}_x F_i = d, 1 \leq i \leq m\} \subseteq C(I).$$

The cone $\mathrm{Max}(I)$ is the closed Hilbert-Samuel stratum of $C(I)$. These two objects and the ridge are considered and connected by H. Hironaka in a more general context. See also [35] [36] [60] for definition and computation of the ridge and Hilbert-Samuel stratum.

Proposition 2.15. ***(Hironaka)****[43] Let $C = C(F)$ be a hypersurface cone. There are inclusions*

$$\mathrm{Dir}(F) \subseteq \mathrm{Max}(F) \subseteq C(F).$$

If k is perfect or if $\dim R \leq p + 1$, the left hand side inclusion is an equality.

Remark 2.4. Counterexamples to the last statement exist for nonperfect k and $\dim R > p + 1$. For $\dim R \leq 4$, such counterexamples exist only if $\dim R = 4$ and $p = 2$. For applications to the proof of theorem 1.4, we only have to deal with this difficulty for the initial form polynomial ($\dim R = 4$) which is of the form

$$\mathrm{in} h = Z^2 - \lambda U_1 Z + F_{2,Z},\ F_{2,Z} \in S/m_S[U_1, U_2, U_3]_2,\ \lambda \in S/m_S.$$

By [43], the polynomial $\mathrm{in} h$ is a counterexample to the last statement in proposition 2.15 if and only if $\lambda = 0$ and, up to a linear change of variables,

$$\mathrm{in}_{m_S} h = Z^2 + \lambda_2 U_1^2 + \lambda_1 U_2^2 + \lambda_1 \lambda_2 U_3^2 \tag{2.45}$$

with λ_1, λ_2 2-independent, i.e. $[(S/m_S)^2(\lambda_1, \lambda_2) : (S/m_S)^2] = 4$. This very special case is dealt with in proposition 5.3.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ (definition 2.8). In case $\epsilon(x) > 0$, we have $\eta^{-1}(m_S) = \{x\}$, $k(x) = S/m_S$ (proposition 2.3) and the initial form polynomial has the form

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z] = S/m_S[U_1, \ldots, U_n][Z] \tag{2.46}$$

by theorem 2.14 applied to $\alpha = \mathbf{1} \in \mathbb{R}^n_{>0}$. There is an associated integer $i_0(x) = p - 1$ (resp. $i_0(x) = p$) if $G \neq 0$ (resp. if $G = 0$). We denote by $H \subseteq G(m_S)_d$ the initial form vector space of the ideal $H(x)$, $d = \sum_{j=1}^{e} H_j$ (definition 2.10). If $i_0(x) = p - 1$, we have

$$H^{-1}G^p =< \prod_{j=1}^{e} U_j^{pB_j} >, \ B_j \in \frac{1}{p}\mathbb{N} \text{ and } \sum_{j=1}^{e} pB_j = \epsilon(x). \tag{2.47}$$

We can restate previous material as follows:

Proposition 2.16. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ and assume that $\epsilon(x) > 0$. The following holds:*

(i) the vector space $V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1}$ satisfies

$$V(F_{p,Z}, E, m_S) = 0 \Leftrightarrow F_{p,Z} \in S/m_S[U_1, \ldots, U_e][U_{e+1}^p, \ldots U_n^p];$$

(ii) the vector space $J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)}$ satisfies

$$J(F_{p,Z}, E, m_S) = 0 \Leftrightarrow F_{p,Z} \in (S/m_S[U_1, \ldots, U_n])^p\,;$$

(iii) if $i_0(x) = p$, the vector space $V(F_{p,Z}, E, m_S)$ is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$; if $i_0(x) = p$ and $V(F_{p,Z}, E, m_S) = 0$, the vector space $J(F_{p,Z}, E, m_S)_{\epsilon(x)}$ is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$.

Proof. The first statement follows from (2.39) and (2.44), while (ii) follows from (2.38). Assume now that $i_0(x) = p$, i.e. $G = 0$.

To begin with, the situation in (ii) does not occur because the polyhedron $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. If $Z' = Z - \theta$, $\theta \in \hat{S}$ with $\mathrm{ord}_{m_S}\theta \geq \delta(x)/p$, we have $F_{p,Z'} = F_{p,Z} + \Theta^p$ for some $\Theta \in S/m_S[U_1, \ldots, U_n]_{\delta(x)/p}$ (so $\Theta = 0$ if $\delta(x) \notin \mathbb{N}$). Hence $D \cdot F_{p,Z'} = D \cdot F_{p,Z}$ for every $D \in \mathrm{Der}(G(m_S))$.

By elementary calculus, the vector space

$$V(F_{p,Z}, E, m_S) = H^{-1} < \left\{ \frac{\partial F_{p,Z}}{\partial U_j} \right\}_{e+1 \leq j \leq n} >$$

is unchanged by adapted coordinate change (more generally by changes stabilizing the vector space $< U_1, \ldots, U_e >$) and this proves the first statement in (iii). If $V(F_{p,Z}, E, m_S) = 0$, the vector space

$$J(F_{p,Z}, E, m_S) = H^{-1} < \left\{ U_j \frac{\partial F_{p,Z}}{\partial U_j} \right\}_{1 \leq j \leq e}, \left\{ \frac{\partial F_{p,Z}}{\partial \lambda_l} \right\}_{l \in \Lambda_0} > .$$

is not affected either by changes of coordinates fixing each $< U_j >$, $j \leq e$. $\square$

We now turn to the version of proposition 2.16(iii) for $i_0(x) = p - 1$. The problem is elementary, though more technical, and the remaining part of this section is devoted to it.

Let $(\mathbf{e}_j)_{1 \leq j \leq n}$ be the standard basis of $\mathbb{R}^n$ and let

$$\mathbb{E} := \{\mathbf{x} \in \mathbb{R}^n : x_{e+1} = \cdots = x_n = 0\} \simeq \mathbb{R}^e.$$

Given $d \in \frac{1}{p}\mathbb{N}$ and $\mathbf{H} \in \mathbb{N}^n \cap \mathbb{E}$, we denote

$$\Delta_{\mathbf{H}}(d) := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} :| \mathbf{x} |= d \text{ and } x_j \geq \frac{H_j}{p}, 1 \leq j \leq e\}$$

and

$$\mathcal{V}_{\mathbf{H}}(pd) := (U^{\mathbf{H}}) \cap G(m_S)_{pd} \subseteq G(m_S)_{pd}. \tag{2.48}$$

We fix once and for all $\mathbf{b} \in (\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)) \cap \mathbb{E}$. Note that $\mathcal{V}_{\mathbf{H}}(pd) \neq (0)$ only if $H_1 + \cdots + H_e \leq pd$ and that such $\mathbf{b}$ as above exists only if $d \in \mathbb{N}$. By convention, we take $\{\mathbf{b}\} = \emptyset$ if $d \notin \mathbb{N}$ in the following formulæ. For applications, we will take $d = \delta(x_0)$, $\mathbf{H}$ as in definition 2.10 and $\mathbf{b}$ will be defined by $< G >=:< U_1^{b_1} \cdots U_e^{b_e} >$.

Notation 2.4. Any homogeneous polynomial $F \in \mathcal{V}_{\mathbf{H}}(pd)$ has a unique expansion of the form

$$F := \sum_{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)} \lambda(\mathbf{x}) U^{p\mathbf{x}}, \lambda(\mathbf{x}) \in S/m_S.$$

We denote

$$\Delta(F) := \mathrm{Conv}(\{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \lambda(\mathbf{x}) \neq 0\} \cup \{\mathbf{b}\}) \subseteq \Delta_{\mathbf{H}}(d).$$

According to theses conventions, we have $\Delta(0) = \{\mathbf{b}\}$.

Definition 2.14. With notations as above, let $T : \mathcal{V}_{\mathbf{H}}(pd) \to \mathcal{V}_{\mathbf{H}}(pd)$ be the S/m_S-linear truncation operator defined as follows: let

$$A := \{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \mathbf{b} + p(\mathbf{x} - \mathbf{b}) \in \Delta_{\mathbf{H}}(d)\}. \tag{2.49}$$

and

$$TF := \sum_{\mathbf{x} \notin A} \lambda(\mathbf{x}) U^{p\mathbf{x}} \in \mathcal{V}_{\mathbf{H}}(pd). \tag{2.50}$$

For $d \notin \mathbb{N}$, we have $A = \emptyset$ and T is the identity map.

The construction of the previous section associates two vector spaces $V(TF, E, m_S)$ and $J(TF, E, m_S)$. Explicitly, we have:

$$V(TF, E, m_S) = U^{-\mathbf{H}} < \frac{\partial TF}{\partial U_j}, e+1 \leq j \leq n > \subseteq G(m_S)_{pd-1-|\mathbf{H}|}$$

for the former one. If $V(TF, E, m_S) = 0$ (and only in this case), we will use the latter one, given explicitly by and

$$J(TF, E, m_S) = U^{-\mathbf{H}} < \{U_j \frac{\partial TF}{\partial U_j}\}_{1 \leq j \leq e}, \{\frac{\partial TF}{\partial \lambda_l}\}_{l \in \Lambda_0} > \subseteq G(m_S)_{pd-|\mathbf{H}|},$$

with notations as in the previous section. We can now state:

Lemma 2.17. *Assume that $d \in \mathbb{N}$. With notations as above, we have*

$$\mathrm{Ker} T = U^{(p-1)\mathbf{b}} \mathcal{V}_{\lceil \frac{\mathbf{H}}{p} \rceil}(d),$$

where $\lceil \frac{\mathbf{H}}{p} \rceil := (\lceil \frac{H_1}{p} \rceil, \ldots, \lceil \frac{H_e}{p} \rceil, 0, \ldots, 0)$.

Let $G := \mu U^{\mathbf{b}}$, $\mu \in S/m_S$, $\Phi \in \mathcal{V}_{\lceil \frac{\mathbf{H}}{p} \rceil}(d)$ and $F \in \mathcal{V}_{\mathbf{H}}(pd)$. Then

$$V(T(F + \Phi^p - G^{(p-1)}\Phi), E, m_S) = V(TF, E, m_S).$$

If $V(TF, E, m_S) = 0$, then

$$J(T(F + \Phi^p - G^{(p-1)}\Phi), E, m_S) = J(TF, E, m_S),$$

Proof. We analyze the definition of T in (2.50). The kernel of T is generated by those monomials $U^{p\mathbf{x}} \in \mathcal{V}_{\mathbf{H}}(pd)$ such that

$$\mathbf{y} := p\mathbf{x} - (p-1)\mathbf{b} \in \Delta_{\mathbf{H}}(d).$$

Since $\mathbf{x} \in \frac{1}{p}\mathbb{N}^n$, $\mathbf{b} \in \mathbb{N}^n$, we have $\mathbf{y} \in \mathbb{N}^n$ for such $\mathbf{y}$. Therefore Ker$T$ is generated by

$$\mathrm{Ker}T =< \{U^{(p-1)\mathbf{b}}U^{\mathbf{y}} : \mathbf{y} \in \mathbb{N}^n, \mid \mathbf{y} \mid = d \text{ and } y_j \geq \frac{H_j}{p}, 1 \leq j \leq e\} > .$$

This proves the first statement. For the second part, we have proved that

$$T(F + \Phi^p - G^{(p-1)}\Phi) = TF + T\Phi^p.$$

Hence $D \cdot T(F + \Phi^p - G^{(p-1)}\Phi) = D \cdot TF$ for every $D \in \mathrm{Der}(G(m_S))$. $\square$

We now study invariance properties of $V(F, E, m_S)$ and $J(F, E, m_S)$ under changes of adapted coordinates. Given two r.s.p.'s $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{u}' = (u'_1, \ldots, u'_n)$ adapted to E, there exists a matrix $M \in \mathcal{M}(S)$,

$$\mathcal{M}(S) := \{(m_{ij}) \in \mathrm{GL}(n, S) : m_{jj'} = 0, (j, j') \in \{1, \ldots, e\} \times \{1, \ldots, n\}, j \neq j'\}$$

such that $\mathbf{u} = M\mathbf{u}'$. The set $\mathcal{M}(S)$ is the set of S-points of an affine S-scheme $\mathcal{M} \subset \mathrm{GL}(n, S)$. Denote by

$$\mathrm{GL}(n, S) \to \mathrm{GL}(n, S/m_S), \ M \mapsto \overline{M}$$

the canonical surjection. Each such $\overline{M}$ induces a graded S/m_S-automorphism of $\mathrm{gr}_{m_S}(S) \simeq S/m_S[U_1, \ldots, U_n]$. By (2.48), this automorphism restricts to an automorphism of $\mathcal{V}_{\mathbf{H}}(pd)$ for each $d \in \frac{1}{p}\mathbb{N}$ still denoted by $\overline{M}$.

Given a homogeneous polynomial $F \in \mathcal{V}_{\mathbf{H}}(pd)$ as above and a matrix $\overline{M} \in \mathcal{M}(S/m_S)$, we denote for simplicity the transformed equation $U \mapsto \overline{M}U'$ by

$$F' =: \sum_{\mathbf{x}' \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)} \lambda'(\mathbf{x}')U'^{p\mathbf{x}'}. \tag{2.51}$$

Let $\Delta(F') := \mathrm{Conv}(\{\mathbf{x}' \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \lambda'(\mathbf{x}') \neq 0\} \cup \{\mathbf{b}\}) \subseteq \Delta_{\mathbf{H}}(d)$ be the corresponding polytope and T' be the corresponding operator on $\mathcal{V}_{\mathbf{H}}(pd)$ with variable U'. The linear operator T obviously does not commute with $\overline{M}$ in general (i.e. $(TF)' \neq T'F'$ in general), but the lemma below extracts the

relevant invariant data. We refer to definition 2.13 for the notation $\mathrm{Max}(I)$, $I \subset G(m_S)$ generated by homogeneous polynomial of one and the same degree.

Notation 2.5. We denote by

$$B := \{j,\ 1 \leq j \leq e : pb_j - H_j > 0\} \text{ and } U_B := \{U_j, j \in B\}. \tag{2.52}$$

We denote $U_{B'} := \{U_j, j \notin B\}$ and stick to our former conventions, i.e.

$$B' = \{1, \ldots, n\} \backslash B,\ (B')_E = \{1, \ldots, e\} \backslash B.$$

Lemma 2.18. *With notations as above, there is an equality of sets*

$$\mathrm{Max}(V(TF, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(V(T'F', E, m_S)) \cap \{U'_B = 0\}. \tag{2.53}$$

If $V(TF, E, m_S) = 0$, then $V(T'F', E, m_S) = 0$ and there is an equality of sets

$$\mathrm{Max}(J(TF, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(J(T'F', E, m_S)) \cap \{U'_B = 0\}. \tag{2.54}$$

Proof. The operator T commutes with $\overline{M}$ when $\overline{M}$ stabilizes the vector space $< U_{e+1}, \ldots, U_n >$. In these cases, we have

$$V(T'F', E, m_S) = V((TF)', E, m_S).$$

If $V(TF, E, m_S) = 0$, then

$$V(T'F', E, m_S) = 0 \text{ and } J(T'F', E, m_S) = J((TF)', E, m_S).$$

So the lemma is trivial in this case and we may therefore assume that

$$m_{jj'} = 0, (j, j') \in \{e+1, \ldots, n\} \times \{e+1, \ldots, n\}, j \neq j' \text{ and } m_{jj} = 1, 1 \leq j \leq n.$$

By elementary calculus, this new assumption implies for every $\Phi \in G(m_S)$:

$$\frac{\partial \Phi'}{\partial U'_j} = \left(\frac{\partial \Phi}{\partial U_j}\right)',\ e + 1 \leq j \leq n. \tag{2.55}$$

Let $\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)$. Since $pb_j = H_j$ for $j \in (B')_E$, we have by (2.49):

$$\mathbf{x} \in A \Leftrightarrow \forall j \in B, px_j \geq (p-1)b_j.$$

Expand $TF = \sum_{\mathbf{y}} U_B^{\mathbf{y}} F_{\mathbf{y}}(U_{B'})$, so we have:

$$V(TF, E, m_S) = U^{-\mathbf{H}} < \{\sum_{\mathbf{y}} U_B^{\mathbf{y}} \frac{\partial F_{\mathbf{y}}(U_{B'})}{\partial U_j}\}_{e+1 \leq j \leq n} > .$$

For $P \in \mathrm{Spec}G(m_S)$ such that $(U_B) \subseteq P$, we get:

$$P \in \mathrm{Max}(V(TF, E, m_S)) \Leftrightarrow P \in \bigcap_{\mathbf{y}} \bigcap_{j=e+1}^{n} \mathrm{Max}(G_{\mathbf{y}}), \tag{2.56}$$

where $G_{\mathbf{y}} := U_{B'}^{-\mathbf{H}'} \frac{\partial F_{\mathbf{y}}(U_{B'})}{\partial U_j}$, $\mathbf{H}' := (H_{j'})_{j' \in (B')_E}$.

Suppose furthermore that $\overline{M}$ stabilizes the vector space $< U_{B'} >$. Then T also commutes with $\overline{M}$ and each term $G_{\mathbf{y}}$ in (2.56) is transformed into

$$(G_{\mathbf{y}})' = U_{B'}^{-\mathbf{H}_{B'}} \frac{\partial F'_{\mathbf{y}}(U'_{B'})}{\partial U'_j}$$

by (2.55) and (2.53) follows. Suppose furthermore that $V(TF, E, m_S) = 0$; then $G_{\mathbf{y}} = 0$ for each $\mathbf{y}$ in (2.55) and we get $V(T'F', E, m_S) = 0$. For $1 \leq j \leq e$ and $l \in \Lambda_0$, we have

$$\left(U_j \frac{\partial TF}{\partial U_j}\right)' = U'_j \frac{\partial T'F'}{\partial U'_j}, \ \left(\frac{\partial TF}{\partial \lambda_l}\right)' = \frac{\partial T'F'}{\partial \lambda_l}, \tag{2.57}$$

and (2.54) also follows. Hence we may furthermore assume that

$$m_{jj'} = 0, (j, j') \in \{e+1, \ldots, n\} \times (B')_E.$$

In this situation, T does not commute any longer with $\overline{M}$. However, for each term $G_{\mathbf{y}}$ as above, we have

$$\mathrm{ord}_P(D \cdot G_{\mathbf{y}}) \geq \mathrm{deg} G_{\mathbf{y}} - a \tag{2.58}$$

for any differential operator D on $S/m_S[U_{B'}]$ of order not greater than a. Let

$$(G_{\mathbf{y}})' = \sum_{|\alpha| \leq \mathrm{deg} G_{\mathbf{y}}} (U'_B)^{\alpha} (D^{(\alpha)} \cdot G_{\mathbf{y}}), \ D^{(\alpha)} \cdot G_{\mathbf{y}} \in S/m_S[U'_{B'}]_{\mathrm{deg} G_{\mathbf{y}} - |\alpha|}$$

be the (characteristic free) Taylor expansion, where $D^{(\alpha)}$ is a differential operator of order $\mid \alpha \mid$. Take again $P \in \mathrm{Spec}G(m_S)$ such that $(U_B) \subseteq P$. By (2.58), we have

$$P \in \mathrm{Max}(G_{\mathbf{y}}) \Rightarrow P \in \bigcap_{\alpha} \mathrm{Max}(D^{(\alpha)} \cdot G_{\mathbf{y}}) \Rightarrow P \in \mathrm{Max}((G_{\mathbf{y}})').$$

We deduce from (2.56) that

$$P \in \mathrm{Max}(V(TF, E, m_S)) \Rightarrow P \in \mathrm{Max}(V((TF)', E, m_S)).$$

This proves (2.53). If $V(TF, E, m_S) = 0$, (2.54) follows from (2.57) as above. □

This lemma is the key to our version of proposition 2.16(iii) for $i_0(x) = p - 1$:

Proposition 2.19. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ and assume that $\epsilon(x) > 0$ and $i_0(x) = p - 1$. Let*

$$d := \delta(x), \ \mathbf{H} := (H_1, \ldots, H_e, 0, \ldots, 0) \text{ and } < U_1^{b_1} \cdots U_e^{b_e} > := < G >$$

be defined respectively by definition 2.5, definition 2.10 and (2) of theorem 2.14. With notations as above, the following holds:

(i) the set

$$\mathrm{Max}(V(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} \subseteq \mathrm{Spec}G(m_S)$$

is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$;

(ii) the property $V(TF_{p,Z}, E, m_S) = 0$ is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$; when it holds, the set

$$\mathrm{Max}(J(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} \subseteq \mathrm{Spec}G(m_S)$$

is also independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$.

Proof. For such $(u_1, \ldots, u_n; Z)$, the corresponding initial form is

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z].$$

Since $G \neq 0$, we have $d = \delta(x) = \deg G \in \mathbb{N}$. If $(u'_1, \ldots, u'_n)$ is an adapted r.s.p. of S, there exists $M \in \mathcal{M}(S)$ such that $\mathbf{u} = M\mathbf{u}'$. Let $(u'_1, \ldots, u'_n; Z')$ be well adapted coordinates at x. We have $Z' = Z - \phi$ for some $\phi \in S$, with $\mathrm{ord}_{m_S}\phi \geq d$. We deduce that

$$\mathrm{in}_{m_S} h = Z'^p - G^{p-1}Z' + \Phi^p - G^{p-1}\Phi + F_{p,Z} \in G(m_S)[Z']$$

for some $\Phi := \mathrm{cl}_d\phi \in G(m_S)_d$. We deduce the formula

$$F_{p,Z'} = F_{p,Z} + \Phi^p - G^{p-1}\Phi.$$

By lemma 2.17, we have $V(TF_{p,Z'}, E, m_S) = V(TF_{p,Z}, E, m_S)$; if moreover $V(TF_{p,Z}, E, m_S) = 0$, then $J(TF_{p,Z'}, E, m_S) = J(TF_{p,Z}, E, m_S)$. By lemma 2.18, we have an equality of sets

$$\mathrm{Max}(V(TF_{p,Z'}, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(V(T'F'_{p,Z'}, E, m_S)) \cap \{U'_B = 0\}$$

and this proves (i). If $V(TF_{p,Z'}, E, m_S) = 0$, then $V(T'F'_{p,Z'}, E, m_S) = 0$ by lemma 2.18 and there is an equality of sets

$$\mathrm{Max}(J(TF_{p,Z'}, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(J(T'F'_{p,Z'}, E, m_S)) \cap \{U'_B = 0\}.$$

This concludes the proof. □

Remark 2.5. We consider proposition 2.16(iii) as the special case $B = \emptyset$, $T = \mathrm{id}$ of proposition 2.19.

2.7 Main invariants.

Let $s \in \mathrm{Spec}S$ and $y \in \eta^{-1}(s)$. The purpose of this section is to attach to y a resolution complexity

$$\iota(y) = (m(y), \omega(y), \kappa(y)) \in \{1, \ldots, p\} \times \mathbb{N} \times \{1, \ldots, 4\} \tag{2.59}$$

with certain invariance properties. Auxiliary numbers

$$(\tau(y), \tau'(y)) \in \{1, \ldots, n+1\} \times \{1, \ldots, n\} \tag{2.60}$$

are similarly attached to y.

The pair $(m(y), \tau(y))$ are the standard multiplicity and Hironaka τ-number of $\mathcal{X}$ at y (definition 2.15). The pair $(\omega(y), \tau'(y))$ play the role of a

differential multiplicity and differential τ-number attached to $\eta : \mathcal{X} \to \mathrm{Spec} S$ at y. The behavior of the function ι under blowing up is studied in theorem 3.6 below.

In all definitions that follow it can be assumed without loss of generality that $s = m_S$ by localizing S at s, since our assumptions **(G)** and **(E)** are stable when changing (S, h, E) to (S_s, h_s, E_s) (notation 2.2).

Definition 2.15. (Multiplicity). Let $x \in \eta^{-1}(m_S)$. We have already defined

$$m(x) = \mathrm{ord}_{m_{S[X]_x}} h(X) \leq p.$$

Let $M_x \subset S[X]$ be the ideal of x, $G_x := \mathrm{Spec}(\mathrm{gr}_{M_x} S[X]_{M_x})$ and H_x be the initial form of h in $(G_x)_{m(x)}$. From definition 2.12, we let

$$\tau(x) := \tau(H_x).$$

If $m(x) < p$, we let $\iota(x) := (m(x), 0, 1)$.

Note that $m(y) < p$ whenever $s = \eta(y) \notin E$ (definition 2.11 and following comments). If $m(y) = p$, we have

$$s = \eta(y) \in E, \ \eta^{-1}(s) = \{y\} \text{ and } k(y) = k(s)$$

by proposition 2.10.

Applying proposition 2.16(iii) (resp. proposition 2.19(ii)) to S if $i_0(x) = p$ (resp. if $i_0(x) = p - 1$) proves that $(\omega(x), \kappa(x))$ is well-defined. We recall that $TF_{p,Z} = F_{p,Z}$ whenever $i_0(x) = p$ (see remark 2.5).

Definition 2.16. (Adapted order). Assume that $m(x) = p$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. We let

$$\omega(x) = \begin{cases} \epsilon(x) - 1 & \text{if} \quad V(TF_{p,Z}, E, m_S) \neq 0 \\ \epsilon(x) & \text{if} \quad V(TF_{p,Z}, E, m_S) = 0 \end{cases}.$$

We define:

$$\kappa(x) := 1 \text{ if } (\omega(x) = \epsilon(x) \text{ and } i_0(x) = p - 1).$$

Otherwise, we simply let $\kappa(x) \geq 2$.

Remark 2.6. It is obvious from this definition that $\omega(x)$ is not determined by the characteristic polyhedra $\Delta_S(h; u_1, \ldots, u_n; Z)$, even for unspecified well adapted coordinates $(u_1, \ldots, u_n; Z)$.

For example, take $n = 3$, $p \geq 3$ for simplicity and $k(x)$ algebraically closed of characteristic $p > 0$. Suppose:

$$\mathrm{in}_{m_S} h = Z^p + U_1 U_2 U_3^p + U_1^{p+2} + U_2^{p+2} + c U_3 U_2 U_1^p, \ E = \mathrm{div}(u_1 u_2),$$

where $c \in k(x)$. Let $(u_1', u_2', u_3'; Z')$ be well adapted coordinates such that $\mathrm{div}(u_j) = \mathrm{div}(u_j')$ for $j = 1, 2$. Then

$$\Delta_S(h; u_1', u_2', u_3'; Z') = \mathrm{Conv}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}) \subset \{x_1 + x_2 + x_3 = \delta(x) = 1 + 2/p\}$$

is independent of c, where

$$\mathbf{v}_1 := ((p+2)/p, 0, 0), \ \mathbf{v}_2 := (0, (p+2)/p, 0), \ \mathbf{v}_3 := (1/p, 1/p, 1).$$

But $\omega(x) = p + 2$ (resp. $\omega(x) = p + 1$) for $c = 0$ (resp. for $c \neq 0$).

Remark 2.7. This definition is different from the one used in [27] chapter 1, definition **II.4** when $G \neq 0$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. There is an obvious implication

$$\omega(x) = \epsilon(x) - 1 \Longrightarrow V(F_{p,Z}, E, m_S) \neq 0.$$

The converse is however false, even if it is assumed that $V(F_{p,Z}, E, m_S) \neq 0$ for every possible choice of well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x and this is the reason for this difference. For $n \leq 3$, this phenomenon is easily dealt with, *vid.* [27] chapter 1 **II.3.3.1** and **II.3.3.2**; proof of **II.5.4.2**(iv); theorem **II.5.6**.

In chapter 4, we define the projection number $\kappa(x) \in \{2, 3, 4\}$ when $n = 3$ and state that $\iota(x) = (m(x), \omega(x), \kappa(x))$ can be decreased by Hironaka permissible blowing ups w.r.t. E (projection theorem 5.1 below).

We now turn to the definition of the adapted cone and directrix and the attached invariant $\tau'(x)$. Applying proposition 2.16(iii) (resp. proposition 2.19) if $i_0(x) = p$ (resp. if $i_0(x) = p - 1$) proves that $\mathrm{Max}(x)$, $\mathrm{Dir}(x)$ and $\tau'(x)$ are well defined.

Definition 2.17. (Adapted cone and directrix). Assume that $m(x) = p$ and $\omega(x) > 0$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. We define a reduced subcone $\mathrm{Max}(x) \subseteq \mathrm{Spec}G(m_S)$ by:

$$\mathrm{Max}(x) := \begin{cases} \mathrm{Max}(V(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} & \text{if} \quad \omega(x) = \epsilon(x) - 1 \\ \mathrm{Max}(J(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} & \text{if} \quad \omega(x) = \epsilon(x) \end{cases}.$$

We define an affine subspace $\mathrm{Dir}(x) \subseteq \mathrm{Spec}G(m_S)$ by

$$\mathrm{Dir}(x) := \begin{cases} \mathrm{Dir}(V(TF_{p,Z}, E, m_S), U_B) & \text{if} \quad \omega(x) = \epsilon(x) - 1 \\ \mathrm{Dir}(J(TF_{p,Z}, E, m_S), U_B) & \text{if} \quad \omega(x) = \epsilon(x) \end{cases}.$$

We let $\mathrm{Vdir}(x)$ to be the underlying vector space of $\mathrm{Dir}(x)$ and

$$\tau'(x) := \dim_{k(x)} \mathrm{Vdir}(x).$$

Remark 2.8. We will use the invariants $\mathrm{Dir}(x)$ and $\tau'(x)$ only when $\mathrm{Dir}(x) = \mathrm{Max}(x)$ (last statement in proposition 2.15 and following remark).

Let $S \subseteq \tilde{S}$ be a regular local base change, $\tilde{S}$ excellent. Recall notation 2.1 and notation 2.2. It has been explained when defining conditions **(G)** and **(E)** that they are stable by such base changes and by localization at a prime. Let $\tilde{s} \in \mathrm{Spec}\tilde{S}$ and $\tilde{y} \in \tilde{\eta}^{-1}(\tilde{s})$. In order to relate $\iota(\tilde{y})$ and $\iota(y)$ (2.59), where $y \in \mathcal{X}$ is the image of $\tilde{y}$, we may thus assume that $s = m_S$, $\tilde{s} = m_{\tilde{S}}$.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. Then $(u_1, \ldots, u_n)$ can be completed to a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$ which is adapted to $\tilde{E}$. There is an inclusion

$$G(m_S) = k(x)[U_1, \ldots, U_n] \subseteq G(m_{\tilde{S}}) = G(m_S) \otimes_{k(x)} \frac{\tilde{S}}{m_{\tilde{S}}}[\tilde{U}_{n+1}, \ldots, \tilde{U}_{\tilde{n}}]. \tag{2.61}$$

Theorem 2.20. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Let $\tilde{x} \in \tilde{\eta}^{-1}(m_{\tilde{S}})$ and $x \in \eta^{-1}(m_S)$ be its image. The following holds:*

(1) we have $(m(\tilde{x}), \omega(\tilde{x})) = (m(x), \omega(x))$;

(2) if $m(x) = p$, then

(i) $H(\tilde{x}) = H(x)\tilde{S}$, $i_0(\tilde{x}) = i_0(x)$, and ($\kappa(\tilde{x}) = 1 \Leftrightarrow \kappa(x) = 1$);

(ii) we have $\epsilon(\tilde{x}) \geq \epsilon(x)$, and $\epsilon(\tilde{x}) > \epsilon(x)$ if and only if

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z},\ F_{p,Z} \in (k(\tilde{x})[U_1, \ldots, U_n])^p$$

where $(u_1, \ldots, u_n; Z)$ are well prepared coordinates at x. When this holds, we have $\tilde{n} > n$, $\epsilon(\tilde{x}) = \epsilon(x) + 1$ and

$$\mathrm{in}_{m_{\tilde{S}}} \tilde{h} = \tilde{Z}^p + \sum_{j=n+1}^{\tilde{n}} U_j \Phi_j(U_1, \ldots, U_n) + \Psi(U_1, \ldots, U_n) \in G(m_{\tilde{S}})[\tilde{Z}],$$

with $\Phi_j \neq 0$ for some $j \geq n+1$ and $\Phi_j \in k(\tilde{x})[U_1^p, \ldots, U_n^p]$ for every $j \geq n+1$, where $(u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ are well prepared coordinates at $\tilde{x}$.

Proof. The theorem is trivial if $m(x) = 1$: then $m(\tilde{x}) = 1$ because $S \subseteq \tilde{S}$ is regular.

Assume that $m(x) \geq 2$ and pick well prepared coordinates $(u_1, \ldots, u_n; Z)$ at x, then complete $(u_1, \ldots, u_n)$ to a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$ which is adapted to $\tilde{E}$. We have $\delta(x) > 0$, so $h \in (Z, u_1, \ldots, u_n)$, and $k(x) = S/m_S$ by proposition 2.10. Applying (2.61) to the local base change $S[Z]_{(m_S,Z)} \subseteq T[Z]_{(m_T,Z)}$ which is also regular gives

$$m(x) = \mathrm{ord}_x h(Z) = \mathrm{ord}_{\tilde{x}} \tilde{h}(Z) = m(\tilde{x}).$$

This concludes the proof when $m(x) < p$ and we assume from now on that $m(x) = p$. In particular we have $\{\tilde{x}\} = \tilde{\eta}^{-1}(m_{\tilde{S}})$, $k(\tilde{x}) = \tilde{S}/m_{\tilde{S}}$. Let

$$\mathrm{in}_{m_S} h = Z^p + \sum_{i=1}^{p} F_{i,Z} Z^{p-i} \in G(m_S)[Z],$$

be the corresponding initial form polynomial. Let $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ be a vertex of the polyhedron $\Delta_S(u_1, \ldots, u_n; Z)$. We denote by

$$\tilde{\mathbf{x}} := (\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$$

the corresponding vertex in $\Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$. Note that $\tilde{\mathbf{x}}$ *may be a solvable vertex* of the latter polyhedron. We have:

$$\tilde{\mathbf{x}} \text{ solvable} \Leftrightarrow \mathrm{in}_{\tilde{\mathbf{x}}} \tilde{h} \in ((\mathrm{gr}_\alpha \tilde{S})[Z])^p$$

with notations as in definition 2.3. Therefore we have

$$\tilde{\mathbf{x}} \text{ solvable} \Leftrightarrow (\mathrm{in}_{\mathbf{x}} h = Z^p + F_{p,Z,\mathbf{x}}, \mathbf{x} \in \mathbb{N}^n,\ F_{p,Z,\mathbf{x}} = \lambda U^{p\mathbf{x}}, \lambda \in k(\tilde{x})^p).$$

We deduce for the initial form polynomial that

$$\delta(\tilde{x}) > \delta(x) \Leftrightarrow (i_0(x) = p \text{ and } F_{p,Z} \in (k(\tilde{x})[U_1, \ldots, U_n])^p). \tag{2.62}$$

Since the fiber ring $\tilde{S}/m_S\tilde{S}$ is geometrically regular over $k(x)$, the ring $\tilde{S}[Y]/(Y^p - l)$ is regular for every unit $l \in S$ with residue $\bar{l} \notin k(x)^p$. Therefore if $\bar{l} \in k(\tilde{x})^p$, we have

$$\forall \tilde{l} \in \tilde{S}, \tilde{v} := \tilde{l}^p - l \in m_{\tilde{S}} \Longrightarrow \tilde{v} \text{ is a regular parameter in } \tilde{S}.$$

Such $\tilde{v}$ restricts to a regular parameter of $\tilde{S}/m_S\tilde{S}$, so the previous formula is refined to:

$$\tilde{v} \text{ is a regular parameter transverse to } \mathrm{div}(u_1 \cdots u_n) \subset \mathrm{Spec}\tilde{S}. \tag{2.63}$$

This equation implies in particular that $\tilde{n} > n$. Let $\xi \in \mathrm{Spec}(\tilde{S}/m_S\tilde{S})$ be the generic point. Applying the above remarks to the regular local base change $S \subset \tilde{S}_\xi$ shows that $k(\xi)^p \cap k(x) = k(x)^p$.

Let $s_j := (u_j) \in \mathrm{Spec}S$, $1 \leq j \leq e$, and apply this remark to the regular local base change $S_{(u_j)} \subseteq \tilde{S}_{(u_j)}$. This proves that the field inclusion $QF(S/(u_j)) \subseteq QF(\tilde{S}/(u_j))$ is inseparably closed.

The polynomial $\mathrm{in}_{(s_j)} h_{s_j} \in QF(S/(u_j))[U_j][Z]$ is not a pth-power by proposition 2.4. Therefore $\mathrm{in}_{(s_j)} h_{s_j}$ is not a pth-power in $QF(\tilde{S}/(u_j))[U_j][Z]$. Turning back to definition 2.9, we get

$$H(\tilde{x}) = H(x)\tilde{S}. \tag{2.64}$$

Definition 2.9 now shows that $\epsilon(\tilde{x}) \geq \epsilon(x)$ and that

$$\epsilon(\tilde{x}) > \epsilon(x) \Leftrightarrow (i_0(x) = p \text{ and } F_{p,Z} \in (k(\tilde{x})[U_1, \ldots, U_n])^p). \tag{2.65}$$

This proves the first part of (2.ii). To go on with the proof, we consider two cases.

Case 1: assume that $i_0(x) < p$. By (2.65), we have $\epsilon(\tilde{x}) = \epsilon(x)$, so the proof of (2.ii) is already complete. Let $\tilde{\phi} \in \tilde{S}$ be such that $\Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ is minimal, with $\tilde{Z} := Z - \tilde{\phi}$ and $\mathrm{ord}_{m_{\tilde{S}}} \tilde{\phi} \geq \delta(x)$. We have

$$\mathrm{in}_{m_{\tilde{S}}} \tilde{h} = \tilde{Z}^p + \sum_{i=i_0}^{p} F_{i,\tilde{Z}} \tilde{Z}^{p-i} \in G(m_{\tilde{S}})[\tilde{Z}],$$

with $F_{i_0,\tilde{Z}} = F_{i_0,Z}$ by proposition 2.9. Therefore $i_0(\tilde{x}) = i_0(x)$ and it is sufficient to prove that $\omega(\tilde{x}) = \omega(x)$ in order to complete the proof of (1) and (2.i) in the theorem (still under the assumption $i_0(x) < p$). This is obvious if $\epsilon(x) = 0$, since

$$0 \leq \omega(\tilde{x}) \leq \epsilon(\tilde{x}) = \omega(x) = 0.$$

Assume that $\epsilon(x) > 0$. We have $i_0(x) = p - 1$ and $-F_{p-1,Z} = G^{p-1}$, with $< G >=< U^{\mathbf{b}} >$ for some $\mathbf{b} \in \mathbb{N}^n \cap \mathbb{E}$ by theorem 2.14(2) (in particular $\delta(x) \in \mathbb{N}$). We have

$$V(TF_{p,Z}, E, m_S) =< \left\{ H^{-1} \frac{\partial TF_{p,Z}}{\partial U_j} \right\}_{e+1 \leq j \leq n} > .$$

Note that the truncation maps T and $\tilde{T}$ associated with the local rings S and $\tilde{S}$ (definition 2.14) commute with the inclusion $G(m_S) \subseteq G(m_{\tilde{S}})$ by (2.64). Since $F_{p,Z} \in G(m_S) = k(x)[U_1, \ldots, U_n]$, we have

$$V(\tilde{T}F_{p,Z}, \tilde{E}, m_{\tilde{S}}) =< \left\{ H^{-1} \frac{\partial \tilde{T}F_{p,Z}}{\partial U_j} \right\}_{j=e+1}^{\tilde{n}} >= V(TF_{p,Z}, E, m_S) \otimes_{k(x)} k(\tilde{x})$$

with obvious notations, taking (2.64) into account. There exists $\tilde{\Theta} \in G(m_{\tilde{S}})$ such that

$$F_{p,\tilde{Z}} = F_{p,Z} + \tilde{\Theta}^p - G^{p-1}\tilde{\Theta}.$$

By lemma 2.17 applied to $F_{p,\tilde{Z}} \in G(m_{\tilde{S}})$, we deduce that

$$V(\tilde{T}F_{p,\tilde{Z}}, \tilde{E}, m_{\tilde{S}}) = V(TF_{p,Z}, E, m_S) \otimes_{k(x)} k(\tilde{x}). \tag{2.66}$$

This completes the proof of the theorem when $\omega(x) = \epsilon(x) - 1$, applying definition 2.16. If $\omega(x) = \epsilon(x)$, (1) and the last statement of (2.i) in the theorem also follow from (2.66) and the proof is complete.

Case 2: assume that $i_0(x) = p$. The proof runs parallel to that of case 1 (with $B = \emptyset$, $\tilde{T} = \mathrm{id}$, *cf.* remark 2.5) *provided that* $\epsilon(\tilde{x}) = \epsilon(x)$. Assume now that $\epsilon(\tilde{x}) > \epsilon(x)$. To complete the proof, we have to show that

$$(i_0(\tilde{x}), \omega(\tilde{x})) = (p, \omega(x)),$$

as well as the last statement in (2.ii). By (2.65), we have $\omega(x) = \epsilon(x)$, $\delta(x) \in \mathbb{N}$ and there is an expansion

$$F_{p,Z} = \sum_{|\mathbf{x}|=\delta(x)} \lambda(\mathbf{x}) U^{p\mathbf{x}} \in (k(\tilde{x})[U_1, \ldots, U_n]_{\delta(x)})^p, \ \lambda(\mathbf{x}) \in k(x).$$

Note that this situation possibly occurs only if $k(x)$ is *not* inseparably closed in $k(\tilde{x})$ (in particular $\tilde{n} > n$). We have $\mathbf{x} \in \mathbb{N}^n$ for every $\mathbf{x}$ such that $\lambda(\mathbf{x}) \neq 0$. Without loss of generality, it can be assumed that $\lambda(\mathbf{x}) \notin k(x)^p$ for every $\mathbf{x}$ such that $\lambda(\mathbf{x}) \neq 0$. Let $l(\mathbf{x}) \in S$ be a preimage of $\lambda(\mathbf{x})$. By (2.63), we may pick for every such $\mathbf{x}$ a unit $\tilde{l}(\mathbf{x}) \in T$ such that $\tilde{v}(\mathbf{x}) := \tilde{l}(\mathbf{x})^p - l(\mathbf{x})$ is a regular parameter of $\tilde{S}$ transverse to $\mathrm{div}(u_1 \cdots u_n)$. Expand

$$h = Z^p + \sum_{i=1}^{p} f_{i,Z} Z^{p-i} \in S[Z], \ \mathrm{ord}_{m_S} f_{i,Z} \geq i\delta(x).$$

For $1 \leq i \leq p-1$, the above inequality is strict, since $i_0(x) = p$. On the other hand, we have $\delta(x) \in \mathbb{N}$, so we deduce that

$$\frac{\mathrm{ord}_{m_S} f_{i,Z}}{i} \geq \delta(x) + \frac{1}{i} > \delta(x) + \frac{1}{p}, \ 1 \leq i \leq p-1. \tag{2.67}$$

Let

$$\tilde{Z} := Z + \sum_{|\mathbf{x}|=\delta(x)} \tilde{l}(\mathbf{x}) u^{\mathbf{x}}.$$

By (2.67), there is an expansion

$$f_{p,\tilde{Z}} = -\sum_{|\mathbf{x}|=\delta(x)} \tilde{v}(\mathbf{x}) u^{p\mathbf{x}} + g + \tilde{g}, \tag{2.68}$$

with $g \in S$, $\mathrm{ord}_{m_S} g \geq p\delta(x) + 1$ and $\tilde{g} \in \tilde{S}$, $\mathrm{ord}_{m_{\tilde{S}}} \tilde{g} > p\delta(x) + 1$. We deduce that

$$\delta(h; u_1, \ldots, u_{\tilde{n}}; \tilde{Z}) = \delta(x) + \frac{1}{p}.$$

Since $\delta(x) + \frac{1}{p} \notin \mathbb{N}$, $\Delta_{\tilde{S}}(h; u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ has no solvable vertex within its initial face $\{\tilde{\mathbf{x}} \in \mathbb{R}^{\tilde{n}}_{\geq 0} :| \tilde{\mathbf{x}} |= \delta(x) + \frac{1}{p}\}$.

Let $(u_1, \ldots, u_{\tilde{n}}; \tilde{Z}_1)$ be well adapted coordinates at $\tilde{x}$. Without loss of generality, it can be assumed that $\tilde{Z}_1 = \tilde{Z} - \tilde{\theta}_1$ with $\mathrm{ord}_{m_{\tilde{S}}} \tilde{\theta}_1 \geq \delta(x) + 1$. By (2.68), we get

$$\mathrm{in}_{m_{\tilde{S}}} \tilde{h} = \tilde{Z}_1^p - \sum_{|\mathbf{x}|=\delta(x)} \tilde{V}(\mathbf{x}) U^{p\mathbf{x}} + G(U_1, \ldots, U_n) \in G(m_{\tilde{S}})[\tilde{Z}_1] \tag{2.69}$$

and (2.ii) is proved. We have $i_0(\tilde{x}) = p$, $\delta(\tilde{x}) = \delta(x) + \frac{1}{p}$ and $\epsilon(\tilde{x}) = \epsilon(x) + 1$. Finally, we have

$$\frac{\partial F_{p,\tilde{Z}_1}}{\partial U_j} = \sum_{|\mathbf{x}|=\delta(x)} \frac{\partial \tilde{V}(\mathbf{x})}{\partial \tilde{V}_j} U^{p\mathbf{x}} \in k(\tilde{x})[U_1, \ldots, U_n],\ n+1 \leq j \leq \tilde{n},$$

so $V(F_{p,\tilde{Z}_1}, \tilde{E}, m_{\tilde{S}}) \neq 0$ and $\omega(\tilde{x}) = \epsilon(\tilde{x}) - 1 = \omega(x)$. □

Remark 2.9. Theorem 2.20 reduces computations of $\omega(x)$ to the case where S is strict Henselian, i.e. Henselian with separably algebraically closed residue field S/m_S by changing S to its strict Henselianization $\tilde{S}$, $\dim \tilde{S} = n = \dim S$.

Applying the theorem to a tower $\tilde{S}$ of smooth local base changes of the form $S \subseteq S[Y]_{(m_S, Y^p - l)}$ with $l \in S$ a unit with residue $\bar{l} \notin (S/m_S)^p$ also reduces computations of $\omega(x)$ to the case of an algebraically closed residue field for some $\tilde{S}$ with $\dim \tilde{S} > n = \dim S$, *vid.* comments before notation 2.1 for the excellent of such $\tilde{S}$.

The cone $\mathrm{Max}(x)$ and directrix $\mathrm{Dir}(x)$ have no such good behavior w.r.t. regular local base changes.

2.8 Resolution when $\omega(x) = 0$.

In this section, we prove that the multiplicity of $\mathcal{X}$ can be reduced at any point x such that $(m(x), \omega(x)) = (p, 0)$. This is achieved by combinatorial blowing ups in a way which is similar to the equal characteristic zero situation. This resolution algorithm does not depend on the choice of a valuation centered at x and we formalize Hironaka's A/B game as follows:

Definition 2.18. Let (S, h, E) be as before, $x \in \mathcal{X}$ and $L = \mathrm{Tot}(S[X]/(h))$. Suppose that for every valuation μ of L centered at x, a composition of local

Hironaka-permissible blowing ups (definition 2.7)

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{2.70}$$

is associated, where $x_i \in \mathcal{X}_i$ is the center of μ, $0 \leq i \leq r$. The sequence (2.70) is said to be *independent* if the blowing up center $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$ does not depend on the chosen valuation μ *having center in* x_i, $0 \leq i \leq r-1$.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. If $\epsilon(x) > 0$, recall that $\eta^{-1}(m_S) = \{x\}$, $k(x) = S/m_S$, and that

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z] = k(x)[U_1, \ldots, U_n][Z]$$

by (2.46). The initial form of $H(x)$ in $G(m_S)$ is denoted H as before.

Lemma 2.21. *Assume that $m(x) = p$ and $\epsilon(x) = 1$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. If*

$$H^{-1}F_{p,Z} \nsubseteq < U_1, \ldots, U_e >,$$

then $\omega(x) = 0$.

Proof. According to definition 2.16, we must show that $V(TF_{p,Z}, E, m_S) \neq 0$. Expand

$$H^{-1}F_{p,Z} =< \sum_{j=1}^{n} \alpha_j U_j > \subseteq G(m_S)_1, \ \alpha_j \in k(x).$$

By assumption, we have $\alpha_{j_0} \neq 0$ for some j_0, $e+1 \leq j_0 \leq n$, so

$$0 \neq H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j_0}} \subseteq V(F_{p,Z}, E, m_S). \tag{2.71}$$

If $i_0(x) = p$, we have $TF_{p,Z} = F_{p,Z}$. If $i_0(x) = p-1$, then $H^{-1}G^p =< U_{j_1} >$ for some j_1, $1 \leq j_1 \leq e$, by theorem 2.14(2). Comparing with definition 2.14, we have $\mathbf{x} \in A \Longrightarrow px_{j_1} > H_{j_1}$, therefore $F_{p,Z} - TF_{p,Z} \in HU_{j_1}$. So (2.71) implies that $V(TF_{p,Z}, E, m_S) \neq 0$. □

Proposition 2.22. *Assume that $(m(x), \omega(x)) = (p, 0)$, $\{x\} := \eta^{-1}(m_S)$. Let $\mathcal{Y} \subset (\mathcal{X}, x)$ be a Hironaka-permissible center w.r.t. E, $\pi : \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$.*

If $W := \eta(\mathcal{Y})$ is an intersection of components of E or if $\epsilon(y) = \epsilon(x)$, then $(m(x'), \omega(x') \leq (p, 0)$.

Proof. According to definition 2.16, there are two different cases to consider:

(1) $\epsilon(x) = 0$;

(2) $\epsilon(x) = 1$, $V(TF_{p,Z}, E, m_S) \neq (0)$.

To begin with, we have $\delta(x) \geq 1$ by proposition 2.3(ii). Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x with $I(W) = (\{u_j\}_{j\in J})$ for some subset $J \subseteq \{1, \ldots, n\}$. By definition 2.9, we have:

$$\epsilon(x) = \min_{1\leq i\leq p} \left\{ \frac{\mathrm{ord}_{m_S}(H(x)^{-i} f_{i,Z}^p)}{i} \right\}. \tag{2.72}$$

Case 1: $\epsilon(x) = 0$. By (2.72), we have

$$\begin{cases} H(x)^{-i} f_{i,Z}^p & \subseteq m_S, \quad 1 \leq i < i_0(x) \\ H(x)^{-i_0(x)} f_{i_0(x),Z}^p & = S, \\ H(x)^{-i} f_{i,Z}^p & \subseteq S, \quad i_0(x) < i \leq p. \end{cases} \tag{2.73}$$

By proposition 2.7, there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec} S & \xleftarrow{\sigma} & \mathcal{S}' \end{array}$$

where $\sigma : \mathcal{S}' \to \mathrm{Spec} S$ is the blowing up along W. Let

$$\eta' : \mathcal{X}' \to \mathcal{S}',\ s' := \eta'(x'),\ S' := \mathcal{O}_{\mathcal{S}',s'},\ E' := (\sigma^{-1}(E)_{\mathrm{red}})_{s'}.$$

Since $W \subseteq E$, it can be assumed after possibly reordering coordinates that

$$(J')_E := \{2, \ldots, e_0\},\ J = \{1, e_0 + 1, \ldots, n_0\},\ 1 \leq e_0 \leq e \leq n_0.$$

Furthermore, it can be assumed that $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$ or that $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$ with $n_0 > e_0$.

We first prove the proposition when $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$. Let

$$h' := u_1^{-p} h = Z'^p + f_{1,Z'} Z'^{p-1} + \cdots + f_{p,Z'} \in S'[Z'],$$

where $Z' := Z/u_1,\ f_{i,Z'} := u_1^{-i} f_{i,Z} \in S'$ for $1 \leq i \leq p$. We have

$$E' = \mathrm{div}(u_1 \cdots u_{e_0} \frac{u_{e_0+1}}{u_1} \cdots \frac{u_e}{u_1}) \tag{2.74}$$

and (S', h', E') satisfies both conditions **(G)** and **(E)** by propositions 2.10 and 2.13. There exists an adapted r.s.p. of S' of the form

$$(u'_1 := u_1, \ldots, u'_{e_0} := u_{e_0}, u'_{e_0+1}, \ldots, u'_{n'_0}, u'_{n_0+1} := u_{n_0+1}, \ldots, u'_n := u_n).$$

Since we do not assume that x' is a closed point, we have $e_0 \leq n'_0 \leq n_0$ in general, with

$$n' := \mathrm{dim} S' = n - (n_0 - n'_0).$$

We emphasize that the number of irreducible components e' of E' satisfies $e_0 \leq e' \leq e$ and that $e' \neq e$ in general because some of the u_j/u_1 in (2.74) may be units. After reordering coordinates, we may also assume that

$$E' = \mathrm{div}(u'_1 \cdots u'_{e'}) \text{ and } u'_j := u_j/u_1,\ e_0 + 1 \leq e' \leq e.$$

Since $\mathcal{Y}$ is Hironaka-permissible at x, we have (see definition 2.10):

$$\mathrm{ord}_W H(x) = p \sum_{j \in J} d_j \geq p.$$

Therefore $I' := u_1^{-p} H(x) \subseteq S'$ and this ideal is monomial in $(u'_1, \ldots, u'_{e'})$, i.e. $I' =: ({u'_1}^{H'_1} \cdots {u'_{e'}}^{H'_{e'}})$. We let:

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^{n'},$$

where

$$H'_1 = p(\sum_{j \in J} d_j - 1) \text{ and } H'_j = H_j = pd_j,\ 2 \leq j \leq e'. \tag{2.75}$$

Then (2.73) gives:

$$\begin{cases} I'^{-i} f^p_{i,Z'} & \subseteq m_S S' \quad 1 \leq i < i_0(x) \\ I'^{-i_0(x)} f^p_{i_0(x),Z'} & = S' \\ I'^{-i} f^p_{i,Z'} & \subseteq S' \quad i_0(x) < i \leq p. \end{cases} \tag{2.76}$$

This shows that

$$\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z') = \mathbf{x}' + \mathbb{R}^{n'}_{\geq 0}. \tag{2.77}$$

If $i_0(x) < p$, or if $\sum_{j\in J_E} d_j \notin \mathbb{N}$ or if $d_{j'} \notin \mathbb{N}$ for some j', $2 \leq j' \leq e'$, then $\mathbf{x}'$ is not solvable (definition 2.3) by (2.77), hence $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ is minimal. Therefore we may compute $\epsilon(x')$ from (2.77) and get $\epsilon(x') = 0$, so the proposition is proved in this case.

If ($i_0(x) = p$, $\sum_{j\in J_E} d_j \in \mathbb{N}$ and $d_{j'} \in \mathbb{N}$ for all j', $2 \leq j' \leq e'$), write $f_{p,Z} = \gamma u^{p\mathbf{x}}$, $\gamma \in S$ a unit and $\mathbf{x} := (d_1, \ldots, d_e, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^n$. We have

$$\mathrm{in}_{\mathbf{x}'} h' = Z'^p + \lambda(\prod_{j=e'+1}^{e} \lambda_j^{H_j}) U'^{p\mathbf{x}'}, \tag{2.78}$$

where $\lambda \in k(x)$ (resp. $\lambda_j \in k(x')$) is the residue of γ (resp. of u_j/u_1). We let:

$$\lambda' := \lambda \prod_{j=e'+1}^{e} \lambda_j^{H_j} \in k(x'),\ \lambda' \neq 0.$$

If $\lambda' \notin k(x')^p$, then $\mathbf{x}'$ is not solvable and we also have $\epsilon(x') = 0$.

If $\lambda' \in k(x')^p$, let

$$C' := \mathrm{Spec}\left(\frac{k(x)[Z, U_1, U_{e_0+1}, \ldots, U_e]}{(\overline{H})}\right),\ \overline{H} := \mathrm{in}_{m_S} h = Z^p + \lambda \prod_{j=e'+1}^{e} U_j^{H_j}.$$

We claim that the affine cone C' is regular away from the torus

$$\mathbb{T} := \mathbb{A}^{e-e_0+2}_{k(x)} \backslash V(Z \prod_{j\in J_E} U_j).$$

To see this, let $(\lambda_l)_{l\in\Lambda_0}$ be an absolute p-basis of $k(x)$. By [54] theorem 30.5, the ideal of the singular locus of C' is:

$$I(\mathrm{Sing}C') = \left(\overline{H}, \{\frac{\partial \overline{H}}{\partial \lambda_l}\}_{l\in\Lambda_0}, \{\frac{\partial \overline{H}}{\partial U_j}\}_{e'+1\leq j\leq e}\right).$$

If $d_j \notin \mathbb{N}$ for some j, $e'+1 \leq j \leq e$, then $\frac{\partial \overline{H}}{\partial U_j}$ does not vanish on $\mathbb{T}$. Otherwise, we have $\lambda \notin k(x)^p$ because $\mathbf{x}$ is a vertex of $\Delta_S(u_1, \ldots, u_n; Z)$ and is not solvable. Therefore $\frac{\partial \overline{H}}{\partial \lambda_l}$ does not vanish on $\mathbb{T}$ for any $l \in \Lambda_0$ such that $\frac{\partial \lambda}{\partial \lambda_l} \neq 0$ and the claim is proved. We deduce that there exists a unit $l' \in S'$ such that

$$v' := l'^p + \gamma \prod_{j=e'+1}^{e} \left(\frac{u_j}{u_1}\right)^{H_j}$$

is a regular parameter of S' transverse to

$$E'_1 := \mathrm{div}(u'_1 \cdots u'_{e'} u'_{n_0+1} \cdots u'_{n'}),\ E'_1 \supseteq E'.$$

We may thus take $u'_{e'+1} := v'$ in our r.s.p. of S' adapted to E'. Let $Z'_1 := Z' - l' u'^{p\mathbf{x}'}$, so the polyhedron $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z'_1)$ has a vertex

$$\mathbf{x}'_1 := (H'_1/p, \ldots, H'_{e'}/p, 1/p, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^{n'} \tag{2.79}$$

which is not solvable, since $\mathbf{x}'_1 \notin \mathbb{N}^{n'}$. Let $Z'_2 := Z'_1 - \theta'$, $\theta' \in S'$, be such that $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z'_2)$ is minimal. We deduce from (2.76) and (2.79) that

$$H(x') = (u'^{p\mathbf{x}'}),\ \epsilon(x') = 1 \text{ and } H'^{-1} F_{p,Z'_2} \not\subseteq < U'_1, \ldots, U'_{e'} > .$$

We get $m(x') = 1$ if $\mathbf{x}' = \mathbf{0}$, and $(m(x'), \omega(x')) = (p, 0)$ otherwise by lemma 2.21 as required.

If $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$, it can be furthermore assumed that $s' \notin \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$, i.e. u_j/u_{n_0} is *not* a unit in S' for $j \in J_E$. The proof is now a simpler variation of the above one: (2.74) is replaced by

$$E' = \mathrm{div}(\frac{u_1}{u_{n_0}} u_2 \cdots u_{e_0} \frac{u_{e_0+1}}{u_{n_0}} \cdots \frac{u_e}{u_{n_0}} u_{n_0}).$$

The polyhedron $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z')$ in (2.77) is minimal except if ($d_j \in \mathbb{N}$ for each j, $1 \leq j \leq e$, and $\lambda \in k(x')^p$) with notations as above. We have $\epsilon(x') = 0$ (resp. $\epsilon(x') = 1$) in the former (resp. in the latter) situation. This concludes the proof in case 1.

Case 2: $\epsilon(x) = 1$. The proof runs parallel to that in case 1 and we only indicate the necessary changes. By assumption, W is an intersection of components of E (case 2a) or $\epsilon(y) = \epsilon(x) = 1$ (case 2b).

To begin with, let $v \in S$ be such that $H(x)^{-1} f_{p,Z} = (v)$. By assumption, we have $V(TF_{p,Z}, E, m_S) \neq (0)$, so v is transverse to E.

In case 2a, we may assume that $(u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n)$ is an adapted r.s.p. of S after renumbering variables. Since $\mathbf{x}_0 := (d_1, \ldots, d_e, \frac{1}{p}, \ldots, 0) \notin \mathbb{N}^n$ is the unique vertex of $\Delta_S(h; u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n; Z)$ induced by $f_{p,Z}$,

this polyhedron has no solvable vertex. In other terms, it can be assumed that $v = u_{e+1}$.

In case 2b, proposition 2.4 implies that $v \in I(W)$, so $(u_1, \ldots, u_e, v)$ can be completed to an adapted r.s.p. of S such that $I(W) = (\{u_j\}_{j\in J})$ for some subset $J \subseteq \{1, \ldots, n\}$. The polyhedron $\Delta_S(h; u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n; Z)$ has no solvable vertex either and it can also be assumed that $v = u_{e+1}$.

We remark in both cases 2a and 2b that, if $\Delta_S(h; u_1, \ldots, u_n; Z)$ has a vertex distinct from $\mathbf{x}_0$, then it has exactly two vertices: this follows from theorem 2.14(2), the other vertex being then given by

$$\mathbf{x}_1 := (\frac{D_1}{p(p-1)}, \ldots, \frac{D_e}{p(p-1)}, 0, \ldots, 0), \ (\mathrm{Disc}_Z(h)) =: (u_1^{D_1} \cdots u_e^{D_e}). \quad (2.80)$$

After blowing up, we obtain a (S', h', E') again satisfying conditions **(G)** and **(E)**.

In case 2a, there exists an adapted r.s.p. of S' of the form

$$(u'_1 := u_1, \ldots, u'_{e_0} := u_{e_0}, u'_{e_0+1}, \ldots, u'_{e_1}, u'_{e+1} := u_{e+1}, \ldots, u'_n := u_n),$$

with $J = \{1, e_0+1, \ldots, e\}$ and $E' = \mathrm{div}(u'_1 \cdots u'_{e'})$ after reordering variables, $1 \leq e_0 \leq e' \leq e_1 \leq e$. Then $\Delta_{S'}(h'; u'_1, \ldots, u'_n; Z')$ has again a vertex

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 0, \ldots, 0, 1/p, 0, \ldots, 0) \notin \mathbb{N}^{n-(e-e_1)},$$

thus $\mathbf{x}'$ is not solvable. We deduce that $\epsilon(x') \leq 1$ and $\omega(x') = 0$ follows from lemma 2.21 if $(m(x'), \epsilon(x')) = (p, 1)$.

In case 2b, it can be assumed after reordering variables that

$$(J')_E := \{2, \ldots, e_0\}, \ J = \{1, e_0+1, \ldots, n_0\}, \ 1 \leq e_0 \leq e, \ e+1 \leq n_0.$$

We let $u'_{j'} := u_{j'}$ for $j' \in J'$ and consider three distinct situations depending on x', up to reordering coordinates:

(1) $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$ and $u_{e+1}/u_1 \in m_{S'}$. We may complete the family $(\{u_{j'}\}_{j'\in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_1 := u_1, u'_{e_0+1}, \ldots, u'_{e_1}, u'_{e_1+1} := u_{e+1}/u_1), \ n' := \dim S' = n - (n_0 - e_1).$$

Then $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ has a vertex

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 1/p, 0, \ldots, 0) \notin \mathbb{N}^{n'},$$

thus $\mathbf{x}'$ is not solvable. We conclude that $\epsilon(x') \leq 1$ and that $\omega(x') = 0$ if $(m(x'), \epsilon(x')) = (p, 1)$ by lemma 2.21.

(2) $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$ and $u_{e+1}/u_{n_0} \in m_{S'}$, where $n_0 > e+1$. After dealing with (1), we may assume furthermore that $u_j/u_{n_0} \in m_{S'}$, $j \in J_E$. We complete the family $(\{u_{j'}\}_{j' \in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_{e_0+1} := u_{e_0+1}/u_{n_0}, \ldots, u'_{e+1} := u_{e+1}/u_{n_0}, u'_{n_1}, \ldots, u'_{n_0-1}, u'_{n_0} := u_{n_0}),$$

with $n' := \dim S' = n - (n_1 - e - 2)$. We conclude as in (1).

(3) $I(W)S' = (u_{e+1})$. We complete the family $(\{u_{j'}\}_{j' \in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_1 := u_{e+1}, u'_{e_0+1}, \ldots, u'_{n_1}),\ n' := \dim S' = n - (n_0 - n_1).$$

Let $E' =: \mathrm{div}(u'_1 \cdots u'_{e'})$ and consider two situations as in case 1:

If $\frac{1}{p} + \sum_{j \in J_E} d_j \notin \mathbb{N}$ or if $d_{j'} \notin \mathbb{N}$ for some j', $2 \leq j' \leq e'$, then the polyhedron $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ is minimal and we have $\epsilon(x') = 0$.

If $(\frac{1}{p} + \sum_{j \in J_E} d_j \in \mathbb{N}$ and $d_{j'} \in \mathbb{N}$ for every j', $2 \leq j' \leq e')$, the initial form polynomial $\mathrm{in}_{\mathbf{x}'} h'$ has the form

$$\mathrm{in}_{\mathbf{x}'} h' = Z'^p - \mu^{p-1} U'^{(p-1)\mathbf{x}'} Z' + \lambda(\prod_{j=e'+1}^{e} \lambda_j^{H_j}) U'^{p\mathbf{x}'},$$

where $\lambda \in k(x)$ (resp. $\lambda_j \in k(x')$) is the residue of γ (resp. of u_j/u_{e+1}), *vid.* (2.78). We have $\mu \neq 0$ in the above formula precisely if

$$U^{p(\mathbf{x}_1 - \mathbf{x}_0)} = U_{j_0}/U_{e+1},\ u_{j_0}/u_{e+1} \in S' \text{ a unit}$$

for some j_0, $e_0 + 1 \leq j_0 \leq e$ with notations as in (2.80). Then μ^{p-1} is the residue in $k(x')$ of

$$\gamma_{p-1,Z} \prod_{j=e'+1}^{e} \left(\frac{u_j}{u_{e+1}}\right)^{A_{p-1,j}}$$

with notations as in theorem 2.14(2). The end of the proof goes along as in case 1.

This completes the proof of (3), hence the proof of the proposition in case 2. □

Remark 2.10. This proposition is a lighter version of theorem 3.6 where it is assumed that $\omega(x) > 0$ and that the blowing up centers are permissible of the first or second kind (definitions 3.1 and 3.2 below).

Theorem 2.23. *Assume that* $(m(x), \omega(x)) = (p, 0)$*, where* $\{x\} = \eta^{-1}(m_S)$*. For every valuation* μ *of* $L = \mathrm{Tot}(S[X]/(h))$ *centered at* x*, there exists a finite and independent composition of local Hironaka-permissible blowing ups (2.70) such that* $m(x_r) < p$*.*

Proof. We will produce a Hironaka-permissible center $\mathcal{Y} \subset (\mathcal{X}, x)$ w.r.t. E satisfying the assumptions of proposition 2.22 and such that the following holds:

(*) let $\pi : \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$. Then

$$\delta(x') < \delta(x).$$

Applying proposition 2.22, the center $x_1 \in \mathcal{X}'$ of a given valuation μ again satisfies the assumptions of the theorem if $m(x_1) = p$. Iterating, any finite sequence (2.70) induces a sequence

$$\delta(x_r) < \delta(x_{r-1}) < \cdots < \delta(x)$$

provided that $m(x_i) = p$, $1 \leq i \leq r-1$. Since $\delta(x_i) \in \frac{1}{p}\mathbb{N}$, we have $\delta(x_r) < 1$ for some $r \geq 1$, hence $m(x_r) < p$ by proposition 2.3(2), so the theorem follows from claim (*). In order to construct $\mathcal{Y}$ with the required properties, we consider two cases as in the proof of proposition 2.22.

Case 1: $\epsilon(x) = 0$. We have $\delta(x) = \sum_{j=1}^{e} d_j \geq 1$. Therefore there exists a subset

$$J \subseteq \{1, \ldots, e\}, \ \sum_{j \in J} d_j \geq 1,$$

with smaller possible number of elements among all subsets of $\{1, \ldots, e\}$ with this property. Let $W := V(\{u_j\}_{j \in J}) \subset \mathrm{Spec} S$ and remark that

$$\mathrm{ord}_W H(x) = p \sum_{j \in J} d_j \geq p.$$

Hence $\mathcal{Y} := \eta^{-1}(W) = V(Z, \{u_j\}_{j \in J})$ is Hironaka-permissible w.r.t. E and W is an intersection of components of E. By (2.75), we have

$$\mathrm{ord}_{m_{S'}} H(x') \leq p(\delta(x) + \sum_{j \in J \setminus \{j_0\}} d_j - 1), \tag{2.81}$$

where $I(W)S' = (u_{j_0})$. The minimality property required of J implies that

$$\sum_{j\in J\setminus\{j_1\}} d_j < 1 \text{ for every } j_1 \in J \text{ (so } \sum_{j\in J} d_j < 2 \text{ if } \mid J \mid \geq 2). \tag{2.82}$$

If $\epsilon(x') = 0$, we deduce from (2.81) that

$$p\delta(x') = \mathrm{ord}_{m_{S'}} H(x') < p\delta(x)$$

as required in (*). Note that if $\mid J \mid = 1$, we have $\lambda = \lambda'$ in (2.78) and $S = S'$, hence $\lambda' \notin k(x')^p = k(x)^p$. Since $\epsilon(x') = 0$ in this situation, we may now assume that $\mid J \mid \geq 2$.

If $\epsilon(x') = 1$, we are in the situation discussed in (2.79). We may then take $j_0 = 1$, $E' = \mathrm{div}(u'_1 \cdots u'_{e'})$ and have

$$\sum_{j\in J} d_j \in \mathbb{N},\ d_j \in \mathbb{N} \text{ for } 2 \leq j \leq e'.$$

By (2.82), we have $\sum_{j\in J} d_j = 1$, $d_j = 0$ for $2 \leq j \leq e'$, so $H(x') = (1)$ and $m(x') = 1$. This concludes the proof in case 1.

Case 2: $\epsilon(x) = 1$. We have $\delta(x) = \frac{1}{p} + \sum_{j=1}^{e} d_j \geq 1$.

If $\delta(x) > 1$, there exists a subset

$$J \subseteq \{1, \ldots, e\},\ \sum_{j\in J} d_j \geq 1,$$

with smaller possible number of elements among all subsets of $\{1, \ldots, e\}$ with this property as in case 1 and we also let $W := V(\{u_j\}_{j\in J}) \subset \mathrm{Spec} S$. The proof goes along as in case 1, with

$$p\delta(x') - p\delta(x) \leq \mathrm{ord}_{m_{S'}} H(x') - \mathrm{ord}_{m_S} H(x) < 0.$$

If $\delta(x) = 1$, we may assume that $H(x)^{-1} f_{p,Z} = (u_{e+1})$ and that (2.80) holds if $\Delta_S(h; u_1, \ldots, u_n; Z)$ has more than one vertex. In this case, this polyhedron has exactly two vertices and we have

$$H(x)^{-(p-1)} f^p_{p-1,Z} = (u_{j_0})^{p-1} \text{ for some } j_0,\ 1 \leq j_0 \leq e$$

by theorem 2.14(2). We deduce that

$$H(x)^{-i} f^p_{i,Z} \subseteq (u_{j_0}, u_{e+1})^i,\ 1 \leq i \leq p \tag{2.83}$$

by definition of $\Delta_S(h; u_1, \ldots, u_n; Z)$. We let $J := \{j : d_j > 0\} \cup \{e+1\}$ and

$$W := V(\{u_j\}_{j\in J}) \subset \mathrm{Spec} S,\ \mathcal{Y} := \eta^{-1}(W) = V(Z, \{u_j\}_{j\in J}).$$

We have $\mathrm{ord}_W H(x) = p$, so $\mathcal{Y}$ is Hironaka-permissible w.r.t. E. Since $H(x)^{-1} f_{p,Z} = (u_{e+1})$, we have $\epsilon(y) = \epsilon(x) = 1$ by (2.83), where $y \in \mathcal{X}$ is the generic point of $\mathcal{Y}$. Thus proposition 2.22 applies and gives $m(x') \leq p-1$ under either assumption (1)(2) or (3) in the proof of proposition 2.22. $\square$

3 Permissible blowing ups.

3.1 Blowing ups of the first and second kind.

In this section, we introduce a notion of permissible blowing up which is well behaved w.r.t. our main resolution invariant $y \mapsto \iota(y)$ on $\mathcal{X}$. *We assume that*

$$m(x) = p,\ \{x\} = \eta^{-1}(m_S) \text{ and } \omega(x) > 0$$

in what follows since theorem 2.23 rules out the case $\omega(x) = 0$.

Definition 3.1. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is *permissible of the first kind* at x if $m(y) = m(x) = p$ and the following conditions hold:

(i) $\mathcal{Y}$ is Hironaka-permissible w.r.t. E at x (definition 2.7);

(ii) $\epsilon(y) = \epsilon(x)$.

If $y \in \mathcal{X}$ satisfies $m(y) = p$, it follows from the definition that $\mathcal{Y} := \overline{\{y\}}$ is permissible of the first kind at y. It also follows from (ii) that a permissible center of the first kind has codimension at least two in $\mathcal{X}$.

The main result of this chapter (theorem 3.6 below) will require comparing the initial form polynomials $\mathrm{in}_W h$ and $\mathrm{in}_{m_S} h$. We keep notations as in section 2.4: given well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x, we let

$$W := \eta(\mathcal{Y}),\ I(W) = (\{u_j\}_{j\in J}). \tag{3.1}$$

We denote:

$$\mathrm{in}_W h = Z^p + \sum_{i=1}^{p} F_{i,Z,W} Z^{p-i} \in G(W)[Z]$$

and (proposition 2.16(i) since $\epsilon(x) > 0$)

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z].$$

There are associated homogeneous submodules

$$H_W \subseteq G(W)_{d_W} \text{ (resp. } H := H_{m_S} \subseteq G(W)_d)$$

by (2.43), with

$$d_W := \sum_{j \in J_E} H_j, \ d = \sum_{j=1}^{e} H_j.$$

A word of caution is required at this point: formula (2.43) *defines* the monomial ideal H_W which is the *initial form* of $H(x)$ in $G(W)$ and is different in general from the ideal $H(\Xi)$ associated to the triple

$$(G(W)_\Xi, \mathrm{in}_W h, E_W), \ \Xi := (\{U_j\}_{j \in J}) + m_{S_W}.$$

Corresponding to the above choice for H_W (resp. to H), there are associated S_W-submodules

$$V(F_{p,Z,W}, E, W) \subseteq G(W)_{\epsilon(y)-1}, \ J(F_{p,Z,W}, E, W) \subseteq \widehat{G(W)}_{\epsilon(y)}$$

(resp. $k(x)$-vector subspaces

$$V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1}, \ J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)})$$

given by (2.44).

Notation 3.1. We first recall notations and definitions from section 2.4. We denote

$$J_E := J \cap \{1, \ldots, e\}, \ J' := \{1, \ldots, n\} \backslash J \text{ and } (J')_E := \{1, \ldots, e\} \backslash J_E.$$

The image $\overline{m}_S$ of m_S in S_W has regular parameters $(\overline{u}_j)_{j \in J'}$, the respective residues of the corresponding parameters of S.

Let now $d \in \mathbb{N}$ be fixed and

$$F = \sum_{|\mathbf{a}|=d} \hat{f}_{\mathbf{a}} U^{\mathbf{a}} \in \widehat{G(W)}_d = \widehat{S_W}[\{U_j\}_{j \in J}]_d.$$

Note that $\mathrm{gr}_{\overline{m}_S}\widehat{G(W)}_d \simeq \mathrm{gr}_{\overline{m}_S}G(W)_d$ and that it has a structure of graded $\mathrm{gr}_{\overline{m}_S}S_W$-module. For any $d_0 \leq \min_{\mathbf{a}}\{\mathrm{ord}_{\overline{m}_S}\hat{f}_{\mathbf{a}}\}$, F has an initial form in $\mathrm{gr}_{\overline{m}_S}G(W)_d$ by taking

$$\overline{F} := \sum_{|\mathbf{a}|=d}(\mathrm{cl}_{d_0}\hat{f}_{\mathbf{a}})U^{\mathbf{a}} \in (\mathrm{gr}_{\overline{m}_S}G(W)_d)_{d_0}. \tag{3.2}$$

This notation requires specifying d_0 to avoid ambiguity. We extend the notation to homogeneous submodules $M \subseteq \widehat{G(W)}_d$ as follows:

$$\overline{M} :=< \overline{F},\ F \in M >\subseteq (\mathrm{gr}_{\overline{m}_S}G(W)_d)_{d_0}$$

for fixed $d_0 \leq \min\{d_0(F), F \in M\}$ with obvious notations. For fixed d, d_0, there is an inclusion of S/m_S-vector spaces:

$$(\mathrm{gr}_{\overline{m}_S}G(W)_d)_{d_0} \subset \frac{G(m_S)_{d+d_0}}{< (\{U_j\}_{j\in J})^{d+1} \cap G(m_S)_{d+d_0} >}. \tag{3.3}$$

Proposition 3.1. *Let $\mathcal{Y}$ be permissible of the first kind at $x \in \mathcal{Y}$. Then for any well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that $I(W) = (\{u_j\}_{j\in J})$, the initial form $\mathrm{in}_{m_S}h \in G(m_S)[Z]$ satisfies*

$$H^{-1} < G^p, F_{p,Z} >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)}.$$

Proof. The existence of well adapted coordinates $(u_1, \ldots, u_n; Z)$ such that $I(W) = (\{u_j\}_{j\in J})$ follows from proposition 2.4. This theorem furthermore implies that the polyhedron

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}_J(\Delta_S(h; u_1, \ldots, u_n; Z)) \text{ is minimal,} \tag{3.4}$$

where $\mathrm{pr}_J :\ \mathbb{R}^n \to \mathbb{R}^J$ denotes the projection on the $(u_j)_{j\in J}$-space.

By (ii) of definition 3.1, we have $\epsilon(x) = \epsilon(y)$. Therefore

$$H^{-i}F^p_{i,Z} = \mathrm{cl}_0(H_W^{-i}F^p_{i,Z,W}) \subseteq G(m_S)_{i\epsilon(x)} = k(x)[U_1, \ldots, U_n]_{i\epsilon(x)}$$

is simply the reduction of $H_W^{-i}F^p_{i,Z,W}$ modulo $\overline{m}_S$ for $1 \leq i \leq p$, i.e. taking $d_0 = 0$ in notation 3.1, via the inclusion (3.3)

$$k(x)[\{U_j\}_{j\in J}]_{i\epsilon(y)} \simeq (\mathrm{gr}_{\overline{m}_S}G(W)_{i\epsilon(y)})_0 \subset G(m_S)_{i\epsilon(y)} \simeq k(x)[U_1, \ldots, U_n]_{i\epsilon(x)}.$$

We get respectively $(H^{-1}G^p)^{p-1}$, $(H^{-1}F_{p,Z})^p$ for $i = p-1, p$ and this completes the proof. $\square$

The following corollary will be required in the proof of the blowing up theorem below. The adapted cone $\mathrm{Max}(x) \subseteq G(m_S)$ is defined in definition 2.17.

Corollary 3.2. *With notations as above, let $\mathcal{Y}$ be permissible of the first kind at x. The defining ideal* $\mathrm{IMax}(x) \subseteq G(m_S)$ *of* $\mathrm{Max}(x)$ *satisfies*

$$\mathrm{IMax}(x) = (\mathrm{IMax}(x) \cap k(x)[\{U_j\}_{j\in J}])G(m_S).$$

Proof. This follows from proposition 3.1 and definition 2.17. Note that the truncation operator T used in the definition of $\mathrm{Max}(x)$ does not affect the conclusion of the corollary since it is obvious from the definitions that:

$$V(F_{p,Z}, E, m_S) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)-1} \Rightarrow V(TF_{p,Z}, E, m_S) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)-1}.$$

The same implication holds for $J(F_{p,Z}, E, m_S)$ and $J(TF_{p,Z}, E, m_S)$. □

We now define a second kind of permissible blowing up.

Definition 3.2. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is *permissible of the second kind* at x if $m(y) = m(x) = p$ and the following conditions hold:

(i) $\mathcal{Y}$ is Hironaka-permissible w.r.t. E at x (definition 2.7);

(ii) $\epsilon(y) = \epsilon(x) - 1$ and $i_0(y) \leq i_0(x)$;

(iii) $\overline{J}(F_{p,Z,W}, E, W) := \mathrm{cl}_0 J(F_{p,Z,W}, E, W) \neq 0$.

The following important example constructs a threefold $\mathcal{X}$ such that every resolution of singularities $\tilde{\mathcal{X}} \to \mathcal{X}$ which is a composition of Hironaka-permissible blowing ups does actually involve blowing up a permissible curve of the second kind.

Example 3.1. Let k be a perfect field of characteristic $p > 0$, $A := k[u_1, u_2, u_3]$, $P \in k[x]\backslash k[x^p]$ and take

$$h := Z^p + P(u_3)u_2^p + u_1^{p+1} \in A[Z],\ E := \mathrm{div}(u_1).$$

Let $\mathcal{Y} := V(Z, u_1, u_2) \subseteq \mathrm{Sing}_p\mathcal{X}$ with generic point y. Let $\pi :\ \tilde{\mathcal{X}} \to \mathcal{X}$ be any composition of Hironaka-permissible blowing ups with $\tilde{\mathcal{X}}$ regular. Since

y is an isolated point of $\mathrm{Sing}_p\mathcal{X}$, the map π factors through the blowing up π_0 along $\mathcal{Y}$ above y. Define a nonempty Zariski open subset $\mathcal{U} \subseteq \mathcal{Y}$ by:

$$x \in \mathcal{U} \Leftrightarrow \begin{cases} \pi \text{ factors through } \pi_0 \text{ above } x \\ \\ \mathrm{ord}_x P'(\overline{u}_3) = 0 \end{cases}.$$

For $x \in \mathcal{U}$, there exist well adapted coordinates $(u_1, u_2, v_x; Z_x := Z - \gamma_x u_2)$ at x, $\gamma_x \in A_{\eta(x)}$ a unit such that

$$h = Z_x^p + v_x u_2^p + u_1^{p+1} \in A_{\eta(x)}[Z_x].$$

Then $\mathcal{Y}$ is permissible of the second kind at every $x \in \mathcal{U}$ since

$$\overline{J}(F_{p,Z_x,W}, E, W) = \frac{\partial F_{p,Z_x,W}}{\partial \overline{v}_x} = U_2^p \neq 0,\ F_{p,Z_x,W} = \overline{v}_x U_2^p \in G(W)_p$$

with notations as in definition 3.2(iii). This is dealt with in the course of the proof of theorem 1.4 in proposition 7.14 when applying lemma 7.13 ($\kappa(x) = 2$ in this example, *cf.* definition 5.1).

When $n = 3$, permissible blowing ups of the second kind only occur in propositions 7.14 and 7.21 ($\kappa(x) = 2$).

Proposition 3.3. *Let $\mathcal{Y}$ be permissible of the second kind at x. For any well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that $I(W) = (\{u_j\}_{j\in J})$, the initial form $\mathrm{in}_{m_S} h \in G(m_S)[Z]$ satisfies*

$$\begin{cases} H^{-1}G^p & \subseteq & U_{j_0} k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)} \text{ for some } j_0 \in (J')_E \\ \\ H^{-1}F_{p,Z} & = & < \sum_{j\in J'} U_{j'} \Phi_{j'}(\{U_j\}_{j\in J}) + \Psi(\{U_j\}_{j\in J}) > \subseteq G(m_S)_{\epsilon(x)} \end{cases}. \tag{3.5}$$

with $\Phi_{j'} \neq 0$ for some $j' \in J' \backslash (J')_E$. In particular $\epsilon(y) = \omega(x)$.

Proof. We argue as in the proof of proposition 3.1 and build up from (3.4). By (ii) of definition 3.2, we have $\epsilon(x) = \epsilon(y) + 1$. Therefore

$$\mathrm{cl}_0(H_W^{-i} F_{i,Z,W}^p) = 0,\ 1 \leq i \leq p.$$

This shows that $H_W^{-i} F_{i,Z,W}^p \subseteq \overline{m}_S S_W[\{U_j\}_{j\in J_E}]_{i\epsilon(y)}$. We have $\epsilon(y) > 0$, so $F_{i,Z,W} = 0$, $1 \leq i \leq p-2$ by theorem 2.14. For $i = p-1$, we have

$-F_{p-1,Z,W} = G_W^{p-1}$ for some $G_W \in G(W)_{\delta(y)}$ (so $G_W = 0$ if $\delta(y) \notin \mathbb{N}$). We deduce that

$$H_W^{-1}(G_W^p, F_{p,Z,W}) \subseteq \overline{m}_S S_W[\{U_j\}_{j\in J_E}]_{\epsilon(y)}. \tag{3.6}$$

If $i_0(x) = p$, we have $H^{-1}G^p = 0$ so the first part of (3.5) is trivial. If $i_0(x) = p-1$, we have $i_0(y) = p-1$ by definition 3.2(ii), so $G_W \neq 0$. The first part of (3.5) then follows from (3.6), i.e.

$$H^{-1}G^p = \mathrm{cl}_1(H_W^{-1}G_W^p) \subseteq U_{j_0}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)},$$

for some $j_0 \in (J')_E$.

Going back to the definition of $J(F_{p,Z,W}, E, W)$ in (2.40), we deduce from (3.6) that

$$\overline{J}(F_{p,Z,W}, E, W) =< \mathrm{cl}_0(H_W^{-1}\frac{\partial F_{p,Z,W}}{\partial \overline{u}_{j'}}), j' \in J'\backslash(J')_E >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Taking classes as in (3.2) with $d_0 = 1$, we get

$$\mathrm{cl}_1(H_W^{-1}F_{p,Z,W}) \subseteq \sum_{j'\in J'} U_{j'}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Since $\mathrm{cl}_1(H_W^{-1}F_{p,Z,W})$ is a homomorphic image of $H^{-1}F_{p,Z} \in G(m_S)_{\epsilon(x)}$ as described in (3.3), there exists an expansion (3.5). For $j' \in J'\backslash(J')_E$, we have

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}} = \mathrm{cl}_0(H_W^{-1}\frac{\partial F_{p,Z,W}}{\partial \overline{u}_{j'}}).$$

Collecting together for all $j' \in J'\backslash(J')_E$, we get

$$\overline{J}(F_{p,Z,W}, E, W) =< H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}}, j' \in J'\backslash(J')_E >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}$$

and the second part of (3.5) follows from definition 3.2(iii).

Note that $\epsilon(y) = \omega(x)$ is an immediate consequence of definition 2.16 if $i_0(m_S) = p$. If $i_0(m_S) = p-1$, we must introduce a truncation operator $T : G(m_S)_{\delta(x)} \to G(m_S)_{\delta(x)}$ in order to compute $\omega(x)$. The first part of (3.5) now shows that there exists $j_0 \in (J')_E$ such that

$$H^{-1}(F_{p,Z} - TF_{p,Z}) \in U_{j_0}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Since $\overline{J}(F_{p,Z,W}, E, W) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}$, we thus have:

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}} = H^{-1}\frac{\partial TF_{p,Z}}{\partial U_{j'}}$$

for every $j' \in J'\backslash(J')_E$. This proves that $\omega(x) = \epsilon(y)$. $\square$

Note that it follows from the above proposition that a permissible center of the second kind has codimension at least two in $\mathcal{X}$, since $\epsilon(y) > 0$. We now introduce the adapted cone associated to a permissible blowing up. Recall the definition of B from (2.52) (*cf.* also definition 2.16). We have $B = \emptyset$ if $i_0(m_S) = p$, and

$$B = \{j : U_j \text{ divides } H^{-1}G^p\} \text{ if } i_0(m_S) = p - 1.$$

Definition 3.3. Let $\mathcal{Y} \subset \mathcal{X}$, with generic point y, be a permissible center at x. We define a subcone

$$C(x, \mathcal{Y}) \subset \mathrm{Spec}(k(x)[\{U_j\}_{j\in J}])$$

as follows: if $\mathcal{Y}$ is of the first kind, we let:

$$C(x, \mathcal{Y}) := \mathrm{Spec}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(\mathrm{IMax}(x) \cap k(x)[\{U_j\}_{j\in J}])}\right);$$

if $\mathcal{Y}$ is of the second kind, we let $B_J := B\backslash\{j_0\}$ with notations as in proposition 3.3 and define:

$$C(x, \mathcal{Y}) := \mathrm{Max}(\overline{J}(F_{p,Z,W}, E, W)) \cap \{U_{B_J} = 0\}.$$

In both cases, we denote the associated projective cone by $PC(x, \mathcal{Y}) \subseteq \mathbb{P}^{|J|-1}_{k(x)}$.

Theorem 3.4. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Let $\tilde{x} \in \tilde{\eta}^{-1}(m_{\tilde{S}})$ and $x \in \eta^{-1}(m_S)$ be its image.*

If $\mathcal{Y} \subset \mathcal{X}$ is a permissible center (of the first or second kind) at x, then

$$\tilde{\mathcal{Y}} := \mathcal{Y} \times_S \mathrm{Spec}\tilde{S} \subseteq \tilde{\mathcal{X}} = \mathcal{X} \times_S \mathrm{Spec}\tilde{S}$$

is permissible (of the first or second kind) at $\tilde{x}$.

Proof. We denote $(\tilde{S}, \tilde{h}, \tilde{E})$ and $(u_1, \ldots, u_{\tilde{n}})$ as in notations 2.1 and 2.2. Since W has normal crossings with E at x, $\tilde{W} := \tilde{\eta}(\tilde{\mathcal{Y}})$ has normal crossings with $\tilde{E}$ at $\tilde{x}$. Since $\mathcal{Y}$ is permissible at x, we have $m(y) = p$. Any generic point $\tilde{y}$ of $\tilde{\mathcal{Y}}$ has $m(\tilde{y}) = p$ by theorem 2.20(1), and $\tilde{\mathcal{Y}}$ itself is irreducible by proposition 2.10. Theorem 2.20(2) applies to $\tilde{y}$ (with $n(y) = \tilde{n}(y)$) and to $\tilde{x}$ and states that

$$\epsilon(\tilde{y}) = \epsilon(y),\ \epsilon(\tilde{x}) \geq \epsilon(x),\ i_0(\tilde{y}) = i_0(y),\ i_0(\tilde{x}) = i_0(x)$$

Cases of inequality $\epsilon(\tilde{x}) > \epsilon(x)$ are classified in *ibid.*(2.ii).

Suppose that $\epsilon(\tilde{x}) > \epsilon(x)$. Then

$$F_{p,Z} \in k(x)[U_1^p, \ldots, U_n^p] \text{ and } i_0(m_S) = i_0(m_{\tilde{S}}) = p.$$

Then $\mathcal{Y}$ is permissible of the first kind since $F_{p,Z} \in k(x)[U_1^p, \ldots, U_n^p]$ is incompatible with the conclusion of proposition 3.3. Note that

$$\epsilon(y) = \epsilon(x) = \epsilon(\tilde{x}) - 1.$$

We claim that $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$.

To prove the claim, note that definition 3.2(i) and $i_0(\tilde{y}) \leq i_0(\tilde{x}) = p$ are already checked. We have

$$H^{-1}\frac{\partial F_{p,\tilde{Z}}}{\partial U_{j'}} = H^{-1}\Phi_{j'}(U_1, \ldots, U_n) \neq 0, \tag{3.7}$$

with notations as in theorem 2.20(2.ii) for some j', $n+1 \leq j' \leq \tilde{n}$. Since $H(\tilde{x}) = H(x)\tilde{S}$ by theorem 2.20(2.i), and $H^{-1}F_{p,Z} \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)}$ by proposition 3.1, we have

$$H^{-1}F_{p,\tilde{Z}} \subseteq \sum_{j=1}^{\tilde{n}} U_j k(\tilde{x})[\{U_j\}_{j\in J}]_{\epsilon(x)}.$$

This proves that definition 3.2(iii) holds for $\tilde{\mathcal{Y}}$ at $\tilde{x}$. On the other hand this implies that $\epsilon(\tilde{y}) = \epsilon(y)$ because

$$H^{-1}F_{p,\tilde{Z}} \nsubseteq k(\tilde{x})[\{U_j\}_{j\in J}]_{\epsilon(\tilde{x})}$$

follows obviously from (3.7). So definition 3.2(ii) is also checked and $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$.

Assume now that $\epsilon(\tilde{x}) = \epsilon(x)$. If $\mathcal{Y}$ is permissible of the first kind at x, we have $\epsilon(\tilde{y}) = \epsilon(\tilde{x})$, so $\tilde{\mathcal{Y}}$ is also permissible of the first kind at $\tilde{x}$.

If $\mathcal{Y}$ is permissible of the second kind at x, definition 3.2(ii) is checked. Finally by proposition 3.3, the polyhedron $\Delta_S(h; u_1, \ldots, u_n; Z)$ has a vertex $\mathbf{x}$ such that $x_{j'} \notin \mathbb{N}$ for some $j' \in J' \backslash J_E$. The corresponding vertex

$$\tilde{\mathbf{x}} := (\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$$

is thus not solvable. We hence get $\tilde{\mathbf{x}} \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ and definition 3.2(iii) is checked. Hence $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$ as required, since $H(\tilde{x}) = H(x)T$. $\square$

3.2 Blowing up theorem.

Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ (of the first or second kind) at $x \in \mathcal{Y}$, $\{x\} = \eta^{-1}(m_S)$. Our objective is to relate $\omega(x')$ to $\omega(x)$ for points $x' \in \pi^{-1}(x)$.

We keep notations as in proposition 2.7 and proposition 2.10. Then $\sigma : \mathcal{S}' \to \mathrm{Spec} S$ denotes the blowing up along W and there is a commutative diagram (2.15). Let

$$\eta' : \mathcal{X}' \to \mathcal{S}', \ s' := \eta'(x') \in \sigma^{-1}(m_S), \ S' := \mathcal{O}_{\mathcal{S}', s'}.$$

We denote by $W' := \sigma^{-1}(W)$ and $E' := \sigma^{-1}(E)_{\mathrm{red}}$. We do not change notations to denote stalks at s', i.e. we will write $\eta' : \mathcal{X}_{s'} \to \mathrm{Spec} S'$ for the stalk at s' of the above map η', and W', E' for the stalks at s' of the corresponding divisors. By proposition 2.10, we have $\eta'^{-1}(s') = \{x'\}$ if x' is not a regular point of X'.

For the purpose of computations, we shall pick well adapted coordinates $(u_1, \ldots, u_n; Z)$ such that

$$I(W) = (\{u_j\}_{j \in J}), \ \mathcal{Y} = V(Z, \{u_j\}_{j \in J}).$$

with notations as in (3.1). We denote by $u \in S'$ a local equation for W', which can be taken to be some u_{j_1}, where $j_1 \in J$ depends on s'. We have $\mathcal{X}' = \mathrm{Spec}(S'[X']/(h'))$, where

$$h' := u^{-p} h = X'^p + f_{1,X'} X'^{p-1} + \cdots + f_{p,X'} \in S'[X'], \tag{3.8}$$

and

$$X' := Z/u,\ \ f_{i,X'} := u^{-i} f_{i,Z} \in S' \text{ for } 1 \leq i \leq p. \tag{3.9}$$

Since $\mathcal{Y}$ is permissible, we have $\epsilon(y) > 0$ so the initial form $\mathrm{in}_W h$ reduces to :

$$\mathrm{in}_W h = Z^p - G_W^{p-1} Z + F_{p,Z,W} \in G(W)[Z], \tag{3.10}$$

with $G_W \in G(W)_{\delta(y)}$ and $F_{p,Z,W} \in G(W)_{p\delta(y)}$ (in particular $G_W = 0$ if $\delta(y) \notin \mathbb{N}$). Since $\sigma^{-1}(W) = \mathbf{Proj} G(W)$, the restriction map

$$G(W)_d = \Gamma(W', \mathcal{O}_{W'}(d)) \to \Gamma(W' \backslash V(U), \mathcal{O}_{W'}(d))$$

gives an inclusion

$$U^{-d} G(W)_d = S_W[\{U_j/U\}_{j \in J}]_{\leq d} \subset \mathcal{O}_{W',s'} = S'/(u) \tag{3.11}$$

for each $d \geq 0$. There is an identification:

$$U^{-d} G(W')_d = (S_W[\{U_j/U\}_{j \in J}])_{s'} = S'/(u). \tag{3.12}$$

Finally, we note that $\mathcal{D}_{W'} = \mathcal{D}(W')$ by (2.41) since W' is a component of E'. These remarks are essential for stating the blow up formula in proposition 3.5(v) below.

Proposition 3.5. *(Blow up formula) Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ at x, $\{x\} = \eta^{-1}(m_S)$ and $x' \in \pi^{-1}(x)$. With notations as above, the following holds:*

(i) there exists a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S' which is adapted to (S', h', E');

(ii) $\mathrm{in}_{W'} h' = X'^p - G_{W'}^{p-1} X' + F_{p,X',W'} \in G(W')[X']$ *and is given by*

$$G_{W'} = U^{-1} G_W \in G(W')_{\delta(y)-1},\ \ F_{p,X',W'} = U^{-p} F_{p,Z,W} \in G(W')_{p(\delta(y)-1)};$$

(iii) the polyhedron $\Delta_{\widehat{S'}}(h'; u; X')$ is minimal;

(iv) we have $H(x') = u^{\epsilon(y)-p} H(x) \subseteq S'$;

(v) there is an equality of ideals of $\hat{S}'/(u)$:

$$\begin{cases} H_{W'}^{-1} G_{W'}^p & = & (U^{-\epsilon(y)} H_W^{-1} G_W^p)_{s'} \\ J(F_{p,X',W'}, E', W') & = & (U^{-\epsilon(y)} J(F_{p,Z,W}, E, W)) \hat{S}'/(u). \end{cases},$$

Proof. Statement (i) is proved in proposition 2.7. The formula in (ii) is obvious from (3.8), (3.9) and (3.10).

If $i_0(W) = p-1$, i.e. $G_W \neq 0$ in (3.10), we have $G_{W'} \neq 0$ by (ii), so $\Delta_{\widehat{S'}}(h'; u; X') \subseteq \mathbb{R}_{\geq 0}$ is minimal.

If $i_0(W) = p$, then $F_{p,Z,W} \notin G(W)^p$, i.e.

$$\delta(y) \notin p\mathbb{N} \text{ or } U^{-\delta(y)} F_{p,Z,W} \notin k(W')^p.$$

Note that $G(W)^p = (k(W')[U, U^{-1}])^p \cap G(W)$ since $G(W)$ is integrally closed. By (ii), $F_{p,X',W'} = U^{-p} F_{p,Z,W}$ so $F_{p,X',W'} \notin G(W')^p$ and this proves (iii).

To prove (iv), first consider those irreducible components $W_j = \mathrm{div}(u_j)$ of E, $1 \leq j \leq e$, whose strict transform W'_j passes through s'. We may pick a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S' which is adapted to (S', h', E'), containing u and $u'_j := u_j/u$ if $j \in J_E$ (resp. t and $u'_j := u_j$ if $j \notin J_E$) for each such j. Let

$$\mathrm{in}_{W_j} h(Z) = Z^p + F_{1,Z,W_j} Z^{p-1} + \cdots + F_{p,Z,W_j} \in S/(u_j)[U_j][Z].$$

We have $\mathrm{in}_{W'_j} h' = \mathrm{in}_{W_j} u^{-p} h(uX') \in S'/(u'_j)[U'_j][X']$, since u is a unit in $S'_{(u'_j)} = S_{(u_j)}$. Since $\Delta_S(h; u_1, \ldots, u_n; Z)$ is minimal, we have

$$\Delta_{S_{(u_j)}}(h; u_j; Z) = \Delta_{S'_{(u'_j)}}(h'; u'_j; X')$$

minimal as well by proposition 2.4, hence $\mathrm{ord}_{(u'_j)} H(x') = \mathrm{ord}_{(u_j)} H(x)$.

By (ii) and (iii), we have $\mathrm{ord}_{(u)} H(x') = p(\delta(y) - 1)$. Therefore

$$\mathrm{ord}_{(u)} H(x') - \mathrm{ord}_{(u)} H(x) = p(\delta(y) - 1) - \mathrm{ord}_W H(x) = \epsilon(y) - p$$

and the conclusion follows.

We now prove (v). The first part of the statement follows immediately from (ii) and (iv). With notations as in (2.42), we have

$$\begin{cases} J(F_{p,Z,W}, E, W) & = & H_W^{-1} \mathcal{J}(F_{p,Z,W}, E, W) & \subseteq & \widehat{G(W)}_{\epsilon(y)}, \\ \\ J(F_{p,X',W'}, E', W') & = & H_{W'}^{-1} \mathcal{J}(F_{p,X',W'}, E', W') & \subseteq & \widehat{G(W')}_0. \end{cases}$$

Applying (ii) and (iv), we get:

$$F_{p,X',W'} = U^{-p} F_{p,Z,W}, \ H_{W'} = H_W U^{\epsilon(y)-p} G(W').$$

Since $D \cdot U^p = 0$ for every $D \in \mathcal{D}_{W'}$, (v) can be written in the following form:

$$U^{-\deg F_{p,Z,W}} \mathcal{J}(F_{p,Z,W}, E', W') = (U^{-\deg F_{p,Z,W}} \mathcal{J}(F_{p,Z,W}, E, W))\hat{S}'/(u). \quad (3.13)$$

We have $G(W') = G(W)[\{V_j\}_{j \in J \backslash \{j_1\}}]_{s'}$, $V_j := U_j/U \in G(W')_0$, $j \in J \backslash \{j_1\}$. Pick an isomorphism $\widehat{S_W} \simeq k(x)[[\{\overline{u}_{j'}\}_{j' \in J'}]]$ (2.36). By (3.11), there are inclusions

$$k(x)[\{U_j\}_{j \in J}] \subset k(x)[U, \{V_j\}_{j \in J \backslash \{j_1\}}] \subset \hat{S}'/(u, \{\overline{u}_{j'}\}_{j' \in J'})[U] \simeq G(\hat{W}')/(\{\overline{u}_{j'}\}_{j' \in J'}).$$

Let $A := k(x)[\{U_j\}_{j \in J}]$, $A' := k(x)[U, \{V_j\}_{j \in J \backslash \{j_1\}}]$. The A'-module

$$\Omega^1_{A'/\mathbb{F}_p} \left(\log(U \prod_{j \in J \backslash \{j_1\}} V_j) \right)$$

is generated by collecting together dU/U, $\{dV_j/V_j\}_{j \in J \backslash \{j_1\}}$ and the pullback of $\Omega^1_{A/\mathbb{F}_p}$. For $F \in A$, we deduce the following standard formulæ in A' up to linear combinations of the $\frac{\partial F}{\partial \lambda_l}$, $l \in \Lambda_0$:

$$U \frac{\partial F}{\partial U} = \sum_{j \in J} U_j \frac{\partial F}{\partial U_j}, \ V_j \frac{\partial F}{\partial V_j} = U_j \frac{\partial F}{\partial U_j}, j \in J \backslash \{j_1\}. \quad (3.14)$$

By (2.40), the $\widehat{G(W)}$-module $\mathcal{D}_W$ is generated by adjoining the family

$$\left(\{U_j \frac{\partial}{\partial U_j}\}_{j \in J_E}, \{U_k \frac{\partial}{\partial U_j}\}_{k \in J, j \in J \backslash J_E} \right) \quad (3.15)$$

together with $(\{\overline{u}_{j'} \frac{\partial}{\partial \overline{u}_{j'}}\}_{j' \in (J')_E}, \{\frac{\partial}{\partial \overline{u}_{j'}}\}_{j' \in J' \backslash (J')_E}, \{\frac{\partial}{\partial \lambda_l}\}_{l \in \Lambda_0})$. Taking $F \in A_d$, $d \in \mathbb{N}$, we have for $j \in J \backslash J_E$,

$$(U^{-d}\{U_k \frac{\partial F}{\partial U_j}\}_{k \in J})A'_{s'} = (U^{-d} U \frac{\partial F}{\partial U_j})A'_{s'}.$$

Collecting together this equation with (3.14) and (3.15), we get

$$U^{-d} \mathcal{J}(F, E', W') = (U^{-d} \mathcal{J}(F, E, W))\hat{S}'/(u)$$

which proves (3.13) as required. □

We now state the main theorem of this section. Recall that the function $y \mapsto \omega(y)$ and $\kappa(y) \in \{1, \geq 2\}$ have been defined for given (S, h, E) and $y \in \mathcal{X}$ (definition 2.15 and definition 2.16). By proposition 2.13, (S', h', E') satisfies again conditions **(G)** and **(E)**. The values of $\epsilon(x')$, $\iota(x')$ are computed w.r.t. the adapted structure (S', h', E').

Notation 3.2. *Choice of coordinates:* by proposition 3.5(i), there exists a r.s.p. $(u'_1, \ldots, u'_{n'})$ which is adapted to (S', h', E') for some $n' \leq n$. We take $u'_1 := u$. Let

$$u'_i := \frac{u_{j_i}}{u},\ 2 \leq i \leq e'_0,\ \text{where } \{j_2, \ldots, j_{e'_0}\} := \{j \in J_E : \frac{u_j}{u} \in m_{S'}\}.$$

Let $\{j_{e'_0+1}, \ldots, j_{e'}\} := (J')_E$, $\{j_{e'+1}, \ldots, j_{n'_0}\} =: J' \backslash (J')_E$. We take

$$u'_i := u_{j_i},\ e'_0 + 1 \leq i \leq n'_0.$$

Let

$$u'_i := \frac{u_{j_i}}{u},\ n'_0 + 1 \leq i \leq n'_1,\ \text{where } \{j_{n'_0+1}, \ldots, j_{n'_1}\} := \{j \in J \backslash J_E : \frac{u_j}{u} \in m_{S'}\}$$

and complete $(u'_1, \ldots, u'_{n'_1})$ to a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S'.

Notation 3.3. Let

$$\overline{S'} := \hat{\mathcal{O}}_{\sigma^{-1}(m_S), s'} = \hat{S}'/(u, \{u_{j'}\}_{j' \in J'}) = k(x)[\{\widehat{U_j/U}\}_{j \in J}]_{\overline{m'}},$$

where $\overline{m'}$ denotes the ideal of the restriction of s' to $\sigma^{-1}(m_S)$:

$$\overline{m'} := (\{\overline{u'_i}\}_{i \in F}),\ F := \{2, \ldots, e'_0\} \cup \{n'_0 + 1, \ldots, n'\}.$$

For $I' \subseteq \hat{S}'/(u)$ an ideal, we denote by

$$\mathrm{ord} I' := \mathrm{ord}_{m_{\hat{S}'/(u)}} I' = \min_{\varphi' \in I'} \{\mathrm{ord}_{m_{\hat{S}'/(u)}} \varphi'\},\ \overline{\mathrm{ord}} I' := \mathrm{ord}_{\overline{m'}} I' \overline{S'}.$$

For every $I' \subseteq \hat{S}'/(u)$, we have $\mathrm{ord} I' \leq \overline{\mathrm{ord}} I' \leq +\infty$. If furthermore d' is given, $d' \leq \overline{\mathrm{ord}} I'$, we write

$$\overline{I'} \subseteq \left(\mathrm{gr}_{\overline{m'}} \overline{S'}\right)_{d'} = k(x')[\{U'_i\}_{i \in F}]_{d'}$$

for the initial part of degree d' of the ideal $I' \overline{S'}$.

The cone $C(x,\mathcal{Y}) \subseteq \mathrm{Spec}(k(x)[\{U_j\}_{j\in J}])$ is given by definition 3.3. For the associated projective cone, there is an embedding

$$PC(x,\mathcal{Y}) \hookrightarrow \sigma^{-1}(m_S).$$

Theorem 3.6. *Assume that $m(x)=p$, $\omega(x)>0$, where $\{x\}=\eta^{-1}(m_S)$. Let $\pi:\mathcal{X}'\to\mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ (of the first kind or second kind) at x, $x'\in\pi^{-1}(x)$ and $\eta':\ \mathcal{X}'\to \mathrm{Spec}S'$ be with notations as above, where $s'=\eta'(x')$. Then*

$$(m(x'),\omega(x'),\kappa(x')) \le (m(x),\omega(x),\kappa(x)). \tag{3.16}$$

If equality holds in (3.16), then $s'\in PC(x,\mathcal{Y})$.

If $\epsilon(x')>\epsilon(x)$, the following holds:

(1) we have $i_0(m_S)=p$, $\epsilon(y)=\epsilon(x)=\omega(x)$, $\delta(y)\in\mathbb{N}$, $H_{j'}\in p\mathbb{N}$ for every $j'\in(J')_E$ and

$$F_{p,Z}\in (k(x')[U_1,\dots,U_n])^p[\{U_j\}_{j\in J_E\setminus\{j_2,\dots,j_{e'_0}\}}];$$

(2) let $(u'_1,\dots,u'_{n'};Z')$ be well adapted coordinates at x'. Then

$$H'^{-1}F_{p,Z'}\not\subseteq k(x')[U'_1,\dots,U'_{n'_1}]_{\epsilon(x')}\oplus(\{U'_i\}_{i\notin F})\cap G(m_{S'})_{\epsilon(x')} \tag{3.17}$$

and there exists $\Phi'\in k(x')[U'^p_1,\dots,U'^{\,p}_{n'_1}][U'_{n'_1+1},\dots,U'_{n'}]_{p\delta(x')}$ such that

$$H'^{-1}(F_{p,Z'}-\Phi')\subseteq(\{U'_i\}_{i\notin F})\cap G(m_{S'})_{\epsilon(x')}. \tag{3.18}$$

Proof. Since $\mathcal{Y}$ is permissible, $\mathcal{Y}$ is Hironaka-permissible at x and this implies that $m(x')\le m(x)=p$ in any case. We are done unless equality holds, so assume that $m(x')=p$.

The polyhedron $\Delta_{S'}(h';u'_1,\dots,u'_{n'};X')$ need not be minimal. We must take $Z'=X'-\theta'$, $\theta'\in S'$ such that the polyhedron $\Delta_{S'}(h';u'_1,\dots,u'_{n'};Z')$ is minimal in order to read off $\epsilon(x')$ and $\omega(x')$ from $\mathrm{in}_{m_{S'}}h'$.

By proposition 3.5(iii), we have $\mathrm{ord}_{(u)}H(x')=p(\delta(y)-1)$. The initial form $H_{W'}$ of $H(x')$ in $G(W')$ is given by proposition 3.5(iv):

$$H_{W'}=<U^{p(\delta(y)-1)}\prod_{i=2}^{e'}\overline{u'_i}^{H_{j_i}}>. \tag{3.19}$$

We have $\theta'^p \in H(x')$ since $f_{p,X'} \in H(x')$. Let $\Theta' \in G(W')_{\delta(y)-1}$ be the initial form of θ' (in particular $\Theta' = 0$ if $\delta(y) \notin \mathbb{N}$). Then

$$\mathrm{in}_{W'} h' = Z'^p - G_{W'}^{p-1} Z' + F_{p,X',W'} + \Theta'^p - G_{W'}^{p-1}\Theta' \in G(W')[Z'] \qquad (3.20)$$

where $G_{W'} = U^{-1} G_W$, $F_{p,X',W'} = U^{-p} F_{p,Z,W}$ by proposition 3.5(ii). According to our notations, we have:

$$F_{p,Z',W'} = F_{p,X',W'} + \Theta'^p - G_{W'}^{p-1}\Theta'.$$

Note that derivatives in $\mathcal{D}_{W'}$ decrease orders by at most one. Since $H_{W'}$ is the initial form of $H(x')$ in $G(W')$, we have:

$$\epsilon(x') \leq \min\{\mathrm{ord}_{m_{S'/(u)}}(H_{W'}^{-1} G_{W'}^p), 1 + \mathrm{ord}_{m_{S'/(u)}} J(F_{p,Z',W'}, E', W')\}. \quad (3.21)$$

Inequality may be strict, since the $H(x')^{-i} f_{i,Z'}^p$, $1 \leq i \leq p$ may acquire terms of lower order not coming from $\mathrm{in}_W h$. Moreover, some derivatives in $\mathcal{D}_{W'}$ do not decrease orders and give a sharper bound in (3.21).

Recall that if $M \subseteq \widehat{G(W)}_d$, $d \in \mathbb{N}$ is a submodule, and d_0 is given, there are associated initial forms

$$\overline{M} \subseteq \left(\mathrm{gr}_{\overline{m}_S} G(W)_d\right)_{d_0} \subset \frac{G(m_S)_{d+d_0}}{< (\{U_j\}_{j\in J})^{d+1} \cap G(m_S)_{d+d_0} >}$$

under the conditions described in (3.2) and (3.3). Note that

$$\left(\mathrm{gr}_{\overline{m}_S} G(W)_d\right)_0 = \Gamma(\sigma^{-1}(m_S), \mathcal{O}_{\sigma^{-1}(m_S)}(d)) = k(x)[\{U_j\}_{j\in J}]_d$$

for $d_0 = 0$.

Since $\theta'^p \in H(x')$, we have $\Theta'^p \in H_{W'}$ in (3.20). We have $\Theta' = 0$ or $\delta(y) \in \mathbb{N}$ and

$$G_{W'}^{p-1}\Theta' \in G_{W'}^{p-1} \left\lceil H_{W'}^{\frac{1}{p}} \right\rceil, \quad \left\lceil H_{W'}^{\frac{1}{p}} \right\rceil :=< U^{\delta(y)-1} \prod_{i=2}^{e'} \overline{u'_i}^{\left\lceil \frac{H_{j_i}}{p} \right\rceil} > .$$

Since $D \cdot \Theta'^p = 0$ for every $D \in \mathcal{D}_{W'}$, we deduce from (3.20) that

$$J(F_{p,Z',W'}, E', W') \equiv J(F_{p,X',W'}, E', W') \bmod H_{W'}^{-1} G_{W'}^{p-1} \left\lceil H_{W'}^{\frac{1}{p}} \right\rceil. \qquad (3.22)$$

Note that if $i_0(m_S) = p$, or if $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$, we have

$$G_W = 0 \text{ or } \mathrm{ord}_{(u_{j'})}(H_W^{-1} G_W^p) > 0 \text{ for some } j' \in (J')_E \tag{3.23}$$

by applying proposition 2.11(iii) in the latter case. In this case, we obtain the following from proposition 3.5(v) and (3.22):

$$(H_{W'}^{-1} G_{W'}^p)\overline{S'} = 0,\ J(F_{p,Z',W'}, E', W')\overline{S'} = J(F_{p,X',W'}, E', W')\overline{S'}. \tag{3.24}$$

Case 1: $i_0(m_S) = p$ and $\mathcal{Y}$ is of the first kind. In order to get an estimate of $\epsilon(x')$ from (3.21), we take:

$$M = J(F_{p,Z,W}, E, W),\ d = \epsilon(y) = \epsilon(x),\ d_0 = 0.$$

Remark 3.1. By proposition 3.1, there is an equality

$$H^{-1} F_{p,Z} = \mathrm{cl}_{\epsilon(x)} H_W^{-1} F_{p,Z,W} \subseteq k(x)[\{U_j\}_{j \in J}]_{\epsilon(x)},$$

but we emphasize that the induced inclusion

$$J(F_{p,Z}, E, m_S) \subseteq \mathrm{cl}_{\epsilon(x)} J(F_{p,Z,W}, E, W). \tag{3.25}$$

is strict in general.

By proposition 2.16(ii) and the remark, we have

$$0 \neq J(F_{p,Z}, E, m_S) \subseteq \overline{M} \subseteq k(x)[\{U_j\}_{j \in J}]_{\epsilon(x)}.$$

Let $I' = J(F_{p,X',W'}, E', W') \subseteq \hat{S}'/(u)$, $d' = \mathrm{ord} I'$. By proposition 3.5(v), we have

$$\big(U^{-\epsilon(x)} J(F_{p,Z}, E, m_S)\big)_{\overline{m'}} \subseteq I'\overline{S'}.$$

Since $i_0(m_S) = p$, we obtain from (3.24) that:

$$\big(U^{-\epsilon(x)} J(F_{p,Z}, E, m_S)\big)_{\overline{m'}} \subseteq I'\overline{S'} = J(F_{p,Z',W'}, E_{W'}, W')\overline{S'}. \tag{3.26}$$

If $\omega(x) = \epsilon(x)$, definition 2.17 gives

$$\mathrm{IMax}(x) = (J(F_{p,Z}, E, m_S))G(m_S).$$

We deduce that $\overline{\mathrm{ord}} I' \leq \omega(x)$ and

$$s' \notin PC(x, \mathcal{Y}) \Longrightarrow \overline{\mathrm{ord}} I' < \omega(x). \tag{3.27}$$

If $\omega(x) = \epsilon(x) - 1$, definition 2.17 gives

$$\mathrm{IMax}(x) = (V(F_{p,Z}, E, m_S))G(m_S).$$

Since $U_{j_1} V(F_{p,Z}, E, m_S) \subseteq J(F_{p,Z}, E, m_S)$ (recall that $u = u_{j_1}$), we also deduce that $\overline{\mathrm{ord}} I' \leq \omega(x)$ and (3.27) holds. We have:

$$\epsilon(x') \leq 1 + \mathrm{ord} I' = 1 + d' \leq 1 + \overline{\mathrm{ord}} I',$$

by (3.21). We have proved that

$$\epsilon(x') \leq 1 + \overline{\mathrm{ord}} I' \leq 1 + \omega(x) \tag{3.28}$$

with strict inequality on the right hand side under the assumption of (3.27). The proof is now an easy consequence of the following claim:

$$\epsilon(x') = 1 + \overline{\mathrm{ord}} I' \Longrightarrow \omega(x') = \epsilon(x') - 1.$$

Namely, assuming the claim, we have $\omega(x') \leq \omega(x)$ and this inequality is strict under the assumption of (3.27). The first part of the proof is complete since $i_0(m_S) = p$ implies $\kappa(x) \geq 2$. To prove the claim, let

$$\mathrm{in}_{m_{S'}} h = Z'^p - G'^{p-1} Z' + F_{p,Z'} \in G(m_{S'})[Z']$$

be the initial form polynomial. Since it is assumed that $\epsilon(x') = 1 + \overline{\mathrm{ord}} I'$, we have $\overline{I'} \neq 0$ and:

$$\overline{I'} =< \left\{ H'^{-1} \frac{\partial F_{p,Z'}}{\partial U'_j} \right\}_{j=n'_0+1}^{n'} > \ \mathrm{mod}(\{U'_{j'}\}_{j' \notin F}) \cap G(m_{S'})_{d'}. \tag{3.29}$$

To compute $\omega(x')$, we must introduce a truncation operator

$$T' : G(m_{S'})_{p\delta(x')} \to G(m_{S'})_{p\delta(x')}$$

as in definition 2.16. By (3.19), we have

$$H' := \mathrm{cl}_{p\delta(x') - \epsilon(x')} H(x') =< U^{p(\delta(y)-1)} \prod_{i=2}^{e'} U_i'^{H_{j_i}} > \in G(m_{S'}).$$

Going back to definition 2.14, we have

$$F_{p,Z'} - T' F_{p,Z'} \in < G'^{p-1} U^{\delta(y)-1} \prod_{i=2}^{e'} U_i'^{\left\lceil \frac{H_{j_i}}{p} \right\rceil} > .$$

Since $i_0(m_S) = p$, (3.24) applies and implies that

$$H'^{-1}(F_{p,Z'} - T'F_{p,Z'}) \subseteq (\{U'_i\}_{i \notin F}) \cap G(m_{S'})_{\epsilon(x')}. \tag{3.30}$$

Comparing with (3.29), there exists i, $n'_0 + 1 \leq i \leq n'$ such that

$$H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \neq 0, \tag{3.31}$$

since $\overline{I'} \neq 0$. This proves that $\omega(x') = \epsilon(x') - 1$ as claimed.

To conclude the proof in case 1, assume that $\epsilon(x') > \epsilon(x)$. If some inequality is strict in (3.27), we have $\epsilon(x') \leq \omega(x) \leq \epsilon(x)$: a contradiction. So $\omega(x') = \omega(x)$ and by the above claim, we get

$$\epsilon(x) = \omega(x) = \omega(x') = \epsilon(x') - 1 = \mathrm{ord} I' = \overline{\mathrm{ord}} I'. \tag{3.32}$$

We use notations as in (2.37). Suppose that there exists $j' \in (J')_E$ such that $H_{j'} \notin p\mathbb{N}$. By proposition 3.1, we have

$$H^{-1}U_{j'}\frac{\partial F_{p,Z}}{\partial U_{j'}} \neq 0.$$

Going back to (3.26), we have

$$\phi_{j'} := \left(U^{-\epsilon(x)}H^{-1}U_{j'}\frac{\partial F_{p,Z}}{\partial U_{j'}}\right)_{\overline{m'}} \subseteq J(F_{p,Z',W'}, E', W')\overline{S'}.$$

Applying the transformation rule in proposition 3.5(v), we have

$$\phi_{j'} = (H^{-1}_{W'}\overline{u}_{j'}\frac{\partial F_{p,Z',W'}}{\partial \overline{u}_{j'}})\overline{S'}.$$

Since $\overline{\mathrm{ord}}\phi_{j'} \leq \epsilon(x)$, we deduce that

$$\epsilon(x') \leq \overline{\mathrm{ord}}(H^{-1}_{W'}F_{p,Z',W'}) \leq \overline{\mathrm{ord}}(H^{-1}_{W'}\overline{u}_{j'}\frac{\partial F_{p,Z',W'}}{\partial \overline{u}_{j'}}) \leq \epsilon(x).$$

This is a contradiction with (3.32). Hence $H_{j'} \in p\mathbb{N}$ for every $j' \in (J')_E$.

Suppose that $\delta(y) \notin \mathbb{N}$. Similarly, by proposition 3.1, we have:

$$H^{-1}D \cdot F_{p,Z} \neq 0,\ D := \sum_{j \in J} U_j \frac{\partial}{\partial U_j} \in \mathrm{Der}(G(W)).$$

Note that we have $\Theta' = 0$ in (3.20) since $\delta(y) \notin \mathbb{N}$. We deduce from (3.14) that

$$\phi_D := \left(U^{-\epsilon(x)} H^{-1} D \cdot F_{p,Z}\right) \hat{S}'/(u) = H_{W'}^{-1} U \frac{\partial F_{p,Z',W'}}{\partial U}.$$

Arguing as above, we get a contradiction from:

$$\epsilon(x') \leq \mathrm{ord}(H_{W'}^{-1} F_{p,Z',W'}) \leq \mathrm{ord}(H_{W'}^{-1} U \frac{\partial F_{p,Z',W'}}{\partial U}) \leq \epsilon(x).$$

Let now $i \in \{2, \ldots, e_0'\}$. By (3.26), we have

$$\phi_i := \left(U^{-\epsilon(x)} H^{-1} U_{j_i} \frac{\partial F_{p,Z}}{\partial U_{j_i}}\right)_{\overline{m'}} \subseteq J(F_{p,Z',W'}, E_{W'}, W')\overline{S'}.$$

Applying once again (3.14) and since $\epsilon(x') > \epsilon(x) = \omega(x)$, we get

$$\mathrm{cl}_{\epsilon(x)}(\{H_{W'}^{-1} \overline{u}_i \frac{\partial F_{p,Z,W'}}{\partial \overline{u}_i}\}_{2 \leq i \leq e_0'}) \equiv \mathrm{cl}_{\epsilon(x)}(\{\phi_i\}_{2 \leq i \leq e_0'}) \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}.$$

If $\phi_i \neq 0$ for some i, $2 \leq i \leq e_0'$, we get

$$\epsilon(x') \leq \mathrm{ord}(H_{W'}^{-1} F_{p,Z',W'}) \leq \mathrm{ord}(H_{W'}^{-1} \overline{u}_i \frac{\partial F_{p,Z,W'}}{\partial \overline{u}_i}) \leq \epsilon(x),$$

again a contradiction. Since $\epsilon(x) = \omega(x)$, we have $\frac{\partial F_{p,Z}}{\partial U_j} = 0$ for every $j \in J \backslash J_E$.

Finally, assume that $F_{p,Z} \notin k(x')^p[U_1, \ldots, U_n]$. With notations as in (2.37), we pick a maximal subset $\Lambda_1 \subseteq \Lambda_0$ such that the family of elements $(d\overline{\lambda}_l)_{l \in \Lambda_1}$ in $\Omega^1_{k(x')/\mathbb{F}_p}$ is linearly independent over $k(x')$. Let $(d\overline{\lambda}_{l'})_{l' \in \Lambda_0'}$ be a basis of $\Omega^1_{k(x')/\mathbb{F}_p}$, $\Lambda_1 \subseteq \Lambda_0'$, and pick a preimage $\lambda_{l'} \in \hat{S}'/(u)$ of $\overline{\lambda}_{l'}$ for $l' \in \Lambda_0' \backslash \Lambda_1$.

By assumption, there exists $l \in \Lambda_1$ such that $\frac{\partial F_{p,Z}}{\partial \lambda_l} \neq 0$. Arguing as above, we get

$$\mathrm{cl}_{\epsilon(x)}(H_{W'}^{-1} \frac{\partial F_{p,Z,W'}}{\partial \lambda_l}) \equiv \mathrm{cl}_{\epsilon(x)} \left(U^{-\epsilon(x)} H^{-1} \frac{\partial F_{p,Z}}{\partial \lambda_l}\right)_{\overline{m'}} \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)},$$

a contradiction and the proof of (1) in the theorem is complete.

We now proceed to prove (2). By proposition 3.5(i), we have

$$H_{W'}^{-1} F_{p,X',W'} \overline{S'} = (U^{-\epsilon(x)} H_W^{-1} F_{p,Z,W})_{\overline{m'}} = (U^{-\epsilon(x)} H^{-1} F_{p,Z})_{\overline{m'}}.$$

By (1) in the theorem and proposition 3.1, there is an expansion

$$F_{p,Z} = \left(\prod_{i=e_0'+1}^{e'} U_{j_i}^{H_{j_i}} \right) \sum_{\mathbf{a}\in A} F_{p,Z,\mathbf{a}}(\{U_j\}_{j\in J_1'}) \prod_{j\in J_1} U_j^{pa_j},\ A \subset \mathbb{N}^{J_1},$$

with $J_1 := \{j_2, \ldots, j_{e_0'}, j_{n_0'+1}, \ldots, j_{n_1'}\}$, $J_1' := J \backslash J_1$, $F_{p,Z,\mathbf{a}} \in k(x')^p[\{U_j\}_{j\in J_1'}]$. We deduce that

$$(U^{-\epsilon(x)} H^{-1} F_{p,Z})_{\overline{m'}} = \overline{H'}^{-1} \left(\sum_{\mathbf{a}\in A} F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J_1'}) \prod_{j\in J_1} (\frac{U_j}{U})^{pa_j} \right), \tag{3.33}$$

with $\overline{H'} := (\prod_{i=2}^{e_0'} \left(\frac{U_{j_i}}{U}\right)^{H_{j_i}}) \subseteq \overline{S'}$. Since $(H_{W'}^{-1} G_{W'}^p)\overline{S'} = 0$ by (3.24), there exists $\theta' \in S'/(u)$ such that

$$H_{W'}^{-1} F_{p,Z',W'} \overline{S'} = H_{W'}^{-1} (F_{p,X',W'} + \theta'^p) \overline{S'}. \tag{3.34}$$

We deduce from (3.33) that there exists a finite subset $A' \subset \mathbb{N}^{J_1}$, $A \subseteq A'$ and elements

$$\theta'_{\mathbf{a}} \in k(x)[\{\frac{U_j}{U}\}_{j\in J_1'}] \text{ for every } \mathbf{a} \in A'$$

such that (letting $F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J_1'}) = 0$ for $\mathbf{a} \in A' \backslash A$) we have:

$$H_{W'}^{-1} F_{p,Z',W'} \overline{S'} = \overline{H'}^{-1} \left(\sum_{\mathbf{a}\in A'} (F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J_1'}) + \theta_{\mathbf{a}}'^{\,p}) \prod_{j\in J_1} (\frac{U_j}{U})^{pa_j} \right).$$

Let $d_{\mathbf{a}} := \epsilon(x') + \sum_{i=2}^{e_0'} H_{j_i} - p \mid \mathbf{a} \mid$ for $\mathbf{a} \in A'$. Since $\overline{\mathrm{ord}}(H_{W'}^{-1} F_{p,Z',W'}) = \epsilon(x')$ we have

$$\overline{\mathrm{ord}}(F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J_1'}) + \theta_{\mathbf{a}}'^{\,p}) \geq d_{\mathbf{a}}$$

for every $\mathbf{a} \in A'$. Taking classes in $G(\overline{m'})$, we define:

$$\Phi'_{\mathbf{a}} := \mathrm{cl}_{d_{\mathbf{a}}}(F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J_1'}) + \theta'^p_{\mathbf{a}}) \in k(x')[U'_{n_1'+1}, \ldots, U'_{n'}]_{d_{\mathbf{a}}}.$$

To conclude the proof, let $I_1 := \{2, \ldots, e_0', n_0'+1, \ldots, n_1'\}$. We take

$$\Phi' := U_1'^{p(\delta(y)-1)} \left(\prod_{i=e_0'+1}^{e'} U_i'^{H_{j_i}} \right) \sum_{\mathbf{a}\in A'} \Phi'_{\mathbf{a}} \prod_{i\in I_1} U_i'^{pa_{j_i}}$$

and claim that Φ' satisfies (2) in the theorem. By the above definition and (1) in the theorem, we have $\Phi' \in k(x')[U_1'^p, \ldots, U_{n_1'}'^p][U_{n_1'}', \ldots, U_{n'}']_{p\delta(x')}$. Also (3.18) follows immediately from (3.34).

With notations as in the above proof of (1), we have

$$J(F_{p,Z}, E, m_S) = H^{-1} < \{U_j \frac{\partial F_{p,Z}}{\partial U_j}\}_{j \in J_E \setminus \{j_2, \ldots, j_{e_0'+1}\}}, \{\frac{\partial F_{p,Z}}{\partial \lambda_l}\}_{l \in \Lambda_0 \setminus \Lambda_1}.$$

Applying once more (3.14), we get

$$\begin{aligned} &\mathrm{cl}_{\epsilon(x)}(\{H_{W'}^{-1} \tfrac{\partial F_{p,Z,W'}}{\partial \overline{u'}_i}\}_{n_1' \le i \le n'}) \\ &\equiv \mathrm{cl}_{\epsilon(x)}(U^{-\epsilon(x)} J(F_{p,Z}, E, m_S))_{\overline{m'}} \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}. \end{aligned}$$

Since $J(F_{p,Z}, E, m_S) \neq 0$, we obtain that

$$H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \notin (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}$$

for some i, $n_1' \le i \le n'$, and the conclusion follows. This concludes the proof of (2).

Case 2: $i_0(m_S) = p-1$ (so $\mathcal{Y}$ is of the first kind). We first take $d = \epsilon(y)$ and

$$M := H_W^{-1} G_W^p, \ d_0 = 0.$$

By proposition 3.1, there is an expansion $H^{-1}G^p =< \prod_{j \in J} U_j^{pB_j} >$. With notations as in definition 2.16, we have

$$pb_j - H_j = pB_j, \ j \in J \text{ and } B = \{j \in J : B_j > 0\}. \tag{3.35}$$

We deduce:

$$(0) \neq \overline{M} = (\prod_{j \in B} U_j^{pB_j}) \subseteq k(x)[\{U_j\}_{j \in J}]_{\epsilon(x)}.$$

Let $I_0' = H_{W'}^{-1} G_{W'}^p$, $d_0' = \mathrm{ord} I_0'$. We have:

$$I_0' \overline{S'} = \left(U^{-\epsilon(x)} \prod_{j \in B} U_j^{pB_j} \right)_{\overline{m'}}. \tag{3.36}$$

This proves that $\epsilon(x') \le \overline{\mathrm{ord}} I_0' \le \epsilon(x)$ and equality holds only if

$$s' \in \mathbf{Proj} \left(\frac{k(x)[\{U_j\}_{j \in J}]}{(U_B)} \right). \tag{3.37}$$

Suppose that $\epsilon(x') < \epsilon(x)$. Then :

$$\omega(x') \leq \epsilon(x') \leq \epsilon(x) - 1 \leq \omega(x).$$

If $\omega(x') = \omega(x)$, then $\omega(x) = \epsilon(x) - 1$, so $\kappa(x) \geq 2$. On the other hand, we have $\omega(x') = \epsilon(x')$ and therefore $\kappa(x') = 1$ by definition 2.16. Hence inequality is strict in (3.16). In other terms, it can be assumed from now on that (3.37) holds and that

$$\epsilon(x') = \epsilon(x). \tag{3.38}$$

We now resume the argument used in case 1 by taking

$$M = J(F_{p,X,W}, E_W, W),\ d = \epsilon(y) = \epsilon(x),\ d_0 = 0.$$

To begin with, (3.26) holds whenever (3.24) applies, i.e. if $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$ or if $\delta(y) \notin \mathbb{N}$. Suppose that $\delta(y) \in \mathbb{N}$ and $H_{j'} \in p\mathbb{N}$ for every $j' \in (J')_E$. In this case, (3.22) reduces to

$$J(F_{p,Z',W'}, E', W') \equiv J(F_{p,X',W'}, E', W') \bmod K' \frac{\hat{S}'}{(u)}, \tag{3.39}$$

$$K' := (\prod_{i=2}^{e'_0} u_i'^{(p-1)b_{j_i} - H_{j_i} + \left\lceil \frac{H_{j_i}}{p} \right\rceil}) \subseteq S'$$

with notations as in (3.35). We let :

$$k' := \sum_{j \in J} \left((p-1)b_j - H_j + \left\lceil \frac{H_j}{p} \right\rceil \right) = \mathrm{ord}_{m_{S'}} K'.$$

Going back to definition 2.16, we have

$$F_{p,Z} - TF_{p,Z} \in (\prod_{j \in J} U_j^{(p-1)b_j + \left\lceil \frac{H_j}{p} \right\rceil} G(m_S))_{p\delta(x)}$$

and we deduce now from (3.39) that

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} \equiv \left(U^{-\epsilon(x)} J(TF_{p,Z}, E, m_S)\right)_{\overline{m'}} \bmod K'\overline{S'}. \tag{3.40}$$

Note that the previous equation remains valid when $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$ or when $\delta(y) \notin \mathbb{N}$. The proof now goes on as in case 1 and we

deduce that $\overline{\mathrm{ord}} I' \leq \omega(x)$; joining (3.37) and (3.40), we obtain that (3.27) holds, i.e.

$$s' \notin \mathbf{Proj}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(\mathrm{IMax}(x) \cap k(x)[\{U_j\}_{j\in J}])}\right) \Longrightarrow \overline{\mathrm{ord}} I' < \omega(x).$$

Equation (3.28) now follows, while (3.29) gets replaced by

$$\overline{I'} =< \left\{ H'^{-1} \frac{\partial F_{p,Z'}}{\partial U'_j} \right\}_{j=n'_0+1}^{n'} > \mathrm{mod}((\{U'_{j'}\}_{j'\notin F}) + (\mathrm{cl}_{k'} K')) \cap G(m_{S'})_{d'}. \tag{3.41}$$

Finally, we obtain that

$$H'^{-1}(F_{p,Z'} - T' F_{p,Z'}) \subseteq ((\{U'_i\}_{i\notin F}) + (\mathrm{cl}_{k'} K')) \cap G(m_{S'})_{\epsilon(x')}$$

and this concludes the proof of the claim, hence of the theorem, as in case 1.

Case 3: $\mathcal{Y}$ is of the second kind. First recall from proposition 3.3 that $\epsilon(x) - 1 = \omega(x)$, so $\kappa(x) \geq 2$ in particular. Let $I'_0 := H_{W'}^{-1} G_{W'}^p$, $d'_1 = \mathrm{ord} I'_0$.

Suppose that $i_0(m_S) = p - 1$. By proposition 3.3, there exists an expansion

$$H^{-1} G^p =< U_{j_1} \prod_{j\in B_J} U_j^{pB_j} >, \ j_1 \in (J')_E, \ B_j > 0 \text{ for } j \in B_J,$$

with notations as in definition 3.3. By proposition 3.5(v), we have:

$$I'_0 S'/(u) = \overline{u}_{j_1} \left(U^{-\epsilon(y)} \prod_{j\in B} U_j^{pB_j} \right)_{m_{S'/(u)}}. \tag{3.42}$$

This proves that $\epsilon(x') \leq \mathrm{ord} I'_0 \leq \epsilon(x)$ and equality holds only if

$$s' \in \mathbf{Proj}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(U_{B_J})}\right). \tag{3.43}$$

Suppose furthermore that $\epsilon(x') < \epsilon(x)$. We have:

$$\omega(x') \leq \epsilon(x') \leq \epsilon(x) - 1 = \omega(x).$$

If $\omega(x') = \omega(x)$, then $\omega(x') = \epsilon(x')$ and therefore $\kappa(x') = 1$ by definition 2.16, so inequality is strict in (3.16). Therefore if $i_0(m_S) = p - 1$, it can be

assumed that $\epsilon(x') = \epsilon(x)$ and in particular that (3.43) holds.

Going back to the general situation of case 3, we now take

$$M = J(F_{p,X,W}, E_W, W),\ d = \epsilon(y),\ d_0 = 0.$$

Note that (3.24) is always valid in this case 3: we either have $i_0(m_S) = p$ or (3.23) holds for $j' = j_0$. Applying proposition 3.5(v) gives:

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} = \left(U^{-\epsilon(y)}\overline{J}(F_{p,Z,W}, E_W, W)\right)_{\overline{m'}}.$$

With notations as in proposition 3.3, we have

$$(0) \neq \overline{J}(F_{p,Z,W}, E_W, W) =< \{\Phi_{j'}(\{U_j\}_{j\in J})\}_{j'\in J'\backslash(J')_E} > .$$

We deduce that

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} =< \{\left(U^{-\epsilon(y)}\Phi_{j'}(\{U_j\}_{j\in J})\right)_{\overline{m'}}\}_{j'\in J'\backslash(J')_E} > . \quad (3.44)$$

Since definition 3.3 gives

$$C(x, \mathcal{Y}) := \mathrm{Max}(\overline{J}(F_{p,Z,W}, E, W)) \cap \{U_{B_J} = 0\},$$

we deduce that $\overline{\mathrm{ord}}J(F_{p,Z',W'}, E_{W'}, W') \leq \omega(x)$ and equality holds only if $s' \in PC(x, \mathcal{Y})$. We obtain:

$$\epsilon(x') \leq 1 + \mathrm{ord}J(F_{p,Z',W'}, E_{W'}, W') \leq 1 + \overline{\mathrm{ord}}J(F_{p,Z',W'}, E_{W'}, W') \leq \epsilon(x). \quad (3.45)$$

Suppose that $s' \notin PC(x, \mathcal{Y})$ and $\omega(x') \geq \omega(x)$. Formula (3.45) shows that $\epsilon(x') = \omega(x') = \omega(x)$. If $i_0(m_{S'}) = p - 1$, we get $\kappa(x') = 1$ so inequality is strict in (3.16). If $i_0(m_{S'}) = p$, we may pick $j' = j_i \in J'\backslash(J')_E$, $e'+1 \leq i \leq n'_0$, such that

$$\overline{\mathrm{ord}}\left(U^{-\epsilon(y)}\Phi_{j'}(\{U_j\}_{j\in J})\right)_{\overline{m'}} < \omega(x).$$

By (3.44), we have $H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \neq 0$. This is a contradiction with the assumption $\epsilon(x') = \omega(x')$. Thus it can be assumed that $s' \in PC(x, \mathcal{Y})$.

We get $\omega(x') \leq \epsilon(x') \leq \omega(x)$ unless all inequalities in (3.45) are equalities. In this case, we claim that $\omega(x') = \epsilon(x') - 1$ and this will conclude the proof. To prove the claim, we may pick $j_i \in J'\backslash(J')_E$, $e' + 1 \leq i \leq n'_0$, such that $\Phi_{j_i}(\{U_j\}_{j\in J}) \neq 0$ by proposition 3.3. Arguing as above, we have

$$H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \equiv < \mathrm{cl}_{\omega(x)}\left(U^{-\epsilon(y)}\Phi_{j_i}(\{U_j\}_{j\in J})\right)_{\overline{m'}} > \mathrm{mod}((\{U'_{j'}\}_{j'\notin F})\cap G(m_{S'})_{\omega(x)}, \quad (3.46)$$

and this proves that $H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \neq 0$. If $i_0(m_{S'}) = p$, we get $\omega(x') = \omega(x)$.

If $i_0(m_{S'}) = p-1$, we must introduce a truncation operator

$$T' : G(m_{S'})_{p\delta(x')} \to G(m_{S'})_{p\delta(x')}$$

as in definition 2.16 in order to compute $\omega(x')$. In any case, we have

$$H'^{-1}G'^p \subseteq (U'_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}, \tag{3.47}$$

which follows from the identity $I'_0 S'/(u) = 0$ (resp. from (3.42)) if $i_0(m_S) = p$ (resp. if $i_0(m_S) = p-1$), *cf.* beginning of the proof of case 3.

Going back to definition 2.14, we have

$$H'^{-1}(F_{p,Z'} - T'F_{p,Z'}) \subseteq (\{U'_i\}_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}.$$

It now follows from (3.46) that

$$H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \equiv < \mathrm{cl}_{\omega(x)}\left(U^{-\epsilon(y)}\Phi_{j_i}(\{U_j\}_{j\in J})\right)_{\overline{m'}} > \mathrm{mod}((\{U'_{j'}\}_{j'\notin F}) \cap G(m_{S'})_{\omega(x)}.$$

This proves at last that $H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \neq 0$, so $\omega(x') = \epsilon(x') - 1$ and this concludes the proof of the claim, hence of the theorem. □

3.3 Consequences of the blowing up theorem and constructibility.

In this section, we prove some basic properties of our main invariant

$$y \mapsto (m(y), \omega(y), \kappa(y))$$

and of our notion of permissibility. The following theorem expresses the persistence of permissibility under permissible blowing ups.

Theorem 3.7. *Assume that $m(x) = p$, $\omega(x) > 0$, where $\{x\} = \eta^{-1}(m_S)$. Let $\mathcal{Y}_0 \subset \mathcal{Y}_1$ with respective generic point y_0, y_1 be permissible centers at x and $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along $\mathcal{Y}_1$.*

The strict transform $\mathcal{Y}'_0$ of $\mathcal{Y}_0$ is permissible at every $x' \in \pi^{-1}(x)$.

Proof. By definition of permissibility, we have $m(y_0) = m(y_1) = p$. Let $W_i = \eta(\mathcal{Y}_i)$, $i = 0, 1$ be with notations as in the previous theorem. There exist associated subsets $J_0 \subset J_1 \subseteq \{1, \ldots, n\}$ such that $I(W_i) = (\{u_j\}_{j \in J_i})$ for a certain choice of an adapted r.s.p. $(u_1, \ldots, u_n)$ of S. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. By proposition 2.4, the polyhedron

$$\Delta_{\hat{S}}(h; \{u_j\}_{j \in J_i}; Z) = \mathrm{pr}_{J_i}(\Delta_S(h; u_1, \ldots, u_n; Z)) \text{ is minimal,}$$

where $\mathrm{pr}_{J_i} : \mathbb{R}^n \to \mathbb{R}^{J_i}$ denotes the projection on the $(u_j)_{j \in J_i}$-space, $i = 0, 1$. In particular, we have $\mathcal{Y}_i = V(Z, \{u_j\}_{j \in J_i})$, $i = 0, 1$. The strict transform W_0' of W_0 at s' has normal crossings with $E' := \sigma^{-1}(E)_{\mathrm{red}}$. Since $m(x') \geq m(y_0)$ for every $x' \in \mathcal{Y}_0'$, this proves that $\mathcal{Y}_0'$ is Hironaka-permissible w.r.t. E'.

Applying again proposition 2.4, we have

$$\epsilon(y_0) \leq \epsilon(y_1) \leq \epsilon(x), \ \epsilon(y_0) \leq \epsilon(x'). \tag{3.48}$$

On the other hand, theorem 3.6 applied to π gives $\epsilon(x') \leq \epsilon(x) + 1$ while classifying equality cases in (1) and (2). Thus $\mathcal{Y}_0'$ is permissible of the first kind except possibly in the following two cases:

Case 1: $\mathcal{Y}_1$ is of the first kind and $\epsilon(x') = \epsilon(x) + 1$;
Case 2: $\mathcal{Y}_0$ is of the second kind and $\epsilon(x') = \epsilon(x)$.

Since $x' \in \mathcal{Y}_0'$, we have, with notations as in theorem 3.6 (*cf.* notation 3.2):

$$(J_0)_E \subseteq \{j_i, \ 2 \leq i \leq e_0'\}, \ J_0 \backslash (J_0)_E \subseteq \{j_i, \ n_0' + 1 \leq i \leq n_1'\}. \tag{3.49}$$

Also, letting $F_0 := \{2, \ldots, e_0'\} \cup \{n_0' + 1, \ldots, n_1'\}$, we have (*cf.* notation 3.3):

$$J_0 \subseteq F_0 \subseteq F = F_0 \cup \{n_1' + 1, \ldots, n'\}. \tag{3.50}$$

Proof in case 1: an immediate consequence of theorem 3.6(1) is that :

$$i_0(m_S) = p, \ \frac{\partial F_{p,Z}}{\partial U_j} = 0, \ j \in J_0 \text{ or } j \geq e + 1.$$

This is incompatible with definition 3.3(iii) applied to $\mathcal{Y}_0$, so $\mathcal{Y}_0$ is also of the first kind. By proposition 3.1 we deduce that

$$H^{-1} G^p = 0, \ H^{-1} F_{p,Z} \subseteq k(x)[\{U_j\}_{j \in J_0}]_{\epsilon(x)}. \tag{3.51}$$

Since $\epsilon(y_0) = \epsilon(x') - 1$, we also have

$$H'^{-1} < G'^p, F_{p,Z'} > \subseteq (\{U'_i\}_{j_i \in J_0})^{\epsilon(y_0)} \cap G(m_{S'})_{\epsilon(x')}. \tag{3.52}$$

We claim that $\mathcal{Y}'_0$ is permissible of the second kind at x'. To prove the claim, note that (3.51) implies that

$$H_{W_1}^{-1} G_{W_1}^p \subseteq (\overline{u}_{j'}) G(W_1)_{\epsilon(x)} \text{ for some } j' \in (J'_1)_E.$$

Since $\mathcal{Y}_0$ is permissible of the first kind at x, we actually have

$$H_{W_1}^{-1} G_{W_1}^p \subseteq (\overline{u}_{j'}) S/(\{u_j\}_{j \in J_1})[\{U_j\}_{j \in J_0}]_{\epsilon(x)}.$$

Letting $j' =: j_{i'}$, $e'_0 + 1 \leq i' \leq e$, proposition 3.5(ii) then shows that

$$H_{W'_1}^{-1} G_{W'_1}^p \subseteq (\overline{u}_{i'}) S'/(u'_1)[\{U'_i\}_{j_i \in J_0}]_{\epsilon(x)}, \; W'_1 := \sigma^{-1}(W_1).$$

In other terms, we have

$$H'^{-1} G'^p \subseteq (U'_1, U_{i'}) k(x')[\{U'_i\}_{j_i \in J_0}],$$

and this proves that $\mathcal{Y}'_0$ satisfies property (ii) of definition 3.2. Finally, applying (3.52) gives an expansion

$$H'^{-1} F_{p,Z'} = < \sum_{i=1}^{n'} U'_i \Phi_i(\{U'_{i'}\}_{j_{i'} \in J_0}) > .$$

Then definition 3.2(iii) is equivalent to:

$$\exists i \in J'_0 \cap \{e' + 1, \ldots, n'\} : \Phi_i \neq 0.$$

By equation (3.17) in theorem 3.6(2), there exists $i \geq n'_1 + 1$ (hence $i \in J'_0$) such that $\Phi_i \neq 0$, since $j_{i'} \in J_0 \Longrightarrow i' \leq n'_1$ by (3.49) and this completes the proof in case 1.

Proof in case 2. Since $\mathcal{Y}_0$ is permissible of the second kind, the initial form $\mathrm{in}_{m_S} h \in G(m_S)[Z]$ satisfies (3.5). The corresponding integer j_0 satisfies $j_0 \notin J'_0$ and the corresponding family $(\Phi_{j'}(\{U_j\}_{j \in J_0}))_{j' \in J'_0}$ is such that $\Phi_{j'} \neq 0$ for some $j' \in J'_0 \backslash (J'_0)_E$. In order to prove that $\mathcal{Y}'_0$ is of the second kind at x', we consider two subcases:

Case 2a: $\mathcal{Y}_1$ is of the second kind at x. Then $j_0 \in J_1'$ and $\Phi_{j'} \neq 0$ for some $j' \in J_1' \backslash (J_1')_E$. By assumption $\epsilon(x') = \epsilon(x)$, and we deduce from (3.42) (resp. from (3.47)) if $i_0(m_S) = p-1$ (resp. if $i_0(m_S) = p$) that the initial form $\mathrm{in}_{m_{S'}} h' \in G(m_{S'})[Z']$ satisfies

$$H'^{-1} G'^p \subseteq U_{j_0'} k(x')[\{U_i'\}_{j_i \in J_0}]_{\epsilon(y_0)} \text{ for some } j_0' \in \{1, e_0'+1, \ldots, e'\} \quad (3.53)$$

and definition 3.2(ii) is checked for $\mathcal{Y}_0'$ at x'. Similarly, definition 3.2(iii) is checked from (3.46): we have $H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \neq 0$ for any i, $e'+1 \leq i \leq n_0'$ such that $j_i \in J_1' \backslash (J_1')_E$ and $\Phi_{j_i} \neq 0$; take $j_i = j'$ with notations as above.

Case 2b: $\mathcal{Y}_1$ is of the first kind at x. Then $j_0 \in J_1$ and $\Phi_{j'} = 0$ for any $j' \in J_1'$. By proposition 3.3 and our assumption $\epsilon(x') = \epsilon(x)$, we have

$$\omega(x) = \epsilon(y_0) = \epsilon(x) - 1 = \epsilon(x') - 1 \leq \omega(x').$$

Therefore theorem 3.6 implies that $\omega(x') = \omega(x)$. We have $\kappa(x), \kappa(x') \geq 2$ since $\omega(x) = \epsilon(x) - 1$, $\omega(x') = \epsilon(x') - 1$. This is the equality case $(m(x'), \omega(x'), \kappa(x')) = (m(x), \omega(x), \kappa(x))$ discussed in theorem 3.6.

If $i_0(m_S) = p$, we are in the equality case of (3.28). Then (3.53) holds and there exists i, $n_1'+1 \leq i \leq n'$ or ($n_0'+1 \leq i \leq n_1'$ and $\Phi_{j_i} \neq 0$) such that

$$H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \neq 0 \quad (3.54)$$

by (3.31). We may take here $j_i := j' \in J_0' \backslash (J_0')_E$. This checks definition 3.2(ii) and (iii) respectively.

If $i_0(m_S) = p-1$, the initial form $\mathrm{in}_{m_{S'}} h' \in G(m_{S'})[Z']$ satisfies

$$H'^{-1} G'^p \subseteq U_{i_1}' k(x')[\{U_i'\}_{j_i \in J_0}]_{\epsilon(y_0)},$$

where $j_{i_1} := j_0 \in J_0'$, $2 \leq i_1 \leq e_0'$ and definition 3.2(ii) is checked. Equation (3.54) also remains valid for some i, $n_0'+1 \leq i \leq n'$, in this case: this follows from (3.31) which is still valid (end of the proof of case 2 of theorem 3.6 where (3.41) replaces (3.29). This checks definition 3.2(iii) and the proof is complete. □

Remark 3.2. The conclusion of the above theorem fails in general if it is only assumed that $\mathcal{Y}_0 \subset \mathcal{Y}_1$ is such that $\mathcal{Y}_0$ is permissible at x, $\mathcal{Y}_1$ Hironaka-permissible at x w.r.t. E.

A counterexample with $n = 4$ is given for $\mathrm{char} S = p > 0$ by taking:

$$h = Z^p + u_4 u_1^p + u_3 u_2^p,\ E = \mathrm{div}(u_1 u_2 u_3),\ \mathrm{Sing}_p \mathcal{X} = V(Z, u_1, u_2).$$

Then $(u_1, \ldots, u_4; Z)$ are well adapted coordinates. Taking

$$\mathcal{Y}_0 = V(Z, u_1, u_2) \subset \mathcal{Y}_1 = V(Z, u_1, u_2, u_4) \subset \{x\} = V(Z, u_1, u_2, u_3, u_4),$$

we have $\epsilon(y_0) = \epsilon(y_1) = \epsilon(x) - 1 = \omega(x) = p$. Note that $\mathcal{Y}_1$ does not satisfy definition 3.2(iii). There is a unique point

$$x' = (Z', u_1', u_2', u_3, u_4') := (Z/u_4, u_1/u_4, u_2/u_4, u_3, u_4) \in \mathcal{Y}_0' = V(Z', u_1', u_2').$$

A local equation for the strict transform $\mathcal{X}'$ of $\mathcal{X}$ at x is:

$$h' = Z'^p + u_4' u_1'^p + u_3 u_2'^p,\ E' = \mathrm{div}(u_1' u_2' u_3 u_4').$$

Thus $\epsilon(x') = \omega(x') = p + 1 > \omega(x)$ and $\mathcal{Y}_0'$ is not permissible at x' since $\epsilon(y_0) = p < \epsilon(x')$.

It is easily seen that such counterexamples exist only for $\mathcal{Y}_0$ of the second kind and $n \geq 4$.

We now turn to formal arcs on $\mathcal{X}$ and their image. Recall that it is assumed all along this chapter that $m(x) = p$, $\omega(x) > 0$ and $\{x\} = \eta^{-1}(m_S)$.

Definition 3.4. A *formal arc* on $(\mathcal{X}, x)$ is a local morphism $\varphi :\ \mathrm{Spec}\mathcal{O} \to (X, x)$, where $(\mathcal{O}, N, l)$ is a complete discrete valuation ring. We denote the closed (resp. generic) point of $\mathrm{Spec}\mathcal{O}$ by O (resp. ξ) and call *support of* φ the subscheme $Z(\varphi) := \overline{\{\varphi(\xi)\}} \subseteq (\mathcal{X}, x)$.

The arc φ is said to be *well parametrized* if the inclusion

$$\mathcal{O}_\xi := \mathcal{O} \cap k(\varphi(\xi)) \subseteq \mathcal{O}$$

induces an isomorphism $\widehat{\mathcal{O}_\xi} \simeq \mathcal{O}$. The arc φ is said to be *nonconstant* if $\varphi(\xi) \neq x = \varphi(O)$.

Given a nonconstant formal arc on $(\mathcal{X}, x)$, and $\pi :\ \mathcal{X}' \to \mathcal{X}$ a blowing up along a permissible center $\mathcal{Y} \subset \mathcal{X}$ at x such that $\mathcal{Y} \subsetneq Z(\varphi)$, there exists a unique lifting $\varphi' :\ \mathrm{Spec}\mathcal{O} \to \mathcal{X}'$. Let

$$x' := \varphi'(O),\ (\mathcal{X}_1, x_1) := (\mathcal{X}', x') \text{ and } \varphi_1 :\ \mathrm{Spec}\mathcal{O} \to (\mathcal{X}_1, x_1)$$

be the induced morphism. The arc φ_1 is again nonconstant, so the process can be iterated. Let

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \tag{3.55}$$

be a sequence of such local blowing ups and centers with

$$x_r \in \mathcal{Y}_r \subsetneq Z_r(\varphi) := \overline{\{\varphi_r(\xi)\}} \subset \mathcal{X}_r. \tag{3.56}$$

Note that the local ring $\mathcal{O}_{\mathcal{X}_r, \varphi_r(\xi)}$ is independent of $r \geq 0$. In particular, $m(\varphi_r(\xi))$, $\epsilon(\varphi_r(\xi))$ and $\omega(\varphi_r(\xi))$ are independent of $r \geq 0$. An important case of such sequences is when taking $\mathcal{Y}_r = \{x_r\}$ for every $r \geq 0$; then (3.55) is called the *quadratic sequence along* φ.

In any case, given a sequence (3.55), we let

$$d(\varphi) := \min_{r \geq 0} \{\dim \mathcal{O}_{\mathcal{X}_r, x_r}\}.$$

If $m(x) = p$ and $\omega(x) > 0$, theorem 3.6 implies that

$$(m(x_1), \omega(x_1), \kappa(x_1)) \leq (m(x), \omega(x), \kappa(x)).$$

If $m(x_r) = p$ and $\omega(x_r) > 0$ for every $r \geq 0$, we let

$$m(\varphi) := p, \ \omega(\varphi) := \min_{r \geq 0} \{\omega(x_r)\} > 0.$$

Proposition 3.8. *With notations as above, let $\varphi : \operatorname{Spec}\mathcal{O} \to (\mathcal{X}, x)$ be a nonconstant well parametrized formal arc on $(\mathcal{X}, x)$ whose quadratic sequence is such that $m(\varphi) = p$ and $\omega(\varphi) > 0$. Then $l|k(x_r)$ is algebraic for $r >> 0$.*

Assume that $l|k(x_r)$ is algebraic with finite inseparable degree for some $r \geq 0$. Then there exists $r_0 \geq 0$ such that the following holds: the support $Z_r(\varphi)$ is Hironaka-permissible at x_r and $\epsilon(x_r) = \epsilon(x_{r_0})$ for every $r \geq r_0$; furthermore exactly one of the following conditions is satisfied:

(1) $Z_r(\varphi)$ is permissible of the first kind at x_r for every $r \geq r_0$;

(2) there exists a finite sequence (3.55):

$$(\mathcal{X}_{r_0}, x_{r_0}) =: (\mathcal{X}', x') \leftarrow (\mathcal{X}'_1, x'_1) \leftarrow \cdots \leftarrow (\mathcal{X}'_{r_1}, x'_{r_1}) =: (\tilde{\mathcal{X}}, \tilde{x})$$

of local blowing ups with permissible centers of the first kind contained in and of codimension one in the successive strict transforms of $Z_{r_0}(\varphi)$, such that the quadratic sequence along φ:

$$(\tilde{\mathcal{X}}, \tilde{x}) =: (\tilde{\mathcal{X}}_0, \tilde{x}_0) \leftarrow (\tilde{\mathcal{X}}_1, \tilde{x}_1) \leftarrow \cdots \leftarrow (\tilde{\mathcal{X}}_r, \tilde{x}_r) \leftarrow \cdots$$

has the following properties for every $r \geq 0$:

(a) $\epsilon(\tilde{x}_r) = \epsilon(x_{r_0})$;

(b) $\dim\mathcal{O}_{\tilde{Z}_r(\varphi),\tilde{x}_r} = \dim\mathcal{O}_{Z_{r_0}(\varphi),x_{r_0}} \geq 2$;

(c) $\tilde{Z}_r(\varphi)$ *is permissible of the second kind at* $\tilde{x}_r$ *(resp.* $\omega(\tilde{x}_r) = 0$*) if* $\epsilon(x_{r_0}) \geq 2$ *(resp. if* $\epsilon(x_{r_0}) = 1$*).*

Proof. It can be assumed without loss of generality that

$$d(\varphi) = \dim\mathcal{O}_{\mathcal{X},x},\ m(x) = p \text{ and } \omega(x) = \omega(\varphi) > 0.$$

Since $m(\varphi) = p$ and $\omega(\varphi) > 0$, we let $\eta_r :\ (\mathcal{X}_r, x_r) \to \mathrm{Spec}S_r$ be the corresponding projection, $I_r(\varphi) \subseteq S_r$ be the ideal of $W_r(\varphi) := \eta_r(Z_r(\varphi))$. We drop the reference to φ in what follows in order to avoid cumbersome notations.

For $f \in m_{S_0}$, $f \notin I_0$ we denote by $\overline{f} \in \mathcal{O}$, $\overline{f} \neq 0$ its image by $\varphi^\sharp$. Let v be the discrete valuation associated with $\mathcal{O}$ and let

$$M_r := \{v(\overline{f}),\ f \in S_r \backslash I_r\}$$

be the semigroup of values of S_r w.r.t. v. The group generated by M_r is the value group of the restriction $v_{|\overline{K}}$ to $\overline{K} = QF(S/I_0)$, hence independent of $r \geq 0$, and is denoted by $a\mathbb{Z} \subseteq v(N)\mathbb{Z}$, $a \in \mathbb{N}$.

Suppose that $M_0 \neq a\mathbb{N}$. Let $\alpha \geq 2$, $\beta \in \mathbb{N}\backslash\alpha\mathbb{N}$ be defined by:

$$a\alpha := \min\{M_0\backslash(0)\},\ a\beta := \min\{M_0\backslash a\alpha\mathbb{N}\}. \tag{3.57}$$

We pick $u, w \in m_{S_0}$ such that $v(\overline{u}) = a\alpha$, $v(\overline{w}) = a\beta$. Obviously u is a regular parameter of S and $wu^{-1} \in m_{S_1}$. Suppose $M_1 \neq a\mathbb{N}$. There are associated integers α_1, β_1 as in (3.57) which satisfy $(\alpha_1, \beta_1) < (\alpha, \beta)$ for the lexicographical ordering. This can repeat only finitely many times so we get $M_r = a\mathbb{N}$ for some $r \geq 0$. W.l.o.g. it can be assumed that $M_0 = a\mathbb{N}$.

Let $(u_1, \ldots, u_n)$ be a r.s.p. of $S = S_0$ which is adapted to $E = \mathrm{div}(u_1 \cdots u_e)$. Without loss of generality, it can be assumed that $v(\overline{u}_e) = a$. Up to renumbering coordinates, there exists $e(\varphi)$, $0 \leq e(\varphi) < e$ such that

$$(u_1, \ldots, u_{e(\varphi)}) \subseteq I := I_0,\ u_j \notin I \text{ for } e(\varphi) + 1 \leq j \leq e.$$

For j, $e(\varphi) + 1 \leq j \leq e - 1$, let $v(\overline{u}_j) =: a\alpha_j$, $\alpha_j \geq 1$. Note that $u_j u_e^{-\alpha_j}$ is a unit in S_{α_j}; in other terms, replacing S by $S_{\max\{\alpha_j\}}$, it can be assumed that

$e(\varphi) = e - 1$.

Let $f \in m_{S_0} \backslash I_0$ and write $f = u_e^{\alpha_r(f)} f_r \in S_r$, where u_e does not divide f_r in S_r and note that

$$f_r \in m_{S_r} \Longrightarrow v(\overline{f}) > \alpha_r(f) v(\overline{u}_e) \geq ar.$$

Since $M_0 = a\mathbb{N}$, there exists $r \geq 0$ such that f_r is a unit. This implies that for every ideal $\overline{J} \subseteq S_0/I_0$, $\overline{J}S_r/I_r$ is a principal ideal for $r >> 0$. This is a well known characterization of valuation rings, i.e.

$$\mathcal{O}_{v_{|\overline{K}}} = \bigcup_{r \geq 0} S_r/I_r. \tag{3.58}$$

Let l_0 be the residue field of the valuation $v_{|\overline{K}}$. Then $l|l_0$ is algebraic (of degree at most p) and $l_0|k(x_r)$ is algebraic for $r >> 0$ by (3.58). This proves the first statement in the theorem. We thus may assume from now on, again by (3.58), that

$$l_0|k(x_0) \text{ is separable algebraic.} \tag{3.59}$$

Let S^{sh} be the strict Henselization of S, so $l^{\mathrm{sh}} := S^{\mathrm{sh}}/m_{S^{\mathrm{sh}}}$ is the separable algebraic closure of l. The residue action induces an isomorphism

$$\mathrm{Gal}(S^{\mathrm{sh}}|S^{\mathrm{h}}) \simeq \mathrm{Gal}(l^{\mathrm{sh}}|k(x))$$

where S^{h} is the Henselization of S. Let $\tilde{S}$ be the fixed subring of S^{sh} by the inverse image of $\mathrm{Gal}(l^{\mathrm{sh}}|l_0)$ under the previous group morphism. Then $S \subset \tilde{S}$ is a local ind-étale map such that $l_0 = \tilde{S}/m_{\tilde{S}}$. In particular $S \subset \tilde{S}$ is regular [47] theorem I.8.1(iv). Since $\mathcal{O}$ is Henselian and $l_0 \subseteq l = \mathcal{O}/N$, the morphism φ factors through $\tilde{S}$.

Recall notation 2.1 and notation 2.2 for the regular local base change $S \subset \tilde{S}$. We apply theorem 2.20 with $\tilde{s} := m_{\tilde{S}}$ and get:

$$m(\tilde{x}) = m(x) = p,\ \omega(\tilde{x}) = \omega(\varphi) > 0 \text{ and } \epsilon(\tilde{x}) = \epsilon(x) > 0,$$

the right hand side equality holding because $\tilde{n} = n$. Applying theorem 2.14, $\tilde{\mathcal{X}} = \mathrm{Spec}(\tilde{S}[X]/(\tilde{h}))$ is irreducible, so in the separable case (case (b) of assumption (**G**)), the $G = \mathbb{Z}/p$-action extends uniquely to $\tilde{\mathcal{X}}$ and (**G**) holds for $(\tilde{S}, \tilde{h}, \tilde{E})$. This proves that $(\tilde{S}, \tilde{h}, \tilde{E})$ satisfies the assumption of the proposition, all other assumptions being trivially satisfied.

Now $W_0 \times_{k(x_0)} \mathrm{Spec} l_0$ may be reducible, but $W_r \times_{k(x_r)} \mathrm{Spec} l_0$ is irreducible for $r >> 0$. After possibly changing indices, it can be assumed that $W := W_0 \times_{k(x_0)} \mathrm{Spec} l_0$ is irreducible. Then W has normal crossings with E at x if and only if $\tilde{W} := W \times_S \mathrm{Spec}\tilde{S}$ has normal crossings with $\tilde{E}$ at $\tilde{x}$. Let $\tilde{Z} := Z \times_S \mathrm{Spec}\tilde{S}$ and $\tilde{z}$ be the generic point of a component of $\tilde{Z}$. By theorem 2.20, we have $m(\tilde{z}) = m(z)$, so $\tilde{Z}$ is Hironaka-permissible at $\tilde{x}$ w.r.t. $\tilde{E}$ if and only if Z is Hironaka-permissible at x w.r.t. E. In other terms, we may replace S by $\tilde{S}$ and thus assume that $l_0 = k(x_0)$ in order to prove the second statement.

Let now

$$e_r := \dim_{k(x_r)} \frac{I_r + m_{S_r}^2}{m_{S_r}^2} \geq e - 1, \ t_r := e_r - (e-1) \geq 0$$

for $r \geq 0$. It can be assumed w.l.o.g. that $(u_{e+1}, \ldots, u_{e+t_0}) \subseteq I_0$. We have $e_{r+1} \geq e_r$ for every $r \geq 0$ and let $e_\infty := \max_{r \geq 0}\{e_r\}$. It can be assumed w.l.o.g. that $e_0 = e_\infty$.

Since $l_0 = k(x_r)$ and $M_r = a\mathbb{N}$ for every $r \geq 0$, the ring morphism $S_r \to \widehat{\mathcal{O}_{v|K}}$ factors through $\hat{S}_r$ to a *surjective* morphism

$$\hat{\varphi}_r : \ \hat{S}_r \to \widehat{\mathcal{O}_{v|K}}.$$

Let $\hat{I}_r$ be the kernel of $\hat{\varphi}_r$, so we have

$$I_r \hat{S}_r \subseteq \hat{I}_r \text{ and } I_r = \hat{I}_r \cap S_r. \tag{3.60}$$

After possibly replacing S_0 by S_r for some $r \geq 0$, it can be assumed that the curve $\mathrm{Spec}(\hat{S}_0/\hat{I}_0)$ is transverse to $\hat{E} = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec}\hat{S}_0$. We claim that

$$I_0 = (u_1, \ldots, u_{e-1}, u_{e+1}, \ldots, u_{e+t_0}). \tag{3.61}$$

To prove the claim, suppose that $I_0 \neq J_0 := (u_1, \ldots, u_{e-1}, u_{e+1}, \ldots, u_{e+t_0})$. We let $\hat{u}_j := u_j$, $1 \leq j \leq e + t_0$ and pick a basis

$$\hat{I}_0 = J_0 + (\hat{u}_{e+t_0+1}, \ldots, \hat{u}_n) \tag{3.62}$$

of $\hat{I}_0$. Since S_0 is excellent, the ring $(\hat{S}_0/I_0)_{\hat{I}_0}$ is regular, hence reduced. By assumption, $I_0 \neq J_0$, so there exists $f \in I_0 \backslash J_0$ such that f restricts to a regular parameter $\overline{f}$ in $\overline{S} := (\hat{S}_0/J_0)_{\hat{I}_0}$:

$$\mathrm{ord}_{\hat{I}_0} f = 1, \ \mathrm{ord}_{m_{\overline{S}}} \overline{f} = 1. \tag{3.63}$$

Let $F \in \mathrm{gr}_{\hat{I}_0}(\hat{S}_0) \simeq \hat{S}_0/\hat{I}_0[\{\hat{U}_j\}_{j\neq e}]$ be the initial form of f. There is an expansion

$$F = \sum_{j\neq e} F_j \hat{U}_j,\ F_j \in \hat{S}_0/\hat{I}_0.$$

By (3.63) we have $F_j \neq 0$ for some j, $1 \leq j \leq e + t_0$. Suppose that

$$\exists j_0,\ 1 \leq j_0 \leq e + t_0 \mid\ m := \min_{j\neq e}\{\mathrm{ord}_{(\overline{u}_e)} F_j\} = \mathrm{ord}_{(\overline{u}_e)} F_{j_0}.$$

Replacing f with $f - \gamma_{j_0} u_{j_0} u_e^m$ for some unit $\gamma_{j_0} \in S_0$ preserves (3.63) while increasing $\mathrm{ord}_{(\overline{u}_e)} F_{j_0}$. Applying finitely many times this procedure, it can be assumed that

$$m := \min_{j\neq e}\{\mathrm{ord}_{(\overline{u}_e)} F_j\} < \min_{j_0 \leq e+t_0}\{\mathrm{ord}_{(\overline{u}_e)} F_{j_0}\}. \tag{3.64}$$

By lemma 3.10 below, there exists $r \geq 1$ and a writing

$$f_r = u_e^{m+r} g_r,\ g_r \notin (u_e) S_r,\ \mathrm{ord}_{m_{S_r}} g_r = 1.$$

Furthermore the last statement in *ibid.* shows that $\mathrm{in}_{\hat{I}_r} g_r \in (\mathrm{gr}_{\hat{I}_r} \hat{S}_r)_1$ is transverse to the initial forms $\overline{u}_e^{-r} U_j$, $1 \leq j \leq e+t_0$, $j \neq e$ by (3.64). Since $g_r \in I_r$, this implies that $e_r > e_0$: a contradiction, so claim (3.61) is proved. Since (3.61) is stable by further blowing ups, this proves that W_r is transverse to the reduced preimage of $\mathrm{div}(u_1 \cdots u_e)$ for every $r >> 0$.

Let $(\hat{u}_1, \ldots, \hat{u}_n; Z)$ be well adapted coordinates at x. There is an associated expansion

$$h = Z^p + f_{1,Z} Z^{p-1} + \cdots + f_{p,Z},\ f_{1,Z}, \ldots, f_{p,Z} \in \hat{S}_0.$$

We factor out $f_{i,Z} = u_e^{m_i} g_{i,Z}$, $1 \leq i \leq p$, with $g_{i,Z} = 0$ or (u_e does not divide $g_{i,Z}$, $m_i \in \mathbb{N}$). The *formal completion* $\hat{S}_1$ of the local blowing up S_1 has a r.s.p. $(\hat{u}'_1, \ldots, \hat{u}'_n)$ given by

$$\hat{u}'_e = \hat{u}_e = u_e \text{ and } \hat{u}'_j = \hat{u}_j/u_e,\ j \neq e.$$

Let $Z' := Z/u_e$, $h' := u_e^{-p} h \in S_1[Z']$ define the strict transform $(\mathcal{X}_1, x_1)$, since $m(\varphi) = p$. We thus have

$$f_{i,Z'} = u_e^{-i} f_{i,Z},\ 1 \leq i \leq p. \tag{3.65}$$

By proposition 2.6, the polyhedron $\Delta_{\hat{S}_1}(h'; \hat{u}'_1, \ldots, \hat{u}'_n; Z')$ is minimal. Applying again lemma 3.10 below, it can be assumed w.l.o.g. that

$$\mathrm{ord}_{m_{\hat{S}_0}} g_{i,Z} = \mathrm{ord}_{\hat{I}_0} g_{i,Z},\ 1 \leq i \leq p. \tag{3.66}$$

Let $\hat{Z}_0 := V(Z', \hat{I}_0) \subset (\hat{\mathcal{X}}_0, \hat{x})$ and $\hat{z}$ be its generic point. Suppose that $\delta(\hat{z}) < 1$ and let i_0 such that $i_0\delta(\hat{z}) = \mathrm{ord}_{\hat{I}_0} f_{i_0,Z} < i_0$. Applying (3.65) gives

$$\mathrm{ord}_{m_{\hat{S}_1}} f_{i_0,Z'} = m_{i_0} + i_0(\delta(\hat{z}) - 1) < m_{i_0}.$$

This can repeat only finitely many times, a contradiction with $m(\varphi) = p$. Hence $\delta(\hat{z}) \geq 1$, i.e. $m(\hat{z}) = p$. By excellence, this implies that $m(z) = p$. Therefore Z_r is Hironaka-permissible at x_r for every $r >> 0$.

Similarly, replacing S_0 by S_r for some $r \geq 0$ and arguing as above, it can be assumed that

$$\epsilon(\hat{z}) = \min_{1 \leq i \leq p} \left\{ \frac{\mathrm{ord}_{\hat{I}_0}(H(x)^{-i} f_{i,Z}^p)}{i} \right\} = \epsilon(\hat{x}).$$

This proves that $\hat{Z}_0$ is permissible of the first kind at $\hat{x}$. Note that this furthermore implies that $\epsilon(x_r) = \epsilon(\hat{z})$ for every $r \geq 0$ and the second statement of the proposition is proved.

In order to prove that alternative (1) in the last statement holds, we may also replace S by $\tilde{S}$ as above and thus assume that $l_0 = k(x_0)$. If $\epsilon(z) = \epsilon(\hat{z})$, then Z_r is permissible of the first kind at x_r (definition 3.1(ii)). This proves that alternative (1) in the proposition is fulfilled or $\epsilon(\hat{z}) > \epsilon(z)$ which we may assume from now on.

By theorem 2.20(2.ii), we have $\dim Z_r \geq 2$ (statement $\tilde{n} > n$ of *ibid.* applied under the assumption $l_0 = k(x_0)$) and

$$\epsilon(\hat{z}) - 1 = \omega(z) = \epsilon(z) = \epsilon(\hat{x}) - 1 = \epsilon(x) - 1,\ i_0(\hat{z}) = i_0(z) = p. \tag{3.67}$$

We pick again well adapted coordinates $(\hat{u}_1, \ldots, \hat{u}_n; \hat{Z})$ at $\hat{x}$. Since $\hat{Z}_0$ is permissible of the first kind at $\hat{x}$, proposition 3.1 (with notations as therein) gives the following property for the initial form $\mathrm{in}_{m_{\hat{S}_0}} h \in G(m_{\hat{S}_0})[\hat{Z}]$:

$$H_0^{-1} G_0^p \in k(\hat{x})[\hat{U}_1, \ldots, \hat{U}_{e-1}, \hat{U}_{e+1}, \ldots \hat{U}_n]_{\epsilon(\hat{x})}.$$

Since $i_0(\hat{z}) = p$, we have $G_0 = 0$, i.e. $i_0(\hat{x}) = p$. This proves that definition 3.2(ii) is satisfied in any case.

To prove that alternative (2) in the proposition is fulfilled, we first assume that $l_0 = k(x_0)$ as before, then push down the result from $\tilde{S}$ to S. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and consider the initial form $\mathrm{in}_{W_0} h = Z^p + F_{p,Z,W_0} \in G(W_0)[Z]$. Let

$$J := \{1, \ldots, e-1, e+1, \ldots, e+t_0\}.$$

Since $\epsilon(\hat{z}) > \epsilon(z)$, we have $\delta(z) \in \mathbb{N}$ and

$$G(W_0) = S_0/I_0[\{U_j\}_{j\in J}],\ F_{p,Z,W_0} \in (\hat{S}_0/\hat{I}_0[\{U_j\}_{j\in J}]_{\delta(z)})^p \tag{3.68}$$

by theorem 2.20(2.ii). By proposition 2.4, the polyhedron

$$\Delta_{\hat{S}_0}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}_J(\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)) \text{ is minimal,}$$

where $\mathrm{pr}_J :\ \mathbb{R}^n \to \mathbb{R}^J$ denotes the projection on the $(u_j)_{j\in J}$-space. Let

$$\Phi_j := H_{W_0}^{-1} \frac{\partial F_{p,Z,W_0}}{\partial \overline{u}_j} \subseteq G(W_0)_{\epsilon(z)},\ \mathrm{cl}_0 \Phi_j = 0,\ j \notin J, j \neq e, \tag{3.69}$$

since $\epsilon(x) = \epsilon(z) + 1$. The local blowing up S_1 has a r.s.p. $(u'_1, \ldots, u'_n)$ given by

$$\begin{cases} u'_j &= u_j/u_e \quad \text{if} \quad j \in J \\ u'_e &= u_e \\ u'_j &= u_j/u_e - \delta_j \quad \text{if} \quad j \notin J, j \neq e \end{cases}$$

where $\delta_j \in S_0$ is a unit or zero since we are assuming that $l_0 = k(x_0)$. Let

$$Z' := Z/u_e - \theta,\ \theta \in S_1,\ h' := u_e^{-p} h \in S_1[Z']$$

define the strict transform $(\mathcal{X}_1, x_1)$, with $\Delta_{S_1}(h'; u'_1, \ldots, u'_n; Z')$ minimal and consider the initial form

$$\mathrm{in}_{W_1} h = Z'^p + F_{p,Z',W_1} \in G(W_1)[Z'],\ G(W_1) = S_1/I_1[\{U'_j\}_{j\in J}].$$

It is easily derived from (3.68)(3.69) that

$$\Phi'_j := H_{W_1}^{-1} \frac{\partial F_{p,Z',W_1}}{\partial \overline{u}'_j} = \overline{u}_e^{-\epsilon(x)} \Phi_j \subseteq G(W_1)_{\epsilon(z)},\ j \notin J, j \neq e.$$

Applying again lemma 3.10 below, it can be assumed w.l.o.g. that

$$(\Phi_j = \overline{u}_e^{m_j}\Psi_j,\ \mathrm{cl}_0\Psi_j \neq 0) \text{ or } \Phi_j = 0,\ j \notin J, j \neq e. \tag{3.70}$$

This equation is valid when $l_0 = k(x_0)$ and holds for S if and only if it holds for $\tilde{S}$. We may therefore replace S by $\tilde{S}$ as before.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ be a vertex of $\Delta_{S_0}(h; u_1, \ldots, u_n; Z)$ mapping to a vertex of $\Delta_{S_0}(h; \{u_j\}_{j\in J}; Z)$ with $\sum_{j\in J} x_j = \delta(y)$. By (3.68) we have $x_j \in \mathbb{N}$ for $j \in J$. Suppose that $x_j \in \mathbb{N}$ for every $j \neq e$. Since $\hat{S}_0/\hat{I}_0 \simeq k(x)[[\overline{u}_e]]$, (3.68) implies that $\mathbf{x}$ is solvable: a contradiction. Taking j such that $x_j \notin \mathbb{N}$, there exists $j \notin J$, $j \neq e$ such that $\Phi_j \neq 0$. This proves that

$$r_1 := \min\{m_j, j \notin J, j \neq e : \Phi_j \neq 0\}$$

is well defined and that we have

$$\Phi_{p,Z,W_0} := \overline{u}_e^{-r_1} H_{W_0}^{-1} F_{p,Z,W_0} \subseteq G(W_0)_{\epsilon(z)},\ \mathrm{cl}_1\Phi_{p,Z,W_0} \notin (\overline{u}_e)G(W_0)_{\epsilon(z)}. \tag{3.71}$$

If $r_1 = 0$, then alternative (2) is fulfilled (definition 3.2(iii)) since

$$\overline{J}(F_{p,Z,W_0}, E, W_0) =< \{\mathrm{cl}_0\Phi_j\}_{j\notin J, j\neq e} >\neq 0.$$

by (3.71). Note that this situation does not occur if $\epsilon(x_{r_0}) = 1$, since $\omega(\varphi) > 0$.

Otherwise, we define $V_0 := V(u_e, I_0)$ and $\mathcal{Y}_0 := \eta_0^{-1}(V_0) \subset Z_0$. Then $\mathcal{Y}_0$ is Hironaka-permissible at x_0 and its generic point y_0 has $\epsilon(y_0) = \epsilon(x)$ by (3.71). Let $\tilde{\mathcal{X}}_1$ be the blowing up of $\mathcal{X}_0$ along $\mathcal{Y}_0$ and note that φ lifts to the point $\tilde{x}_1$ on the strict transform $\tilde{Z}_1$ of Z_0. Let $\tilde{h} := u_e^{-p}h \in \tilde{S}_1[\tilde{Z}]$ define the strict transform $(\tilde{\mathcal{X}}_1, \tilde{x}_1)$ of $(\mathcal{X}, x)$, $\tilde{W}_1 := \tilde{\eta}_1(\tilde{Z}_1)$. By proposition 2.6, the initial form

$$\mathrm{in}_{\tilde{W}_1}\tilde{h} = \tilde{Z}^p + F_{p,\tilde{Z},\tilde{W}_1} \in G(\tilde{W}_1)[\tilde{Z}],\ G(\tilde{W}_1) = \tilde{S}_1/\tilde{I}_1[\{\tilde{U}_j\}_{j\in J}]$$

satisfies a relation (3.71) with associated integer $\tilde{r}_1 = r_1 - 1$. Iterating r_1 times this procedure, we get some $(\tilde{\mathcal{X}}_{r_1}, \tilde{x}_{r_1})$ with initial form

$$\mathrm{in}_{\tilde{W}_r}\tilde{h}_r = \tilde{Z}_r^p + F_{p,\tilde{Z}_r,\tilde{W}_r} \in G(\tilde{W}_r)[\tilde{Z}_r],\ G(\tilde{W}_r) = \tilde{S}_r/\tilde{I}_r[\{\tilde{U}_{j,r}\}_{j\in J}]$$

with $\tilde{U}_{j,r} = \overline{u}_e^{-r_1}U_j$, $j \in J$. We have

$$\tilde{\Phi}_r := H_{\tilde{W}_r}^{-1} F_{p,\tilde{Z}_r,\tilde{W}_r}) \subseteq G(\tilde{W}_r)_{\epsilon(z)},\ \mathrm{cl}_1\tilde{\Phi}_r \notin (\overline{u}_e)G(W_0)_{\epsilon(z)}. \tag{3.72}$$

By proposition 3.3, we now have $\omega(\tilde{x}_{r_1}) = \epsilon(z) = \epsilon(x_{r_0}) - 1 \geq 0$. Thus $\omega(\tilde{x}_{r_1}) > 0$ if $\epsilon(x_{r_0}) \geq 2$ and we are done by the former case $r_1 = 0$. Otherwise, $\epsilon(x_{r_0}) = 1$ and $\omega(\tilde{x}_{r_1}) = 0$ and the conclusion follows. □

Example 3.2. Take $S = k[u_1, u_2, u_3, u_4]_{(u_1,u_2,u_3,u_4)}$ with k a field of characteristic $p > 0$. We let:

$$h = Z^p + u_2^p u_4 u_3^p + u_3 u_1^p \in S[Z].$$

Then (u_1, u_2, u_3, u_4) are adapted to (S, h, E), $E := \mathrm{div}(u_1 u_2)$ (definition 2.6) and $(u_1, u_2, u_3, u_4; Z)$ are well adapted coordinates at the closed point $x = (Z, u_1, u_2, u_3, u_4)$ of $\mathcal{X} = \mathrm{Spec}(S[Z]/(h))$ (definition 2.8). Indeed, it is easily seen that:

$$\mathrm{Sing}_p\mathcal{X} := \{y \in \mathcal{X} : m(y) = p\} = V(Z, u_1, u_2) \cup V(Z, u_1, u_3),\ \omega(x) = p.$$

Let $\vartheta(t) := \sum_{i\geq 1} \lambda_i t^i \in k[[t]]$ be a power series which is *transcendental* over $k(t)$. We define a nonconstant well-parametrized k-linear formal arc on $(\mathcal{X}, x)$ by:

$$\varphi(Z) = \varphi(u_1) = \varphi(u_3) = 0,\ \varphi(u_2) = u_2,\ \varphi(u_4) = \vartheta(t)^p.$$

Let $u_j^{(0)} := u_j$, $1 \leq j \leq 4$. For $r \geq 1$, well adapted coordinates at x_r are $u_j^{(r)} := u_j^{(r-1)}/u_2$, $j = 1, 3$, $u_2^{(r)} := u_2$ and

$$v_4^{(r)} := u_2^{-r}(u_4 - \sum_{ip\leq r} \lambda_i^p u_2^{ip}),\ T_r := u_2^{-r}(Z + (u_3^{(r)})^p \sum_{ip\leq r} \lambda_i^p u_2^{ip}).$$

Then φ lifts through

$$(\mathcal{X}_r, x_r) = (\mathrm{Spec}(S_r[T_r]/(h_r), x_r),\ S_r = S[u_1^{(r)}, u_3^{(r)}, u_4^{(r)}]_{(u_1^{(r)},\ldots,v_4^{(r)})},$$

and the strict transform h_r of h is given by

$$h_r := T_r^p + (u_2^{(r)})^r \left((u_2^{(r)})^p v_4^{(r)} (u_3^{(r)})^p + u_3^{(r)} (u_1^{(r)})^p\right).$$

We have $Z_r := V(T_r, u_1^{(r)}, u_3^{(r)})$ for every $r \geq 1$. Note that Z_r is not permissible at x_r. Therefore φ fulfills alternative (2) of proposition 3.8.

Remark 3.3. We do not know if the conclusion of proposition 3.8 is still valid for $n \geq 4$ when removing the assumption "$l|k(x_r)$ is algebraic with finite inseparable degree for some $r \geq 0$".

When $n = 3$, it can be proved that the above assumption is actually implied by "$m(\varphi) = p$ and $\omega(\varphi) > 0$". This is a (very) special case of the proof of theorem 5.1. The following elementary corollary will be used repeatedly.

Corollary 3.9. *Assume that $n = 3$. Let (S, h, E) be as before and $x \in \mathcal{X}$. Let*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \tag{3.73}$$

be a (possibly infinite) composition of local blowing ups at closed points with ($m(x_r) = p$, $\omega(x_r) > 0$ and $k(x_r) = k(x)$) for every $r \geq 0$. With notations as in proposition 2.7 and notation 2.2, assume that (S_r, E_r, h_r) is such that E_r is irreducible for every $r \geq 0$. Then (3.73) is finite.

Proof. Let $E = \mathrm{div}(u_1)$ and $(u_1, u_2^{(0)}, u_3^{(0)}; Z^{(0)})$ be well adapted coordinates at x. Since $k(x_r) = k(x)$ and E_r is irreducible for every $r \geq 1$, S_r has well adapted coordinates

$$(u_1, u_2^{(r)} := u_2^{(r-1)}/u_1 - \gamma_2^{(r)}, u_3^{(r)} := u_3^{(r-1)}/u_1 - \gamma_3^{(r)}; Z^{(r)} := Z^{(r-1)}/u_1 - \phi^{(r)})$$

where $\gamma_2^{(r)}, \gamma_3^{(r)}, \phi^{(r)} \in S$. Suppose that (3.73) is infinite. We let

$$\hat{u}_j := u_2 - \sum_{r \geq 1} \gamma_j^{(r)} u_1^{(r)} \in \hat{S},\ j = 2, 3, \text{ and } \hat{Z} := Z - \hat{\phi},\ \hat{\phi} := \sum_{r \geq 1} \phi^{(r)} u_1^{(r)} \in \hat{S}.$$

The induced morphism

$$\varphi :\ \mathrm{Spec}(\hat{S}[Z]/(\hat{u}_2, \hat{u}_3, \hat{Z})) \longrightarrow (\mathcal{X}, x)$$

is a nonconstant well parametrized formal arc on $(\mathcal{X}, x)$ with $l = k(x)$ and whose associated quadratic sequence is (3.73). By proposition 3.8, $Z_r(\varphi)$ is Hironaka-permissible for some $r \geq 0$: a contradiction with **(E)**, since $Z_r(\varphi) \nsubseteq E_r$. □

The following lemma is elementary and well-known.

Lemma 3.10. *Let S be a regular local ring (not necessarily excellent) of dimension $n \geq 1$ with r.s.p. $(u_1, \ldots, u_n)$ and*

$$C := V(u_1, \ldots, u_{n-1}) \subset (\mathcal{S}_0, s_0) := \mathrm{Spec} S$$

be a regular curve. Let

$$(\mathcal{S}_0, s_0) \leftarrow (\mathcal{S}_1, s_1) \leftarrow \cdots \leftarrow (\mathcal{S}_i, s_i) \leftarrow \cdots$$

be the composition of local blowing ups such that $\mathcal{S}_i$ is the blowing up of $\mathcal{S}_{i-1}$ along s_{i-1} and $s_i \in \mathcal{S}_i$ is the point on the strict transform C_i of C for $i \geq 1$.

Let $f \in S$, $f \neq 0$ and denote $d := \mathrm{ord}_C f$. There exists $m, i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, there is a decomposition

$$f = u_n^{m+di} g_i,\ g_i \in S_i := \mathcal{O}_{\mathcal{S}_i, s_i} \text{ and } \mathrm{ord}_{C_i} g_i = \mathrm{ord}_{s_i} g_i = d.$$

Furthermore, the initial form $\mathrm{in}_{C_i} g_i \in (\mathrm{gr}_{I_{C_i}} S_i)_d$ is the strict transform of

$$\mathrm{in}_C f \in (\mathrm{gr}_{I_C} S)_d \simeq S/(u_1, \ldots, u_{n-1})[U_1, \ldots, U_{n-1}]_d.$$

Proof. We have $S_i = S_{i-1}[u_1^{(i)}, \ldots, u_{n-1}^{(i)}]_{(u_1^{(i)}, \ldots, u_n^{(i)})}$, where $u_j^{(i)} := u_j^{(i-1)}/u_n^{(i-1)}$, $1 \leq j \leq n-1$, $u_n^{(i)} := u_n^{(i-1)}$ for every $i \geq 1$, with $u_j^{(0)} := u_j$, $1 \leq j \leq n$. Then $C_i = V(u_1^{(i)}, \ldots, u_{n-1}^{(i)})$ with these notations. There is an expansion

$$f = (u_n^{(i-1)})^{m_{i-1}} g_{i-1},\ g_{i-1} := \sum_{\mathbf{x} \in \mathbf{S}} \gamma(\mathbf{x})^{(i-1)} (u_1^{(i-1)})^{x_1} \cdots (u_n^{(i-1)})^{x_n} \in S_{i-1},$$

where $\gamma(\mathbf{x})^{(i-1)} \in S_{i-1}$ is a unit for each $\mathbf{x} \in \mathbf{S}$, $\mathbf{S} \subset \mathbb{N}^n$ a finite set, $m_{i-1} \in \mathbb{N}$, $g_{i-1} \notin (u_n^{(i-1)})$. Since $\mathrm{ord}_C f = d$, it can be assumed without loss of generality that

$$d = \min_{\mathbf{x} \in \mathbf{S}} \{x_1 + \cdots + x_{n-1}\}.$$

Therefore

$$d = \mathrm{ord}_{C_{i-1}} g_{i-1} \leq d_{i-1} := \mathrm{ord}_{s_{i-1}} g_{i-1} = \min_{\mathbf{x} \in \mathbf{S}} \{| \mathbf{x} |\}.$$

Note that the initial form $\mathrm{in}_{C_{i-1}} f$ is given by

$$\mathrm{in}_{C_{i-1}} f = \sum_{x_1 + \cdots + x_{n-1} = d} \overline{\gamma}(\mathbf{x})^{(i-1)} (\overline{u}_n^{(i-1)})^{x_n} (U_1^{(i-1)})^{x_1} \cdots (U_{n-1}^{(i-1)})^{x_{n-1}},$$

where $\overline{\gamma}(\mathbf{x})^{(i-1)}, \overline{u}_n^{(i-1)} \in S_{i-1}/(u_1^{(i-1)}, \ldots, u_{n-1}^{(i-1)})$ denote the classes of the corresponding elements in S_{i-1}. After blowing up, we get an expansion

$$f = (u_n^{(i)})^{m_{i-1} + d_{i-1}} g_i,\ g_i := \sum_{\mathbf{x} \in \mathbf{S}} \gamma(\mathbf{x})^{(i-1)} (u_1^{(i)})^{x_1} \cdots (u_{n-1}^{(i)})^{x_{n-1}} (u_n^{(i)})^{|\mathbf{x}| - d_{i-1}} \in S_i.$$

Let $A_{i-1} := \{\mathbf{x} \in \mathbf{S} : x_1 + \cdots + x_{n-1} < d_{i-1}\}$. For each $\mathbf{x} \in A_{i-1}$, we have $| \mathbf{x} | - d_{i-1} < x_n$. We deduce:

$$0 \leq \min_{\mathbf{x} \in A_i} \{x_n\} < \min_{\mathbf{x} \in A_{i-1}} \{x_n\}.$$

This proves that there exists $i_0 \geq 0$ such that $A_i = \emptyset$ for every $i \geq i_0$. Then $d_i = d$ for $i \geq i_0$. This proves the first statement in the lemma, taking $m := m_{i_0} - di_0 \geq 0$. Finally, this construction preserves the initial form $\mathrm{in}_C f$, i.e.

$$\mathrm{in}_{C_i} f = \overline{u}_n^{-(m+di)} (\mathrm{in}_C f) \left(\overline{u}_n^i U_1^{(i)}, \ldots, \overline{u}_n^i U_n^{(i)} \right),$$

and this concludes the proof. □

Theorem 3.11. *Let $\mathcal{Y} \subset (\mathcal{X}, x)$ be an integral closed subscheme with generic point y. The set*

$$\Omega(\mathcal{Y}) := \{y' \in \mathcal{Y} : (m(y'), \omega(y'), \kappa(y')) = (m(y), \omega(y), \kappa(y))\} \subseteq \mathcal{Y}$$

contains a nonempty Zariski open subset of $\mathcal{Y}$.

Let furthermore $\mathcal{Z} \supset \mathcal{Y}$ be an integral closed subscheme with generic point z such that $\mathcal{Z}$ is permissible (of the first or second kind) at y. The set

$$\mathrm{Perm}(\mathcal{Y}, \mathcal{Z}) := \{y' \in \mathcal{Y} : \mathcal{Z} \text{ is permissible at } y'\} \subseteq \mathcal{Y}$$

contains a nonempty Zariski open subset of $\mathcal{Y}$.

Proof. Our function (m, ω, κ) refines the multiplicity function m on $\mathcal{X}$, and our notion of permissible blowing up refines the Hironaka-permissibility. We may thus apply the well known openness of these properties. It is therefore sufficient to prove the first statement when $m(y) = p$. For the second statement, we take a nonempty Zariski open set $\mathcal{U}_1 \subseteq \mathcal{Y}$ such that $\mathcal{Z}$ is Hironaka permissible at every $y' \in \mathcal{U}_1$.

Let $W := \eta(\mathcal{Y})$, $s := \eta(y)$, $W_{\mathcal{Z}} := \eta(\mathcal{Z})$ for the second statement. We pick an adapted r.s.p. $(u_1, \ldots, u_{n_s})$ of S_s, where $E_s = \mathrm{div}(u_1 \cdots u_{e_s})$. For every $y' \in \mathcal{U}_1$ there exists an adapted r.s.p. $(u_1, \ldots, u_{n_{y'}})$ of $S_{\eta(y')}$ (i.e. $E_{\eta(y')} = \mathrm{div}(u_1 \cdots u_{e_{y'}})$, $e_{y'} \geq e_s$) such that S_s is the localization of $S_{\eta(y')}$ at some prime

$$I(W_{y'}) = (\{u_j\}_{j \in J_{y'}}), \ J_{y'} \subseteq \{1, \ldots, n_{y'}\}.$$

After possibly shrinking $\mathcal{U}_1 \subseteq \mathcal{Y}$, it can be assumed without loss of generality that $e_{y'} = e_s$ for every $y' \in \mathcal{U}_1$.

We now *choose* any point $y_0 \in \mathcal{U}_1$. Let $(u_1, \ldots, u_{n_0}; Z)$ be well adapted coordinates at y_0, $s_0 := \eta(y_0)$, $S_0 := S_{s_0}$. There is a corresponding expansion

$$h = Z^p + f_{1,Z} Z^{p-1} + \cdots + f_{p,Z} \in S_0[Z], \ f_{1,Z}, \ldots, f_{p,Z} \in S_0.$$

After possibly restricting again $\mathcal{U}_1$, we may assume that the rational functions $u_1, \ldots, u_{n_0}, f_{1,Z}, \ldots, f_{p,Z}$ are regular at $\eta(y')$ for every $y' \in \mathcal{U}_1$. Moreover, we have in $S_{\eta(y')}$

$$I(W) = (\{u_j\}_{j\in J}) \text{ (and } I(W_{\mathcal{Z}}) = (\{u_j\}_{j\in J_{\mathcal{Z}}}) \text{ for the second statement)}$$

with $J_{\mathcal{Z}} \subseteq J = \{1, \ldots, n\}$, $n_{y'} \geq n$, subsets which do not depend on y'. We fix an associated expansion at s_0:

$$f_{i,Z} = \sum_{\mathbf{x}\in\mathbf{S}_i} \gamma(i,\mathbf{x}) u_1^{ix_1} \cdots u_{n_0}^{ix_{n_0}} \in S_0,\ 1 \leq i \leq p,$$

with $\mathbf{S}_i \subset (\frac{1}{i}\mathbb{N})^{n_0}$ finite and $\gamma(i,\mathbf{x}) \in S_0$ a unit for each $\mathbf{x} \in \mathbf{S}_i$. After possibly restricting again $\mathcal{U}_1$, it may also be assumed that each $\gamma(i,\mathbf{x})$ appearing in some $f_{i,Z}$, $1 \leq i \leq p$, is a regular function at $\eta(y')$. By proposition 2.4, the polyhedra

$$\Delta_{S_0}(h; \{u_j\}_{j\in J}; Z) \text{ (and } \Delta_{S_0}(h; \{u_j\}_{j\in J_{\mathcal{Z}}}; Z)) \text{ are minimal.} \tag{3.74}$$

We define $A_i \subset (\frac{1}{i}\mathbb{N})^J$ (and $A_{i,\mathcal{Z}} \subset (\frac{1}{i}\mathbb{N})^{J_{\mathcal{Z}}}$ for the second statement) to be the respective images of $\mathbf{S}_i$ by the projections $\mathrm{pr}_J : \mathbb{R}^{n_0} \to \mathbb{R}^J$ and $\mathrm{pr}_{J_{\mathcal{Z}}} : \mathbb{R}^{n_0} \to \mathbb{R}^{J_{\mathcal{Z}}}$. Given $\mathbf{a} \in A_i$, we let:

$$\gamma(i,\mathbf{a}) := \sum_{\mathrm{pr}_J(\mathbf{x})=\mathbf{a}} \gamma(i,\mathbf{x}) \prod_{j\notin J} u_j^{ix_j} \in S_0.$$

By definition of $\epsilon(y)$, we have:

$$\epsilon(y) = p \min_{1\leq i\leq p} \min_{\mathbf{a}\in A_i} \{|\ \mathbf{a}\ |: \gamma(i,\mathbf{a}) \neq 0\} - \sum_{j=1}^{e_s} H_j. \tag{3.75}$$

Let $B \subset \mathbb{Q}^n$ be the set of $(i,\mathbf{a})$ achieving equality on the right hand side of (3.75). The initial form polynomial $\mathrm{in}_{m_{S_s}} h$ is thus of the form

$$\mathrm{in}_{m_{S_s}} h = Z^p + \sum_{(i,\mathbf{a})\in B} \overline{\gamma}(i,\mathbf{a}) \prod_{j\in J} U_j^{ia_j} Z^{p-i} \in G(m_{S_s})[Z], \tag{3.76}$$

where $\overline{\gamma}(i,\mathbf{a})$ denotes the image in $k(y)$. Let

$$B_0 := \{(i,\mathbf{a}) \in B : \exists (i,\mathbf{a}) \in B, i \neq p \text{ or } (i = p \text{ and } \mathbf{a} \notin \mathbb{N}^J)\}.$$

Case 1. Suppose that $B_0 \neq \emptyset$. We define:

$$\mathcal{U} := \{y' \in \mathcal{U}_1 : \forall (i, \mathbf{a}) \in B_0, \overline{\gamma}(i, \mathbf{a}) \text{ is a unit in } S_{\eta(y')}\}.$$

Since $\gamma(i, \mathbf{a})$ is nonzero for $(i, \mathbf{a}) \in B$ by (3.75), $\mathcal{U}$ is a nonempty Zariski open subset of $\mathcal{Y}$. To $y' \in \mathcal{U}$, we associate $\mathbf{x} \in \Delta_{S_{\eta(y')}}(h; u_1, \ldots, u_{n_{y'}}; Z)$ (depending on $(i, \mathbf{a})$) by

$$\begin{cases} x_j &= a_j \quad \text{if} \quad j \in J \\ x_j &= 0 \quad \text{if} \quad j \notin J \end{cases}$$

Computing initial forms from definition 2.2 with $\alpha_{y'} := (1, \ldots, 1) \in \mathbb{R}^{n_{y'}}$, $\delta_{\alpha_{y'}}(h; u_1, \ldots, u_{n_{y'}}; Z) = \delta(y)$, the corresponding initial form polynomial

$$\mathrm{in}_{\alpha_{y'}} h = Z^p + \sum_{i=1}^{p} F_{i,Z,\alpha_{y'}} Z^{p-i} \in G(m_{S_{\eta(y')}})[Z] \tag{3.77}$$

is such that $F_{i,Z,\alpha_{y'}} \neq 0$ for some $i \neq p$ or $F_{p,Z,\alpha_{y'}} \notin k(y')[U_1^p, \ldots, U_{n_{y'}}^p]$. Therefore $\delta(y') = \delta(y)$ and we deduce that

$$\epsilon(y') = \epsilon(y) \text{ for every } y' \in \mathcal{U}. \tag{3.78}$$

To prove the first statement, note that we are already done by (3.78) if $\epsilon(y) = 0$. Assume now that $\epsilon(y) > 0$. If $i_0(y) = p - 1$, there exists some $(p - 1, \mathbf{a}_0) \in B_0$ for some $\mathbf{a}_0 \in \mathbb{N}^J$. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. The corresponding initial form polynomial

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z_{y'}^p - G_{y'}^{p-1} Z_{y'} + F_{p,Z_{y'}} \in G(m_{S_{\eta(y')}})[Z_{y'}]$$

is such that $< G_{y'} >=< U^{\mathbf{a}_0} >$ (resp. $G_{y'} = 0$) if $i_0(y) = p - 1$ (resp. if $i_0(y) = p$). We have

$$F_{p,Z_{y'}} = \sum_{(p,\mathbf{a}) \in B_0} \lambda_{y'}(p, \mathbf{a}) U^{\mathbf{a}} + \Psi_{y'} \subseteq G(m_{S_{\eta(y')}})_{\epsilon(y)},$$

where $\lambda_{y'}(i, \mathbf{a}) \in k(y')$, $\lambda_{y'}(i, \mathbf{a}) \neq 0$, $\Psi_{y'} \in k(y')[\{U_j^p\}_{j \in J}]$ for every $(p, \mathbf{a}) \in B_0$ and every $y' \in \mathcal{U}$. Comparing with definition 2.16, we have $\omega(y') = \omega(y)$, $\kappa(y') = 1$ if $\kappa(y) = 1$ for $y' \in \mathcal{U}$. This proves the first statement in case 1.

For the second statement, we are also done if $\epsilon(z) = \epsilon(y)$, i.e. if $\mathcal{Z}$ is of the first kind at y. Suppose that $\mathcal{Z}$ is permissible of the second kind at y. In particular, we have $\epsilon(y) > 0$. There exist $j_1(y) \in J \backslash J_{\mathcal{Z}}$ and $j'(y) \in J \backslash J_{\mathcal{Z}}$, $j'(y) \geq e_s + 1$, satisfying the conclusion of proposition 3.3. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. The corresponding initial form polynomial (3.78) again satisfies

$$H_{y'}^{-1} G_{y'}^{p} \subseteq U_{j_1(y)} k(y')[U_1, \ldots, U_{n_{y'}}]_{\epsilon(y)}$$

and there is an expansion

$$H_{y'}^{-1} F_{p, Z_{y'}} =< \sum_{j' \in J'} U_{j'} \Phi_{j'}(\{U_j\}_{j \in J}) + \Psi(\{U_j\}_{j \in J}) > \subseteq G(m_{S_{\eta(y')}})_{\epsilon(y)}$$

with $\Phi_{j'(y_0)} \neq 0$, hence $\mathcal{Y}$ is permissible of the second kind at y' and the conclusion follows.

Case 2. Suppose on the contrary that $B_0 = \emptyset$. By (3.76), we have

$$\mathrm{in}_{m_{S_s}} h = Z^p + \sum_{(p, \mathbf{a}) \in B} \overline{\gamma}(p, \mathbf{a}) \prod_{j \in J} U_j^{p a_j} \in G(m_{S_s})[Z] \tag{3.79}$$

and this proves that

$$\delta(y) \in \mathbb{N}, \ \omega(y) = \epsilon(y) \text{ and } \kappa(y) \geq 2. \tag{3.80}$$

Since $(\{u_j\}_{j \in J}; Z)$ are well adapted coordinates at y, there exists a vertex $\mathbf{a}_0 \in \Delta_{S_s}(h; \{u_j\}_{j \in J}; Z)$, $(p, \mathbf{a}_0) \in B$ which is not solvable, i.e. $\overline{\gamma}(p, \mathbf{a}_0) \notin k(y)^p$. Let $B_1 \subseteq B_0$ be the nonempty subset defined by

$$B_1 := \{(p, \mathbf{a}) \in B : \overline{\gamma}(p, \mathbf{a}) \notin k(y)^p\}.$$

Given $(p, \mathbf{a}) \in B_1$, we define a morphism:

$$\eta_{(p, \mathbf{a})} : \ \mathcal{Y}_{(p, \mathbf{a})} := \mathrm{Spec} \left(\frac{\mathcal{O}_{\mathcal{U}_1}[T]}{(T^p - \overline{\gamma}(p, \mathbf{a}))} \right) \longrightarrow \mathcal{U}_1.$$

Note that $\mathcal{Y}_{(p, \mathbf{a})}$ is integral and $\eta_{(p, \mathbf{a})}$ is finite and purely inseparable. We define:

$$\mathcal{U} := \{y' \in \mathcal{U}_1 : \forall (p, \mathbf{a}) \in B_1, \eta_{(p, \mathbf{a})}^{-1}(y')_{\mathrm{red}} \text{ is a regular point of } \mathcal{Y}_{(p, \mathbf{a})}\}.$$

Since $\mathcal{Y}_{(p,\mathbf{a})}$ is excellent, its regular locus is a nonempty Zariski open set. We deduce that $\mathcal{U}$ is a nonempty Zariski open subset of $\mathcal{Y}$.

For $y' \in \mathcal{U}_1$ and $(p, \mathbf{a}) \in B$, we denote by $\lambda_{y'}(p, \mathbf{a}) \in k(y')$ the residue of $\overline{\gamma}(p, \mathbf{a})$. The property

$$\text{“}\eta^{-1}_{(p,\mathbf{a})}(y')_{\mathrm{red}} \text{ is a regular point of } \mathcal{Y}_{(p,\mathbf{a})}\text{ ”}$$

is equivalently characterized as follows: either (a) $\lambda_{y'}(p, \mathbf{a}) \notin k(y')^p$, or (b) there exists $\delta_{y'}(p, \mathbf{a}) \in \mathcal{O}_{\mathcal{Y},y'}$ such that

$$v_{y'}(p, \mathbf{a}) := \overline{\gamma}(p, \mathbf{a}) - \delta_{y'}(p, \mathbf{a})^p$$

is a regular parameter at y'.

We now prove the first statement. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. Let

$$B(y') := \{(p, \mathbf{a}) \in B_1 : \text{(a) is satisfied}\}.$$

Suppose that $B(y') \neq \emptyset$. We get $\delta(y') = \delta(y)$, $i_0(y') = p$ and the initial form polynomial $\mathrm{in}_{m_{S_{\eta(y')}}} h \in G(m_{S_{\eta(y')}})[Z_{y'}]$ is

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z^p_{y'} + \sum_{(p,\mathbf{a})\in B(y')} \lambda_{y'}(p, \mathbf{a}) U^{\mathbf{a}} + \Psi^p_{y'}$$

where $\lambda_{y'}(p, \mathbf{a}) \notin k(y')^p$ and $\Psi_{y'} \in k(y')[\{U^p_j\}_{j\in J}]$. This shows that

$$\omega(y') = \epsilon(y') = \epsilon(y) = \omega(y),$$

the right hand side equality by (3.80). Moreover $\kappa(y') \geq 2$, so $y' \in \Omega(\mathcal{Y})$.

Suppose on the contrary that $B(y') = \emptyset$. We get

$$\delta(y') = \delta(y) + \frac{1}{p},\ i_0(y') = p \text{ (since } \delta(y') \notin \mathbb{N})$$

and the initial form polynomial $\mathrm{in}_{m_{S_{\eta(y')}}} h \in G(m_{S_{\eta(y')}})[Z_{y'}]$ is

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z^p_{y'} + \sum_{(p,\mathbf{a})\in B_1} V_{y'}(p, \mathbf{a}) U^{\mathbf{a}} + \Psi_{y'},$$

where $V_{y'}(p, \mathbf{a}) \in < U_1, \ldots, U_{n_{y'}} > \setminus < \{U_j\}_{j\in J} >$, $\Psi_{y'} \in k(y')[\{U_j\}_{j\in J}]_{p\delta(y)+1}$. This shows that $\omega(y') = \epsilon(y') - 1 = \epsilon(y) = \omega(y)$, applying again (3.80). Moreover $\kappa(y') \geq 2$, so $y' \in \Omega(\mathcal{Y})$. This concludes the proof of the first statement.

For the second statement, note that $\mathcal{Z}$ is necessarily of the first kind at y in case 2, since (3.79) is not compatible with proposition 3.3. With notations as above, $\mathcal{Z}$ is then permissible of the first kind (resp. of the second kind) at y' if $B(y') \neq \emptyset$ (resp. if $B(y') = \emptyset$). $\square$

Corollary 3.12. *With notations as above, the function*

$$\iota : \mathcal{X} \to \{1, \ldots, p\} \times \mathbb{N} \times \{0, 1, \geq 2\},\ y \mapsto (m(y), \omega(y), \kappa(y))$$

is a constructible function on $\mathcal{X}$. In particular, it takes finitely many distinct values.

Proof. This follows from the previous theorem and Noetherian induction on $\mathcal{X}$. $\square$

Remark 3.4. The constructible sets $\mathcal{X}_{p,a} := \{y \in \mathcal{X} : (m(y), \omega(y)) \geq (p, a)\}$, $a \in \mathbb{N}$ are not in general Zariski closed (example 3.3 below). See next proposition for closedness of the set $\mathcal{X}_{p,1}$.

We do not know if the sets $\mathrm{Perm}(\mathcal{Y}, \mathcal{Z})$ as in the theorem are constructible subsets of $\mathcal{Y}$. An important issue about permissibility is addressed below in question 3.1.

Theorem 3.11 is sufficient for the required applications to Local Uniformization. About a possible extension of our methods to a global Resolution of Singularities statement, we remark the following: let $\mathcal{S}$ be an excellent regular domain,

$$\eta :\ \mathcal{X} \to \mathcal{S}$$

be a finite morphism, $x \in \mathcal{X}$ be such that $(\mathcal{X}, x) \to \mathcal{S}_{\eta(x)}$ satisfies the assumption of theorem 3.11. It is easily seen that its conclusion extends to some affine neighbourhood $\mathcal{U}$ of x on $\mathcal{X}$.

Example 3.3. Let $S = k[[u_1, u_2, u_3]]$, k a (nonperfect) field of characteristic $p > 0$ and $\lambda, \mu \in k$ be p-independent. We take:

$$h = Z^p - (u_1 u_2)^{p-1} Z + \lambda u_3^p + u_3 u_1^{p-1} + \mu u_1^p \in S[Z],\ E = \mathrm{div}(u_1 u_2).$$

The coordinates $(u_1, u_2, u_3; Z)$ are well adapted to (S, h, E). Let

$$x := (Z, u_1, u_2, u_3),\ y := (Z, u_1, u_3).$$

We have $H(x) = (1)$, $m(x) = m(y) = p$, and compute:

$$\mathrm{in}_{m_S} h = Z^p + \lambda U_3^p + U_3 U_1^{p-1} + \mu U_1^p,\ i_0(x) = p,\ \omega(x) = \epsilon(x) - 1 = p - 1.$$

On the other hand, we have:

$$\mathrm{in}_{m_{S_{\eta(y)}}} h = Z^p - (U_1\overline{u}_2)^{p-1}Z + \lambda U_3^p + U_3U_1^{p-1} + \mu U_1^p,\ i_0(y) = p-1,\ \epsilon(y) = p.$$

In order to compute $\omega(y)$, we must introduce a truncation operator

$$T_y : k(y)[U_1, U_3]_p \to k(y)[U_1, U_3]_p$$

as in definition 2.16 and get $T_yF_{p,Z,y} = \lambda U_3^p$, so $\omega(y) = p > \omega(x)$. This proves that the set $\mathcal{X}_{(p,p)} := \{z \in \mathcal{X} : (m(z), \omega(z)) \geq (p,p)\}$ is *not* Zariski closed.

Proposition 3.13. *Let $(\mathcal{X}, x)$ be as in the theorem. The set*

$$\Omega_+(\mathcal{X}) := \{y \in \mathcal{X} : (m(y), \omega(y)) > (p, 0)\} \subseteq \mathcal{X}$$

is Zariski closed and of dimension at most $n-2$.

Proof. Let $\xi \in \mathcal{X}$ be the generic point of an irreducible component of $\eta^{-1}(E)$. Then $(m(\xi), \epsilon(\xi)) \leq (p, 0)$, so $\xi \notin \Omega_+(\mathcal{X})$. Therefore it is sufficient to prove that $\Omega_+(\mathcal{X})$ is Zariski closed.

We will use the Nagata Criterion to prove openness of $\mathcal{X}\backslash\Omega_+(\mathcal{X})$. By theorem 3.11, it is sufficient to prove that $\Omega_+(\mathcal{X})$ is stable by specialization. Let $y_0 \rightsquigarrow y_1$ be a specialization in $\mathcal{X}$ and assume that $y_1 \notin \Omega_+(\mathcal{X})$. We must prove that $y_0 \notin \Omega_+(\mathcal{X})$, so we are reduced to the case $m(y_0) = p$. Let $\mathcal{Y}_0 := \overline{\{y_0\}}$.

By localizing η at $\eta(y_1)$, it can be furthermore assumed that $y_1 = x$. Arguing by induction on the dimension of $\mathcal{Y}_0$, it can be furthermore assumed that $\mathcal{Y}_0$ is a curve. Let

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots$$

be a sequence of local blowing ups at closed points belonging to the strict transform of $\mathcal{Y}_0$. We have $m(x_r) \geq m(y_0) = p$, so $m(x_r) = p$ for every $r \geq 0$. Since S is excellent, the strict transform of $\mathcal{Y}_0$ in $\mathcal{X}_r$ is Hironaka permissible for $r >> 0$. By construction, these maps induce local isomorphisms at y_0.

We then have $(m(x_r), \omega(x_r)) \leq (p, 0)$ by proposition 2.22, hence $\omega(x_r) = 0$ since $m(x_r) = p$ for every $r \geq 0$. In other words, after possibly replacing $(\mathcal{X}, x)$ by $(\mathcal{X}_r, x_r)$ for some $r \geq 0$, it can be assumed that $\mathcal{Y}_0$ is Hironaka permissible. Then there exist well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that

$$I(W_0) = (\{u_j\}_{j\in J_0}), W_0 := \eta(\mathcal{Y}_0)$$

with $J_0 = \{1, \ldots, n\} \backslash \{j'\}$ for some j' (since $\mathcal{Y}_0$ is a curve). We let $s_0 := \eta(y_0)$, $S_0 := S_{s_0}$. By proposition 2.4, the polyhedron $\Delta_S(h; \{u_j\}_{j \in J}; Z)$ is minimal, so we deduce that $\epsilon(y_0) \leq \epsilon(x)$.

Since $\omega(x) = 0$ by assumption, we have $\omega(y_0) = 0$ except possibly if $\epsilon(y_0) = \epsilon(x) = 1$. Since $\omega(x) = 0$, the initial form polynomial $\mathrm{in}_{W_0} h \in G(m_S)[Z]$ then satisfies

$$H_{W_0}^{-1} F_{p,Z,W_0} =< \sum_{j \in J_0} \gamma_j U_j > \subseteq G(W_0)_1 = S/I(W_0)[\{U_j\}_{j \in J_0}],$$

and there exists $j_0 \in J_0$, $e + 1 \leq j_0 \leq n$ such that γ_{j_0} is a unit in $S/I(W_0)$. This gives $\omega(y_0) = 0$ if $i_0(y) = p$. If $i_0(y) = p - 1$, we must introduce a truncation operator

$$T_0 : \ G(m_{S_0})_{p\delta(y_0)} \to G(m_{S_0})_{p\delta(y_0)},$$

as in definition 2.16 in order to compute $\omega(y_0)$. However, T_0 proceeds from definition 2.14 in the special case $p\delta(y_0) = 1 + \sum_{j \in J_0} H_j$. Lemma 2.17 then implies that

$$H_{W_0}^{-1} \mathrm{Ker} T_0 \subseteq < \{U_j\}_{j \in J_0, j \leq e} > \subset G(m_{S_0})_{p\delta(y_0)}.$$

Since $j_0 \geq e+1$, we thus have $H_{W_0} U_{j_0} \nsubseteq \mathrm{Ker} T_0$ and this proves that $\omega(y_0) = 0$ as required. $\square$

A very special case of the following question (for μ a discrete valuation with some extra assumption) has been answered in the affirmative in theorem 3.8 above. See also theorem 6.1 for a related result.

Question 3.1. Let $\mathcal{Y} = \mathcal{Y}_0$ be an integral closed subscheme with generic point y, $m(y) = p$, $\omega(y) > 0$, and let μ be a valuation centered at m_S. Does there exist a finite sequence of permissible local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r)$$

with centers $\mathcal{Z}_i \subset (\mathcal{Y}_i, x_i)$, $\mathcal{Y}_i$ denoting the strict transform of $\mathcal{Y}$ in $(\mathcal{X}_i, x_i)$, $0 \leq i \leq r$, such that $\mathcal{Y}_r$ is permissible at x_r?

4 Application to Resolution in dimension three.

In this chapter, we deduce theorem 1.1 from theorem 1.4 and prove the corollaries. Achieving condition **(E)** allows us to use all results from the previous chapters.

We assume that $\dim S = 3$ *from section 4.3 on.*

All results are extensions of [26]. The proofs are based on the following three characteristic free results which can be found respectively in [2] theorem 3, a special case of [25] theorem 0.3 (with $B = \emptyset$) and [26] proposition 4.2:

Proposition 4.1. ***(Abhyankar)*** *Let (R, m) and (R', m') be regular two-dimensional local domains with a common quotient field and such that*

$$R \subseteq R', \ m' \cap R = m.$$

Then R' is an iterated quadratic transform of R.

Proposition 4.2. ***(Cossart-Jannsen-Saito)*** *Let $\mathcal{S}$ be a regular Noetherian irreducible scheme of dimension three which is excellent and $X \hookrightarrow \mathcal{S}$ be a reduced subscheme.*

There exists a composition of blowing ups along integral regular subschemes $\sigma : \mathcal{S}' \to \mathcal{S}$ such that the strict transform $X' \hookrightarrow \mathcal{S}'$ of X has strict normal crossings with the reduced exceptional divisor E of σ. Moreover σ restricts to an isomorphism

$$\pi : \ X' \backslash \sigma^{-1}(\mathrm{Sing} X) \simeq X \backslash \mathrm{Sing} X.$$

Proposition 4.3. ***(Cossart-Piltant)*** *Let $\mathcal{S}$ be a regular Noetherian irreducible scheme of dimension three which is excellent and $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{S}}$ be a nonzero ideal sheaf. There exists a finite sequence*

$$\mathcal{S} =: \mathcal{S}(0) \leftarrow \mathcal{S}(1) \leftarrow \cdots \leftarrow \mathcal{S}(r)$$

with the following properties:

(i) for each j, $0 \leq j \leq r-1$, $\mathcal{S}(j+1)$ is the blowing up along a regular integral subscheme $\mathcal{Y}(j) \subset \mathcal{S}(j)$ with

$$\mathcal{Y}(j) \subseteq \{s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{\mathcal{S}(j),s_j} \text{ is not locally principal}\}.$$

(ii) $\mathcal{I}\mathcal{O}_{\mathcal{S}(r)}$ *is locally principal.*

Proof. The assumption "X/k is quasi-projective" is not used in the proof of [26] proposition 4.2. The equicharacteristic assumption is used only via the power series expansions used for defining E and the characteristic polygon "$\Delta(\mathcal{E}; u_1, u_2; y)$ prepared" on pp.1061-1062 of *ibid.*. But this is also characteristic free by [28] theorem II.3. □

4.1 Reduction to local uniformization and proof of the corollaries.

We now reduce theorem 1.1 to its local uniformization form (LU) below. Let (A, m, k) be a quasi-excellent local domain with quotient field K. Recall that quasi-excellent rings are Noetherian by definition [37] (7.8.2) and remark (7.8.4)(i). We consider the following Local Uniformization problem:

(LU) for every valuation v of K, with valuation ring $(\mathcal{O}_v, m_v, k_v)$ such that

$$A \subset \mathcal{O}_v \subset K,\ m_v \cap A = m,\ k_v|k \text{ algebraic},$$

there exists a finitely generated A-algebra T, $A \subseteq T \subseteq \mathcal{O}_v$, such that T_P is regular, where $P := m_v \cap T$.

Proposition 4.4. *Let* $\mathcal{X}$ *be a reduced and separated Noetherian scheme which is quasi-excellent and of dimension at most three. Let* $\mathcal{X}_1, \ldots, \mathcal{X}_c$ *be the irreducible components of* $\mathcal{X}$*. Assume that (LU) holds for every local ring of the form* $A = \mathcal{O}_{\mathcal{X}_i, x_i}$ *which is of dimension three,* $1 \leq i \leq c$*. Then theorem 1.1 holds.*

Proof. This is an enhanced version of Zariski's Patching Theorem [69] Fundamental theorem on p.539. Suppose that (i) and (ii) in theorem 1.1 have been proved. Apply proposition 4.2 to

$$X := \pi^{-1}(\mathrm{Sing}\mathcal{X})_{\mathrm{red}} \subseteq \mathcal{X}',$$

then blow up along X': we get (iii). There remains to prove (i) and (ii).

Step 1: it can be assumed that $\mathcal{X}$ is irreducible of dimension three.

There is a finite birational morphism

$$f : \coprod_{i=1}^{c} \mathcal{X}_i \to \mathcal{X},$$

isomorphic above $\mathrm{Reg}\mathcal{X}$. The theorem holds for $\mathcal{X}$ if it holds for each $\mathcal{X}_i$. Resolution of singularities is known if $\dim\mathcal{X} \leq 2$ [52], so we may assume that $\dim\mathcal{X} = 3$.

Step 2: it can be assumed that $\mathcal{X} = \mathrm{Spec}A$ is affine.

This is based on lemma 4.5 below. Consider open sets $\mathcal{U} \subseteq \mathcal{X}$ satisfying (i) and (ii) in theorem 1.1, i.e. there exists $\pi_{\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$ proper and birational, such that

$$\mathrm{Reg}\mathcal{U}' = \mathcal{U}' \text{ and } \pi_{\mathcal{U}}^{-1}(\mathrm{Reg}\mathcal{U}) \simeq \mathrm{Reg}\mathcal{U}. \tag{4.1}$$

We assume furthermore that a finite affine covering $\mathcal{U} = U_1 \cup \cdots \cup U_n$ is given such that

$$\pi_{\mathcal{U}}^{-1}(U_i) \to U_i \text{ is projective.} \tag{4.2}$$

Claim: if two open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy (4.1) and (4.2), so does $\mathcal{U}_1 \cup \mathcal{U}_2$ w.r.t. *the union* of their respective coverings. Since $\mathcal{X}$ is Noetherian, this claim completes reduction step 2.

We now prove the claim. Let $\mathcal{V} := \mathcal{U}_1 \cap \mathcal{U}_2$. Denote by $\pi_i : \mathcal{U}_i' \longrightarrow \mathcal{U}_i$ the given resolutions of singularities satisfying (4.1) and (4.2). Let

$$\mathcal{F}_1 \subseteq \mathcal{U}_1' \cap \pi_1^{-1}(\mathcal{V})$$

be the fundamental locus of the birational map

$$\rho : \mathcal{U}_1' \cap \pi_1^{-1}(\mathcal{V}) \cdots \longrightarrow \mathcal{U}_2' \cap \pi_2^{-1}(\mathcal{V}),$$

and $\overline{\mathcal{F}}_1 \subseteq \mathcal{U}_1'$ be its Zariski closure in $\mathcal{U}_1'$. By (4.1), we have:

$$\pi_1(\overline{\mathcal{F}}_1) \subseteq \mathrm{Sing}\mathcal{U}_1.$$

In particular, we may replace $\mathcal{U}_1'$ by any blow up along a regular center contained in $\overline{\mathcal{F}}_1$. We apply lemma 4.5 below to $\pi_i^{-1}(U_{j_1 j_2}) \to U_{j_1 j_2}$, $i = 1, 2$ for each $U_{j_1 j_2} := U_{j_1} \cap U_{j_2}$ with obvious notations.

When some $\mathcal{Z}_i$ in lemma 4.5 is a curve, it can be assumed that $\mathcal{Z}_i$ is regular away from (the inverse image of) $\mathcal{V}$ by blowing up closed points beforehand. Furthermore the sequences (4.6) for distinct $U_{j_1 j_2}$'s glue together, which follows from the definitions (4.7)-(4.8). We may thus assume that

$$\rho \text{ is a morphism.} \tag{4.3}$$

Let $\mathcal{F}_2 \subseteq \mathcal{U}_2' \cap \pi_2^{-1}(\mathcal{V})$ be the fundamental locus of ρ^{-1} and consider the associated sequence (4.6). We will only perform step 1 in the proof of lemma 4.5.

When $\mathcal{Z}_i$ is a closed point mapping to $\mathcal{V}$, we apply proposition 4.3 beforehand to $\mathcal{I}(\mathcal{Z}_i)\mathcal{O}_{\mathcal{U}_1'}$ in order to preserve (4.3).

When $\mathcal{Z}_i$ is an irreducible curve with generic point ξ_i, whose image in $\mathcal{V}$ has dimension one, the ideal $\mathcal{I}(\mathcal{Z}_i)\mathcal{O}_{\mathcal{U}_1'}$ is invertible above ξ_i by proposition 4.1. Applying proposition 4.3 beforehand to $\mathcal{I}(\mathcal{Z}_i)\mathcal{O}_{\mathcal{U}_1'}$, we also preserve (4.3) while $\mathcal{U}_1'$ is unchanged away from the inverse image of finitely many closed points of $\mathcal{V}$. It can be assumed that $\mathcal{Z}_i$ is regular away from the inverse image of $\mathcal{V}$ by blowing up closed points beforehand as above.

Summing up, it can be assumed that (4.3) holds and that ρ^{-1} is a morphism (hence an isomorphism by (4.3)) away from

$$\pi_2^{-1}(x_1), \ldots, \pi_2^{-1}(x_k),\ x_1, \ldots, x_k \in \mathcal{V} \text{ finitely many closed points.} \tag{4.4}$$

We may then glue $\mathcal{U}_1'$ and $\mathcal{U}_2' \backslash \{\pi_2^{-1}(x_1), \ldots, \pi_2^{-1}(x_k)\}$ along

$$\pi_1^{-1}(\mathcal{V} \backslash \{x_1, \ldots, x_k\}) = \pi_2^{-1}(\mathcal{V} \backslash \{x_1, \ldots, x_k\})$$

to some proper morphism $\pi_{\mathcal{W}} :\ \mathcal{W}' \to \mathcal{W} := \mathcal{U}_1 \cup \mathcal{U}_2$. By construction, $\pi_{\mathcal{W}}$ satisfies (4.1) and (4.2) for each $U_{j_1} \subseteq \mathcal{U}_1$. Let $U_{j_2} \subseteq \mathcal{U}_2$ be fixed, so $\pi_2^{-1}(U_{j_2}) \to U_{j_2}$ is projective. Now $\pi_1^{-1}(U_{j_1 j_2}) \to U_{j_1 j_2}$ is projective for each $U_{j_1} \subseteq \mathcal{U}_1$, so $\pi_{\mathcal{W}}(U_{j_2}) \to U_{j_2}$ projective follows from (4.4). This concludes the proof of the claim, hence of step 2.

Step 3: achieving (i) in theorem 1.1 with π projective for $\mathcal{X} = \mathrm{Spec} A$ affine.

The Riemann-Zariski space of valuations

$$\mathrm{Zar}(\mathcal{X}) := \{v \text{ valuation of } K : A \subseteq \mathcal{O}_v\}$$

is quasi-compact by [72] theorem 40 on p.113 and Noetherianity of A. The assumption on v in (LU) means that v is a closed point of $\mathrm{Zar}(\mathcal{X})$. Regularity is a nonempty open property for any reduced $\mathcal{Y}$ which is of finite type over $\mathcal{X}$ because A is excellent. This applies in particular to any projective closure of $\mathrm{Spec} T$, T as in (LU). Hence theorem 1.1(i) is reduced to the following patching problem: let

$$\mathcal{X}_1 \longrightarrow \mathrm{Spec} A,\ \mathcal{X}_2 \longrightarrow \mathrm{Spec} A$$

be projective birational morphisms. There exists $\mathcal{Y} \longrightarrow \mathrm{Spec} A$ projective birational and morphisms $\pi_i :\ \mathcal{Y} \longrightarrow \mathcal{X}_i$, $i = 1, 2$, such that

$$\pi_1^{-1}(\mathrm{Reg}\mathcal{X}_1) \cup \pi_2^{-1}(\mathrm{Reg}\mathcal{X}_2) \subseteq \mathrm{Reg}\mathcal{Y}.$$

As indicated in [69] on p.539, Zariski's Patching Theorem only requires proposition 4.1 and lemma 4.5 (here in our characteristic free context) in order to deduce step 3 from (LU).

Step 4: achieving (ii). Let $\pi : \mathcal{X}' \to \mathcal{X}$ be as in step 3, i.e. projective birational with $\mathrm{Reg}\mathcal{X}' = \mathrm{Reg}\mathcal{X}$. Let $\mathcal{F} \subseteq \mathcal{X}$ be the fundamental locus of π^{-1}. We define

$$\mathcal{F}_1 := \text{Zariski closure in } \mathcal{X} \text{ of } \mathcal{F} \cap \mathrm{Reg}\mathcal{X}.$$

Note that $\mathcal{F}_1$ has dimension at most one. We only sketch the argument and refer to [22] (see also [59] section 6) for the details. There exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X}' & \xleftarrow{e'} & \mathcal{Y}' \\ \downarrow & & \downarrow \\ \mathcal{X} & \xleftarrow{e} & \mathcal{Y} \end{array} \tag{4.5}$$

such that e (resp. e') is a composition of blowing ups with regular centers mapping to $\mathrm{Sing}\mathcal{X}$ (resp. to $\pi^{-1}(\mathrm{Sing}\mathcal{X})$). Let $\pi' : \mathcal{Y}' \to \mathcal{Y}$ be the resulting morphism. This diagram has the following property: let $\mathcal{G} \subset \mathcal{Y}$ be the fundamental locus of ${\pi'}^{-1}$, and $\mathcal{F}_1' \subseteq \mathcal{G}$ be the strict transform of $\mathcal{F}_1$. Then any connected component of $\mathcal{G}$ containing points of $\mathrm{Sing}\mathcal{Y}$ is disjoint from $\mathcal{F}_1'$ (in particular $\mathcal{F}_1' \subset \mathrm{Reg}\mathcal{Y}$). This is achieved as follows:

(a) by iterating finitely many blowing ups of $\mathcal{X}$ at intersection points of $\mathcal{F}_1$ and $\mathrm{Sing}\mathcal{X}$, then applying proposition 4.3, we first obtain e, e' such that $\mathcal{F}_1' \subset \mathrm{Reg}\mathcal{Y}$.

(b) by applying the techniques of step 2 above those irreducible curves $C \subseteq \mathcal{G}$ only such that

$$C \nsubseteq \mathcal{F}_1',\ C \cap \mathcal{F}_1' \neq \emptyset,$$

then applying proposition 4.3 to get e', we disconnect $\mathcal{F}_1'$ from components of $\mathcal{G}$ containing points of $\mathrm{Sing}\mathcal{Y}$.

By (4.5), there exists $\mathcal{U} \subseteq \mathrm{Reg}\mathcal{Y}$ such that the fundamental locus of ${\pi'}^{-1}(\mathcal{U}) \to \mathcal{U}$ is a *projective* subscheme (of dimension at most one) containing $\mathcal{F}_1'$. We define $\mathcal{Z} \subset \mathcal{Y}' \times_{\mathcal{X}} \mathcal{Y}$ by composing the diagonal embedding

$$\Delta_{\mathcal{Y}'} : \mathcal{Y}' \to \mathcal{Y}' \times_{\mathcal{X}} \mathcal{Y}'$$

with the second projection $1 \times \pi'$ above $\mathcal{Y}' \times_{\mathcal{X}} \mathcal{U}$. Then $\mathcal{Z} \to \mathcal{X}$ has the required properties. $\square$

Lemma 4.5. *Let A be a reduced excellent Noetherian domain of dimension three and*

$$\mathcal{X} \longrightarrow \mathrm{Spec}A,\ \mathcal{Y} \longrightarrow \mathrm{Spec}A$$

be projective birational morphisms. Denote by $\rho:\ \mathcal{Y}\cdots\longrightarrow\mathcal{X}$ the birational correspondence and $\mathcal{F}\subset\mathcal{Y}$ its fundamental locus. There exists a sequence

$$\mathcal{Y} =: \mathcal{Y}_0 \leftarrow \mathcal{Y}_1 \leftarrow \cdots \leftarrow \mathcal{Y}_{r+1} = \mathcal{Y}' \tag{4.6}$$

of blowing ups along regular centers $\mathcal{Z}_i \subseteq \mathcal{Y}_i$ such that

(i) $\mathcal{Z}_i$ is fundamental for $\rho_i:\ \mathcal{Y}_i\cdots\longrightarrow\mathcal{X}$, $0\le i\le r$;

(ii) $\rho\circ\pi$ is a morphism on $\pi^{-1}(\mathcal{F}\cap\mathrm{Reg}\mathcal{Y})$, where $\pi:\ \mathcal{Y}'\to\mathcal{Y}$ is the composed map.

Proof. This lemma rephrases [26] proposition 4.7, using the characteristic free proposition 4.3. We denote by

$$\mathcal{F}^\circ := \mathcal{F}\cap\mathrm{Reg}\mathcal{Y},\ \dim\mathcal{F}^\circ\le 1.$$

Let $\overline{\mathcal{F}}\subseteq\mathcal{F}$ be the Zariski closure of $\mathcal{F}^\circ$ in $\mathcal{Y}$ and $\mathcal{G}\subseteq\overline{\mathcal{F}}$ be its one-dimensional component (possibly $\mathcal{G}=\emptyset$). We construct π as a composition of blowing ups along regular subschemes *mapping* to $\overline{\mathcal{F}}$.

Step 1: let

$$\pi_1:\ \mathcal{Y}_{i_1}\to\mathcal{Y} \tag{4.7}$$

be the minimal composition of blowing ups at closed points such that the strict transform $\mathcal{G}'$ of $\mathcal{G}$ is a disjoint union of regular curves, followed by the blowing up along $\mathcal{G}'$. Let

$$\rho_1:\ \mathcal{Y}_{i_1}\cdots\longrightarrow\mathcal{X}$$

denote the composed map $\rho\circ\pi_1$, $\mathcal{F}_1$ its fundamental locus. We now denote

$$\mathcal{F}_1^\circ := \mathcal{F}_1\cap\pi_1^{-1}(\mathrm{Reg}\mathcal{Y})$$

and $\overline{\mathcal{F}}_1\subseteq\mathcal{F}_1$ its Zariski closure in $\mathcal{Y}_{i_1}$. Let furthermore $\mathcal{G}_1\subseteq\overline{\mathcal{F}}_1$ be the union of its one-dimensional irreducible components *whose image in $\mathcal{Y}$ has dimension one.*

We now iterate this construction. Applying a classical result on quadratic sequences in regular local rings of dimension two (e.g. [72] appendix 5, theorem 3 and (E) on p.391), we construct $\pi_n : \mathcal{Y}_{i_n} \to \mathcal{Y}$ such that $\rho \circ \pi_n$ is a morphism away from

$$\pi_n^{-1}((\mathcal{F} \cap \mathrm{Reg}\mathcal{Y})\backslash\{x_1, \ldots, x_k\}),$$

where $x_1, \ldots, x_k$ are finitely many closed points.

Step 2: let $\mathcal{Z}$ be the closure of the graph of $\rho \circ \pi_n$. Since $\mathcal{X}$ is projective, $\mathcal{Z}$ is isomorphic to the blowing up of $\mathcal{Y}_n$ along a certain ideal sheaf $\mathcal{I}_n \subseteq \mathcal{O}_{\mathcal{Y}_{i_n}}$. Since $\pi_n^{-1}(\mathrm{Reg}\mathcal{Y}) \subseteq \mathrm{Reg}\mathcal{Y}_n$, there exists $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{Y}_{i_n}}$ with

$$V(\mathcal{I}) \subseteq \pi_n^{-1}(x_1) \cup \ldots \cup \pi_n^{-1}(x_k), \ \mathrm{dim} V(\mathcal{I}) \leq 1, \tag{4.8}$$

such that $\mathcal{Z}$ is isomorphic to the blowing up of $\mathcal{Y}_{i_n}$ along $\mathcal{I}$ above $\pi_n^{-1}(\mathrm{Reg}\mathcal{Y})$. Applying proposition 4.3 to $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{Y}_{i_n}}$ concludes the proof. □

Proof of corollary 1.2: A is excellent by [37](7.8.3)(iii).

Proof of corollary 1.3: let $\mathcal{Y}$ be any projective $\mathcal{O}$-scheme with generic fiber $\mathcal{Y}_F = \Sigma$, e.g. clearing denominators in Σ. By generic flatness [37](6.9.1), there exists $\mathcal{U} \subseteq \mathrm{Spec}\mathcal{O}$ such that $s^{-1}(\mathcal{U})$ is flat over $\mathcal{U}$. Apply theorem 1.1 to the Zariski closure of $s^{-1}(\mathcal{U})$ in $\mathcal{Y}$, where

$$s : \mathcal{Y} \longrightarrow \mathrm{Spec}\mathcal{O}$$

is the structure morphism.

Remark 4.1. Corollary 1.3 can be strengthened in the obvious way: given any proper and flat $\mathcal{O}$-scheme $\mathcal{Y}$ with generic fiber $\mathcal{Y}_F = \Sigma$ and an open set $\mathcal{U} \subseteq \mathrm{Spec}\mathcal{O}$, there exists a proper and flat $\mathcal{O}$-scheme $\mathcal{X}$ isomorphic to $\mathcal{Y}$ above $\mathcal{U}$ and regular away from $\mathcal{U}$.

4.2 Reduction to cyclic coverings.

In this section, we reduce the local uniformization form (LU) of the previous section to theorem 1.4. This reduction is performed in two steps: first to complete local domains, then to cyclic coverings of degree p in residue characteristic $p > 0$. The first step is adapted from the descent methods of [26] proposition 9.1 for (LU) inside the Henselization of finitely generated algebras of dimension three. Descent from complete local rings to Henselian local rings, i.e. algebraization of (LU), is proved in any dimension in [47] proposition 6.2, but this does not imply proposition 4.6 below.

Proposition 4.6. *Assume that (LU) holds for every complete local domain of dimension three. Then theorem 1.1 holds.*

Proof. By proposition 4.4, it is sufficient to prove that (LU) holds for every quasi-excellent local domain A of dimension three. Let v be a valuation of K as in (LU). Denote by

$$\Gamma_v := K^\times/\mathcal{O}_v^\times,\ r := \dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$$

the value group and rational rank of v. To begin with, we may assume that $\dim\mathcal{O}_v = 1$, i.e. $\Gamma_v \subset (\mathbb{R}, \geq)$, applying [57] theorem 1.1 (valid in all dimensions) or using the dimension three techniques in [26] proposition 5.1. Although this reduction may not preserve the property "$k_v|k$ algebraic", we may assume that it does since transcendental residue extensions provide a reduction in $\dim A$ after blowing up.

Since A is local quasi-excellent, its formal completion $\hat{A}$ w.r.t. m_A is reduced [37](7.8.3)(vii) and remark (7.8.4)(i), so

$$\hat{K} := \mathrm{Tot}(\hat{A}) = \prod_{i=1}^{c} \hat{K}_i,\ \hat{K}_i = QF(\hat{A}/\hat{P}_i)$$

and the $\hat{P}_i$'s are minimal primes. Let $\hat{v}$ be an extension of v to, say $\hat{K}_1$, after possibly renumbering. Note that $\dim\mathcal{O}_{\hat{v}} \geq 1$ and that inequality is strict in general. We have

$$r \leq d := \dim(\hat{A}/\hat{P}_1).$$

Let $\mathcal{X} := \mathrm{Spec}A$, $\hat{\mathcal{X}} := \mathrm{Spec}\hat{A}$ and $f : \hat{\mathcal{X}} \to \mathcal{X}$ be the completion morphism. By assumption in this proposition and proposition 4.4, theorem 1.1 holds for $\hat{\mathcal{X}}$. Let

$$\hat{\pi} :\ \hat{\mathcal{Y}} \to \hat{\mathcal{X}}$$

be the corresponding resolution of singularities. Let $\hat{y} \in \hat{\mathcal{Y}}$ be the center of $\hat{v}$. Since $k_v|k$ is algebraic and $\hat{A}/\hat{P}_1$ is universally catenary, we have

$$d = \dim\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}.$$

By [37](7.8.3)(v), we have $\mathrm{Sing}\hat{\mathcal{X}} = f^{-1}(\mathrm{Sing}\mathcal{X})$. Therefore there exists $g \in A$, $g \neq 0$ such that $\hat{\pi}$ is an isomorphism above $\hat{\mathcal{X}}_g = \mathrm{Spec}\hat{A}_g$ by theorem 1.1(ii). Let also $f_1, \ldots, f_r \in A$ such that $v(f_1), \ldots, v(f_r)$ are $\mathbb{Q}$-linearly independent in Γ_v and set $h := gf_1 \cdots f_r \in A$. We have:

Lemma 4.7. *With notations as above, it can be assumed that*

$$\sqrt{h\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}} = \sqrt{m_{\hat{A}}\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}} = (\hat{u}_1 \cdots \hat{u}_r), \tag{4.9}$$

where $(\hat{u}_1, \ldots, \hat{u}_d)$ *is a r.s.p. of* $\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}$. *In particular*

$$\hat{v}(\hat{u}_1), \ldots, \hat{v}(\hat{u}_r) \in \Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$$

and these values are $\mathbb{Q}$*-linearly independent.*

Proof. This is [26] proposition 6.2, taking into account proposition 4.2. Note that it is not necessary to assume here that $\dim\mathcal{O}_{\hat{v}} = 1$ because $h \in A$. □

We now conclude the proof which is easily adapted from [26] proposition 9.1. By elementary linear algebra, there exists an $r \times r$ matrix $M \in \mathcal{M}(r, \mathbb{Z})$, $a = \det M > 0$ such that

$$g_j := \prod_{i=1}^{r} f_i^{m_{ij}} = \hat{\delta}_j \hat{u}_j^a \in \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}} \cap K, \tag{4.10}$$

where $\hat{\delta}_j \in \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}$ is a unit, $1 \leq j \leq r$. Let

$$\hat{Q}_j := (\hat{u}_j) \cap \hat{A},\ r+1 \leq j \leq d.$$

By construction (4.9), we have $\mathcal{O}_{\hat{\mathcal{Y}},\hat{u}_j} = \hat{A}_{\hat{Q}_j}$, so $(\hat{u}_j)$ is the strict transform of $\hat{Q}_j$ at $\hat{y}$. Since A is dense in $\hat{A}$ for the m_A-adic topology, the right-hand side equality in (4.9) implies: there exists $g'_{r+1}, \ldots, g'_d \in A$ and positive integers m_{ij}, $1 \leq i \leq r$, $r+1 \leq j \leq d$, such that:

$$u'_j := g'_j \prod_{i=1}^{r} \hat{u}_i^{-m_{ij}} \in \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}$$

and $(\hat{u}_1, \ldots, \hat{u}_r, u'_{r+1}, \ldots, u'_d)$ is a r.s.p. of $\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}$. Let now

$$g_j := {g'_j}^a \prod_{i=1}^{r} g_i^{-m_{ij}} = {u'_j}^a \prod_{i=1}^{r} \hat{\delta}_j^{-m_{ij}} \in \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}} \cap K \tag{4.11}$$

and T be the integral closure of $A[g_1, \ldots, g_d]$ in K. By [37] corollary 7.7.3, T is a finitely generated A-algebra. Furthermore, we have

$$A \subseteq T \subseteq \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}} \cap K \subset \mathcal{O}_{\hat{v}} \cap K = \mathcal{O}_v$$

by (4.10)-(4.11). To complete the proof, it must be proved that T_P is regular, where $P := m_v \cap T$. By [37] lemma 7.9.3.1, it is sufficient to prove that $T' := T \otimes_A \hat{A}$ is regular at the center $P' := m_{\hat{y}} \cap T'$ of $\hat{v}$. Since T_P is normal, $T'_{P'}$ is also normal *ibid.* and (7.8.3)(v). There are inclusions

$$\hat{A} \subset T'_{P'} \subseteq \mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}.$$

By (4.10)-(4.11), the right-hand side inclusion satisfies

$$\sqrt{P'\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}}} = m_{\hat{y}},$$

so $\mathcal{O}_{\hat{\mathcal{Y}},\hat{y}} = T'_{P'}$ and the proof is complete. □

Proposition 4.8. *Theorem 1.4 implies theorem 1.1.*

Proof. By proposition 4.6, it is sufficient to prove that (LU) holds for every complete local domain (A, m, k) of dimension three. Let $(\mathcal{O}_v, m_v, k_v)$ be the given valuation ring as in (LU). We may assume here that $\mathrm{char} k_v = p > 0$, the equicharacteristic zero version of theorem 1.1 being known. As in proposition 4.6, it is sufficient to deal with the case $\dim \mathcal{O}_v = 1$.

By Noether normalization [54] theorem 29.4(iii), there exists a complete regular local domain $S \subseteq A$ such that A is a finite S-module, $\dim S = 3$. We will prove that the equal characteristic techniques of [26] extend to our situation. Let F be the quotient field of S, so the field extension $K|F$ is finite algebraic. By [37] corollary 7.7.3, the integral closure of A in any finite extension of F is a finite A-module.

Let $K^{\mathrm{sep}} \subseteq K$ be the separable closure of F, so $K|K^{\mathrm{sep}}$ is trivial ($\mathrm{char} K = 0$) or a tower of purely inseparable extensions of degree $p = \mathrm{char} K$. If (LU) holds for the integral closure A^{sep} of A in K^{sep}, then (LU) holds for A. Namely, it can be assumed that

$$\mathrm{char} K = p,\ K = K^{\mathrm{sep}}(x^{1/p}),\ x \notin (K^{\mathrm{sep}})^p.$$

By proposition 4.3, we may take $x \in T^{\mathrm{sep}}$, where T^{sep} is given by (LU) for A^{sep}. So

$$h := X^p - x \in T^{\mathrm{sep}}[X]$$

satisfies the assumption of theorem 1.4(i). From now on, we assume that $K|F$ is separable.

Let $\overline{K}|K$ be a Galois closure and $\overline{v}$ be an extension of v to $\overline{K}$. Ramification theory of valuations [72] section 12 provides a diagram of fields

$$\begin{array}{ccccccc} K & \subseteq & K^i & \subseteq & K^r & \subseteq & \overline{K} \\ \uparrow & & \uparrow & & \uparrow & & \\ F & \subseteq & F^i & \subseteq & F^r & & \end{array} \tag{4.12}$$

as in the proof of [26] theorem 8.1. The left-hand side (resp. middle) inclusions in this diagram are unramified (resp. totally ramified Abelian of order prime to p). The extension $K^r|F^r$ is a tower of totally ramified Galois extensions of degree p.

Remark 4.2. Theorem 1.4 is actually required only to deal with those ramified extensions of degree p which are immediate (same value group and same residue field) w.r.t. the corresponding restrictions of $\overline{v}$. For extensions of degree p which are not immediate, a much simpler proof is available, *vid.* [26] proposition 6.3 in the equicharacteristic case.

In order to connect ramification theory of valuations and ramification theory of S-algebras essentially of finite type, we restate [26] theorem 7.2 in our context as proposition 4.9. For ramification theory of local rings, we refer to [3] (see also [26] section 2 for a quick summary of the required notions and notations).

Definition 4.1. A normal local model of $\mathcal{O}_v|S$ is the localization B_P of a finitely generated S-algebra B, $S \subseteq B \subseteq \mathcal{O}_v$, $QF(B) = K$ such that B is normal, where $P := m_v \cap B$.

Let $K'|K$ be a finite field extension and v' be an extension of v to K'. Given a normal local model B_P of $\mathcal{O}_v|S$, we define a normal local model B' of $\mathcal{O}_{v'}|S$ by localizing the integral closure $\overline{B}$ of B in K' at $P' := m_{v'} \cap \overline{B}$.

Note that B' is actually a normal local model because S, hence B, is excellent. Also note that if B' is a normal local model of $\mathcal{O}_{v'}|S$ and $K'|K$ is Galois, then $B' \cap K = B'^{\mathrm{Gal}(K'|K)}$ is a normal local model of $\mathcal{O}_v|S$.

Proposition 4.9. ***(Galois Approximation).*** *Let $K'|K$ be a finite Galois extension and v' be an extension of v to K'. There exists a normal local model B_0 of $\mathcal{O}_v|S$ such that for any normal local model B of $\mathcal{O}_v|S$ with $B_0 \subseteq B$, the following holds:*

(1) $G^s(v'|v) = G^s(B'|B)$ and $G^i(v'|v) = G^i(B'|B)$;

(2) the normal model $B^r := B'^{G^r(v'|v)}$ *of* $\mathcal{O}_{v^r}|S$ *satisfies*

$$B^r/m_{B^r} = B^i/m_{B^i},$$

where B^i *is the inertia ring of* B' *over* B*, i.e.* $B^i = B'^{G^i(B'|B)}$*, and* v^r *is the restriction of* v' *to* $K^r := K'^{G^r(v'|v)}$*. Moreover the representation*

$$\rho:\ G^i(v'|v)/G^r(v'|v) \to \mathrm{GL}(m_{B^r}/m_{B^r}^2),\ g \mapsto (\overline{x} \mapsto \overline{g.x})$$

is faithful.

We now prove that theorem 1.4 implies (LU). To emphasize the dependence on v, we say that (LUv) holds if (LU) holds for a particular v. With notations as in (4.12), we denote by $v_0, v_0^i, v_0^r, v^i, v^r$ the respective restrictions of $\overline{v}$ to F, F^i, F^r, K^i and K^r. The strategy is to prove successively the implications

$$(\mathrm{LU}v_0) \Longrightarrow (\mathrm{LU}v_0^i) \Longrightarrow (\mathrm{LU}v_0^r) \Longrightarrow (\mathrm{LU}v^r) \Longrightarrow (\mathrm{LU}v^i) \Longrightarrow (\mathrm{LU}v).$$

Note that (LUv_0) holds by construction since S is regular.

Firstly, (LUv_0^i) holds follows immediately from proposition 4.9 (1) as in [26] corollary 7.3. Then (LUv_0^r) holds because $F^r|F^i$ is a tower of ramified Galois extensions of prime degrees $l \neq p$: the proof relies on the Perron algorithm as in [26] proposition 6.3 and this is characteristic free.

To prove that (LUv^r) holds, we may assume that $K^r|F^r$ is a single Galois extension of degree p. Let $x \in \mathcal{O}_{v^r}$ be a primitive element with minimal polynomial

$$h := X^p + f_1X^{p-1} + \cdots + f_p \in \mathcal{O}_{v_0^r}[X].$$

By proposition 4.3, we may take $f_1, \ldots, f_p \in T^r$, where T^r is a local uniformization, since (LUv_0^r) holds. Theorem 1.4(ii) states that (LUv^r) holds.

Proving that (LUv^i), then (LUv) hold is an easy adaptation of [26] proposition 9.3. This relies on proposition 4.9 (1)(2) and [26] proposition 9.1 (as revisited in proposition 4.6) which are characteristic free. □

4.3 Normal crossings divisors conditions.

In this section, we consider a pair (S, h) satisfying the assumptions of theorem 1.4, i.e. such that **(G)** holds. We construct a sequence $\pi : \mathcal{X}' \to \mathcal{X}$ of blowing

ups along Hironaka-permissible centers in such a way that every $x' \in \pi^{-1}(x)$ has either $m(x') < p$, or ($m(x') = p$ and x' satisfies condition **(E)**). This is proved in corollary 4.13 below. Assumption **(G)** is not required here and we prove a more general version for arbitrary multiplicity in proposition 4.11.

Lemma 4.10. *Let S, $h \in S[X]$ (2.1) and $\eta : \mathcal{X} \to \mathrm{Spec}S$ be given. Assume that $\dim S = 3$ and that h is reduced. There exists a composition of Hironaka-permissible blowing ups (2.15) w.r.t. $E = \emptyset$:*

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}S & \xleftarrow{\sigma} & \mathcal{S}' \end{array}$$

such that $\pi(\mathrm{Sing}_m \mathcal{X}') \subseteq \eta^{-1}(m_S)$.

Proof. This statement means that there exists a diagram

$$\begin{array}{ccccccc} \mathcal{X} =: \mathcal{X}_0 & \xleftarrow{\pi_0} & \mathcal{X}_1 & \xleftarrow{\pi_1} & \cdots & \xleftarrow{\pi_{n-1}} & \mathcal{X}_n =: \mathcal{X}' \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathrm{Spec}S =: \mathcal{S}_0 & \xleftarrow{\sigma_0} & \mathcal{S}_1 & \xleftarrow{\sigma_1} & \cdots & \xleftarrow{\sigma_{n-1}} & \mathcal{S}_n =: \mathcal{S}' \end{array} \tag{4.13}$$

where each morphism π_i, $0 \leq i \leq n-1$, is the blowing up along a Hironaka-permissible center $\mathcal{Y}_i \subset \mathcal{X}_i$ w.r.t. the reduced exceptional divisor E_i of $\pi^{(i)} : \mathcal{X}_i \to \mathcal{X}$. It can be assumed that $\dim(\mathrm{Sing}_m \mathcal{X}) \geq 1$.

Let $y_i \in \mathcal{X}_i$ denote the generic point of such a Hironaka-permissible center $\mathcal{Y}_i \subset \mathcal{X}_i$ w.r.t. E_i. We define:

$$\Delta_i := \{y \in \mathrm{Sing}_m \mathcal{X}_i : \dim \mathcal{O}_{\mathcal{X}_i, y} = \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y)} = 1\},$$

$$\delta_i := \max\{\delta(y), y \in \Delta_i\}, \ N_i := \sharp\{y \in \Delta_i : \delta(y) = \delta_i\}.$$

Let $i \geq 0$. We claim that

$$\left\{ \begin{array}{lcl} (\delta_{i+1}, N_{i+1}) & = & (\delta_i, N_i) \quad \text{if} \quad \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} \geq 2 \\ \\ (\delta_{i+1}, N_{i+1}) & < & (\delta_i, N_i) \quad \text{if} \quad \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} = 1 \end{array} \right. . \tag{4.14}$$

Namely, this is an obvious consequence of the definition if $\dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} \geq 2$. If $\dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} = 1$, let $y \in \mathcal{X}_{i+1}$ with $\pi_i(y) = y_i$. We have

$$(m(y), \delta(y)) \leq (m(y_i), \delta(y_i) - 1)$$

by proposition 2.6 applied for $n = 1$ and the claim follows

Pick $y \in \Delta_i$ with $\delta(y) = \delta_i$ and denote $\mathcal{Y} := \overline{\{y\}} \subset \mathcal{X}_i$. By proposition 4.2, there exists a composition of blowing ups $\mathcal{X}_{i'} \to \mathcal{X}_i$ with regular centers contained in the successive strict transforms of $\mathcal{Y}$ such that $\eta_{i'}(\mathcal{Y}')$ has normal crossings with $E_{i'}$, where $\mathcal{Y}'$ denotes the strict transform of $\mathcal{Y}$ in $\mathcal{X}_{i'}$. Then $\mathcal{Y}'$ itself and each blowing up center in $\mathcal{X}_{i'} \to \mathcal{X}_i$ are Hironaka-permissible w.r.t. $E_{i'}$ because $m(y) = m$.

We have $(\delta_{i'}, N_{i'}) = (\delta_i, N_i)$ by (4.14). Taking as blowing up center $\mathcal{Y}_{i'} := \mathcal{Y}'$ also gives $(\delta_{i'+1}, N_{i'+1}) < (\delta_i, N_i)$ by (4.14). Since Δ_i is a finite set and $\delta_i \in \frac{1}{m}\mathbb{N}$, there exists an index $i_1 > i$ such that $\Delta_{i_1} = \emptyset$ and this is preserved by further Hironaka-permissible blowing ups w.r.t. $E = \emptyset$.

Since $\Delta_{i_1} = \emptyset$, we are done unless $\pi^{(i_1)}(\mathrm{Sing}_m \mathcal{X}_{i_1}) = \mathcal{C}$, where $\mathcal{C}$ has pure dimension one. Let $C \subset \mathrm{Spec} S$ be an irreducible component of $\eta(\mathcal{C})$ and s be its generic point. Note that the stalk $(\mathcal{X}_i)_s$ at s of the S-scheme $\mathcal{X}_i$ is embedded in the regular scheme of dimension three $\mathrm{Spec} S_s[X]$ for $i = 0$ and in an iterated blowing up along regular centers of the former for $i \geq 1$. By proposition 4.2, there exists a composition of Hironaka-permissible blowing ups $\mathcal{X}'_s \to (\mathcal{X}_{i_1})_s$ w.r.t. $(E_{i_1})_s$ such that $\mathrm{Sing}_m \mathcal{X}'_s = \emptyset$.

Let $\mathcal{Y}_s \subseteq (\mathcal{X}_{i_1})_s$ be a Hironaka-permissible center and $\mathcal{Y} \subseteq \mathcal{X}_{i_1}$ be its Zariski closure, so in particular we have $\mathcal{Y} \subseteq \mathrm{Sing}_m \mathcal{X}_{i_1}$. Since $\Delta_{i_1} = \emptyset$, $\mathcal{Y}$ is either (1) a curve mapping onto C, or (2) a surface mapping to some irreducible component of E_{i_1}.

In situation (1), there exists a composition of blowing ups along closed points $\mathcal{X}_{i'_1} \to \mathcal{X}_{i_1}$ such that $\eta_{i'_1}(\mathcal{Y}')$ has normal crossings with $E_{i'_1}$, where $\mathcal{Y}'$ denotes the strict transform of $\mathcal{Y}$ in $\mathcal{X}_{i'_1}$.

In situation (2), $\mathcal{Y}$ itself is Hironaka-permissible w.r.t. E_{i_1} and we let $i'_1 := i_1$.

In both situations, we may blow up $\mathcal{X}_{i'_1}$ along $\mathcal{Y}'$ and iterate: this produces an index $i_2 \geq i_1$ and a composition of Hironaka-permissible blowing ups $\mathcal{X}_{i_2} \to \mathcal{X}_{i_1}$ w.r.t. E_{i_1} such that $\eta^{-1}(s) \cap \pi^{(i_2)}(\mathrm{Sing}_m \mathcal{X}_{i_2}) = \emptyset$. Applying this construction to the finitely many irreducible components of $\eta(\mathcal{C})$ proves the lemma. □

Proposition 4.11. *Let $\mathcal{X}'$ satisfy the conclusion of lemma 4.10 and $E' \subset \mathcal{S}'$ be the reduced exceptional divisor of σ. Let $D \subset \mathcal{S}'$ be a reduced divisor.*

There exists a composition of Hironaka-permissible blowing ups (2.15)

w.r.t. E'*:*

$$\begin{array}{ccc} \mathcal{X}' & \xleftarrow{\pi'} & \mathcal{X}'' \\ \downarrow & & \downarrow \\ \mathcal{S}' & \xleftarrow{\sigma'} & \mathcal{S}'' \end{array}$$

such that the strict transform D'' *of* D *is disjoint from* $\eta''(\mathrm{Sing}_m\mathcal{X}'')$*, where* $\eta'' : (\mathcal{X}'', x'') \to \mathcal{S}''$ *is the local projection at* $x'' \in \mathrm{Sing}_m\mathcal{X}''$*.*

Proof. We take $\mathcal{S}' = \mathrm{Spec}S$. The problem is to find a sequence (4.13) which monomializes $P := \mathrm{I}(D) \subset S$, i.e. such that $P_n := P\mathcal{O}_{\mathcal{S}_n}$ is a monomial with components at normal crossings with E_n.

Let us write $P_i := H_iQ_i$ where H_i is a monomial whose components are components of E_i. At the beginning, $H = H_0 = 1$. The strategy is to get $P_n = H_n$, $Q_n = 1$ at the end.

We consider the idealistic exponents (h, m) and (Q, b) living in $\mathrm{Spec}S[Z]$, where $b = \mathrm{ord}_{m_S}(Q)$. We make a descending induction on b: the case $b = 0$ means that we get the conclusion of 4.11. Each pair of blowing ups π_i, σ_i is locally centered at some Y_i and $\eta(Y_i)$ respectively, and is Hironaka-permissible for h (resp. Q_i) w.r.t. E_i.

Let $P_{i+1} =: H_{i+1}Q_{i+1}$ where Q_{i+1} is the strict transform of Q_i. This means that (Q_{i+1}, b) is the transform of (Q_i, b). When $\mathrm{ord}_{x_{i+1}}(Q_{i+1}) < b$, we have strictly improved and we go on with the new idealistic exponent (Q_{i+1}, b'), with $b' = \mathrm{ord}_{x_{i+1}}(Q_{i+1})$. To define a sequence of σ_i is a consequence of [25] **Theorem 0.3** (Canonical embedded resolution with boundary), the problem is the sequence of π_i, i.e. to define the pair (σ_i, π_i).

To avoid cumbersome notations, from now on, $x_i, S_i, \mathcal{X}_i$,etc.$_i$ are denoted by $x, S, \mathcal{X}$,etc. and $x_{i+1}, S_{i+1}, \mathcal{X}_{i+1}$,etc.$_{i+1}$ by $x', S', \mathcal{X}'$,etc.$'$. Let us define $\mathrm{Vdir}(x, D)$ as $\mathrm{Vdir}(h) + \mathrm{Vdir}(Q)$. This is a vector space of codimension $\tau(x, D)$ in the Zariski's tangent space of $\mathcal{X}$ at x. Of course, $\tau(x, D) \geq 2$.

Lemma 4.12. *Let* π *be the blowing up along* Y *which is permissible for both* (h, m) *and* (Q, b)*. Let* $x' \in \pi^{-1}(x)$ *be such that* $m(x') = m(x) = m$ *and* $ord_{x'}Q' = b$*. Then* x' *is on* $\mathbf{Proj}(S/\mathrm{IDir}(x, D))$*. In particular,* x' *is on the strict transform of* $\mathrm{div}(Z)$*.*

Proof. By proposition 2.15 and remark 2.4, we have $\mathrm{Dir}(F) = \mathrm{Max}(F)$ except if $p = 2$ and

$$F = \lambda(Z^2 + \lambda_2U_1^2 + \lambda_1U_2^2 + \lambda_1\lambda_2U_3^2)^\alpha,\ [k^2(\lambda_1, \lambda_2) : k^2] = 4 \tag{4.15}$$

up to a linear change of variables, $\lambda \neq 0$, $\alpha \geq 1$. Since $m(x') = m(x)$, we have

$$x' := V(U_1^2 + \lambda_1 U_3^2, U_2^2 + \lambda_2 U_3^2, Z^2 + \lambda_1\lambda_2 U_3^2)$$

on $\pi'^{-1}(x) = \mathrm{Proj}(k[Z, U_1, U_2, U_3]/(F))$.

Since $\mathrm{ord}_{x'} Q' = b$, the initial of Q cannot satisfy (4.15) (only the last three variables occur). Therefore

$$x' \in \mathbf{Proj}(S/\mathrm{IDir}(h)) \cap \mathbf{Proj}(S/\mathrm{IDir}(Q)) = \mathbf{Proj}(S/\mathrm{IDir}(x, D)).$$

□

Let us come back to the proof of proposition 4.11. We discuss according to the value of $\tau(x, D)$.

When $\tau(x, D) = 4$, the blowing-up centered at x makes b strictly drop.

When $\tau(x, D) = 2$ or 3, then, if we blow up along x, then $\tau(x', D') \geq \tau(x, D)$. In case $\tau(x, D) = 3$, we make only blowing ups at closed points. Either for some n, $(m(x_n), \mathrm{ord}_{x_n}(Q_n)) <_{\mathrm{lex}} (m, b)$, then we stop at this n; or we have equality for $n \geq 0$. Then, $\tau(x_n, D_n) = 3$, $n \geq 0$, by an usual argument, the x_n are all on the strict transform of a curve $\mathcal{C}_n$ which, for $n >> 0$ is permissible for both (h, m) and (Q, b) and $\eta(\mathcal{C}_n)$ is transverse to E_n. Then at step n in (4.13), we blow up along $\mathcal{C}_n$. By lemma 4.12, $(m(x_{n+1}), \mathrm{ord}_{x_{n+1}}(Q_{n+1})) <_{\mathrm{lex}} (m, b)$.

When $\tau(x, D) = 2$, we can choose Z, u_3 such that

$$\mathrm{Vdir}(Q) =< U_3 >, \ \mathrm{Vdir}(h) \equiv< Z > \ \mathrm{mod}(U_3).$$

Remark 4.3. If there is a component Y of dimension 2 in

$$\mathrm{Sing}(h, m) \cap \mathrm{Sing}(Q, b),$$

then we can choose the parameters so that $I(Y) = (Z, u_3)$. Then $Q \in (z, u_3)^b$, i.e. $Q = u_3^b$, up to multiplication by an invertible. Then, if Y has normal crossing with E, we blow up along Y: π is the blowing up along Y and σ is the identity. In fact in S, we just add $\eta(Y) = \mathrm{div}(u_3)$ to E and we get $b = 0$.

We also note that $(h, m) \cap (Q, b) = (hQ, m + b)$. In other words, we have

$$\mathrm{Sing}(h, m) \cap \mathrm{Sing}(Q, b) = \mathrm{Sing}(hQ, m + b)$$

and permissible centers are the same for $(hQ, m + b)$ and for $(h, m) \cap (Q, b)$.

Then we apply those techniques from [25] **10**, **11**, **12**. More precisely, if for some n_0 the number b just strictly drops, we call "old components" the components of E_{n_0} at x_{n_0} which are components of H and, for $n \geq n_0$, at $x_n, n \geq n_0$ with $b(x_n) = b(x_{n_0})$, the strict transforms of this old components. The first step is to reach the case where x_n is not on the strict transform of this old components: the invariant is $(m, b, o(x))$ where $o(x)$ is the number of these old components. In the language of idealistic exponents, we desingularize $(hQQ_O, mbo(x))$ where Q_O is the equation of the reduced divisor whose components are the old ones. Then we look at the directrix of hQQ_O. When its codimension denoted by $\tau(hQQ_O)$ is 3 or 4, we play the same game that above with $\tau(x, D) = 3$ or 4. We reach the case where $\tau(hQQ_O) = 2$. This means that either $Q_O = 1$ (no old component) or there is one old component which is tangent to Q.

Then we look at the characteristic polyhedron $\Delta(hQQ_0, z, u_3, u_1, u_2)$ as in [25] **Section 7**.

• Case $\Delta(hQQ_0, z, u_3, u_1, u_2) = \emptyset$. This is equivalent to $hQQ_0 \in (z, u_3)^{mbo(x)}$, i.e. this is equivalent to $\dim(\mathrm{Sing}(hQQ_O, mbo(x)) = 2$. So $QQ_O = u_3^{mbo(x)}$, call $Y := \mathbf{V}(z, u_3)$, in fact, at step n_0, as $b(x_0) = b(x)$, Q was a $b(x_0)$ power and, if at x there is one old component, it is a factor of Q: this is impossible, therefore $o(x) = 0$.

So, at x, E is a union of components which are exceptional divisors of the blowing ups σ_n, $n \geq n_0$. By [25]**Theorem 8.3**, they are transverse to u_3: Y is permissible for $(hQQ_O, mbo(x))$ and transverse to E. We apply the first statement of remark 4.3.

• Case where $\dim(\mathrm{Sing}(hQQ_O, mbo(x)) \leq 1$. Then, we apply [25] **Theorem 5.28** which gives the result if $\mathrm{char} k(x) \geq 3$. This hypothesis $p \neq 2$ is used just to get $\mathrm{Dir}(F) = \mathrm{Max}(F)$ at each step, but we showed above in lemma 4.12, that the only case where $\mathrm{Dir}(F) \neq \mathrm{Max}(F)$ stops after blowing up the closed point x. □

Corollary 4.13. *Assume that* $\mathrm{char} S/m_S = p > 0$ *and* (S, h) *satisfies condition* ***(G)****. There exists a composition of Hironaka-permissible blowing ups (2.15) w.r.t.* $E = \emptyset$*:*

$$\begin{array}{ccc} \mathcal{X} = & \xleftarrow{\pi''} & \mathcal{X}'' \\ \downarrow & & \downarrow \\ \mathrm{Spec} S & \xleftarrow{\sigma''} & \mathcal{S}'' \end{array}$$

such that $\eta''(\mathrm{Sing}_p\mathcal{X}'') \subseteq \sigma''^{-1}(m_S)$ *and condition* ***(E)*** *holds at every* $s' \in \eta''(\mathrm{Sing}_p\mathcal{X}'')$*, where* $\eta'' : \mathcal{X}'' \to \mathcal{S}''$ *is the projection.*

Proof. This is a direct application of lemma 4.10 in the purely inseparable case ((iii) of condition **(G)**). If η is separable and $\mathrm{car}S = p$, we apply proposition 4.11 to the strict transform in $\mathcal{S}'$ of $D := \mathrm{div}(\mathrm{Disc}_X(h))$ and the conclusion follows.

Assume that $\mathrm{char}S = 0$. Let D'_1 be the strict transform of $\mathrm{div}(p\mathrm{Disc}_X(h))$ in $\mathcal{S}'$ and D'_2 be the union of those components of E' of characteristic zero. We apply proposition 4.11 to $D := D'_1 \cup D'_2$. Let E'' be the exceptional divisor of σ'' and $s' \in \eta''(\mathrm{Sing}_p\mathcal{X}'')$. Since all blowing up centers of σ' are Hironaka-permissible w.r.t. E', they map to $\eta(x)$ and are thus of characteristic $p = \mathrm{char}S/m_S$. We deduce from proposition 4.11 that any irreducible component of E'' passing through s' has characteristic p and that (ii) of definition 2.11 holds. □

5 Projection number $\kappa(x) \in \{1, 2, 3, 4\}$, projection theorem.

Let (S, h, E) satisfy assumptions **(G)** and **(E)**. In this section, we perform induction on the dimension $\dim S[Z] = 4$ of the ambient space of $\mathcal{X}$, *vid.* introduction. This step is for now far out of reach in higher dimensions and little more than definitions could be stated. We reduce theorem 1.4 to theorem 5.1 below (corollary 5.2) which is proved in the next sections.

5.1 Projection number $\kappa(x)$.

For $y \in \mathcal{X}$, $s := \eta(y) \in \mathrm{Spec}S$, the assignment $\kappa(y) \geq 2$ has sofar been used to express $\kappa(y) \neq 1$; we now distinguish $\kappa(y) = 2, 3, 4$ when ($\omega(y) > 0$, $\kappa(y) \geq 2$). This completes our definition of the complexity function (2.59):

$$\iota : \mathcal{X} \to \{1, \ldots, p\} \times \mathbb{N} \times \{1, \ldots, 4\}, y \mapsto (m(y), \omega(y), \kappa(y)).$$

The projection number $\kappa(y)$ expresses the transverseness of $\mathrm{Vdir}(y)$ w.r.t. E_s. We claim no further invariance property w.r.t. regular local base change than that of theorem 2.20 when $\kappa(y) \geq 2$.

Since our assumptions **(G)** and **(E)** are stable when changing (S, h, E) to (S_s, h_s, E_s) (notation 2.2), we may assume that $s = m_S$. The following definition is for codimension three, the remark afterwards for codimension two. One has $\omega(y) = \epsilon(y) = 0$ in codimension one. We denote $E = \mathrm{div}(u_1 \cdots u_e)$ as before.

Definition 5.1. (Projection Number). Assume that $m(x) = p$, $\omega(x) > 0$ and $\kappa(x) \geq 2$, where $\eta^{-1}(m_S) = \{x\}$. We let

$$\kappa(x) := 4 \text{ if } \mathrm{Vdir}(x) \subseteq < U_1, \ldots, U_e > . \tag{5.1}$$

Assume now that $\kappa(x) \neq 4$. We let $\kappa(x) := 3$ if ($\omega(x) = \epsilon(x) - 1$ and one of the following conditions is satisfied):

(1) $E = \mathrm{div}(u_1)$ and there exists well adapted coordinates $(u_1, u_2, u_3; Z)$ at x such that

$$\mathrm{Vdir}(x) \subseteq < U_1, U_3 > \text{ and } H^{-1}\frac{\partial F_{p,Z}}{\partial U_2} \subseteq < U_1^{\omega(x)} >;$$

(2) $E = \mathrm{div}(u_1 u_2)$.

Finally, we let $\kappa(x) := 2$ if $\kappa(x) \neq 3, 4$.

Remark 5.1. When $\dim \mathcal{O}_{\mathcal{X},y} = 2$, $m(y) = p$, $\omega(y) > 0$ and $\kappa(y) \geq 2$, we define: if $E_s = \mathrm{div}(u_1 u_2)$, let $\kappa(y) := 4$; if $E_s = \mathrm{div}(u_1)$, let:

$$\kappa(y) := \begin{cases} 2 & \text{if} \quad \omega(y) = \epsilon(y) \text{ and } \mathrm{Vdir}(y) \nsubseteq < U_1 > \\ 3 & \text{if} \quad \omega(y) = \epsilon(y) - 1 \\ 4 & \text{if} \quad \omega(y) = \epsilon(y) \text{ and } \mathrm{Vdir}(y) = < U_1 > \end{cases} .$$

5.2 Projection theorem.

We now turn to the statement of the projection theorem. We assume that $\omega(x) > 0$, so $(\mathcal{X}, x)$ is (analytically) irreducible by theorem 2.14. Let μ be valuation of $L = k(\mathcal{X})$ centered at x. We will consider finite sequences of local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{5.2}$$

with Hironaka-permissible centers $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$, where x_i, $0 \leq i \leq r$, denotes the center of μ. We require that our assumptions **(G)** and **(E)** be preserved by such blowing ups and that

$$(m(x_i), \omega(x_i)) \leq (m(x_{i-1}), \omega(x_{i-1})),\ 1 \leq i \leq r.$$

This certainly holds when the blowing up centers are permissible of the first or second kind by proposition 2.13 and theorem 3.6. Another example is blowing up along codimension one centers of the form $V(Z, u_j)$ with $d_j \leq 1$, $1 \leq j \leq e$. In chapter 8, we will use another kind of Hironaka-permissible blowing up with the same property. We recall that all permissibility conditions (definitions 2.7, 3.1 and 3.2) always refer to the reduced total transform E_i of E in S_i, where there are projections

$$\eta_i : (\mathcal{X}_i, x_i) \longrightarrow \mathrm{Spec} S_i,\ 0 \leq i \leq r.$$

Similarly, $\omega(x_i), \epsilon(x_i), \kappa(x_i)$ are always computed w.r.t. E_i.

Finally, we emphasize that we do *not* require any particular behavior about the numbers $\kappa(x_i)$ *along* the process (5.2). Our goal is to *eventually* achieve $\kappa(x_r) < \kappa(x)$ and we may have $\kappa(x_i) > \kappa(x)$ for some i, $1 \leq i < r$.

Definition 5.2. Assume that $m(x) = p$ and $\omega(x) > 0$. Given any finite sequence (5.2), we say that x_r is *very near* x if $\iota(x_r) \geq \iota(x)$.

Let $a \in \{1, \ldots, 4\}$. We say that x is *resolved for* $(p, \omega(x), a)$ (resp. *resolved for* $m(x) = p$) if for every valuation μ of $L = k(\mathcal{X})$ centered at x, there exists a finite and independent sequence (5.2) such that $\iota(x_r) < (p, \omega(x), a)$ (resp. $m(x_r) < p$). We simply say that x is *good* if x is resolved for $\iota(x)$.

The following projection theorem is proved in the next sections: corollary 6.2, theorem 7.18, theorem 9.6, *ibid.*, for $\kappa(x) = 1, 2, 3, 4$ respectively.

Theorem 5.1. ***(Projection Theorem).*** *Assume that (S, h, E) satisfies assumption **(G)** and **(E)**, with $m(x) = p$ and $\omega(x) > 0$.*

For every valuation μ of $L = k(\mathcal{X})$ centered at x, there exists a finite and independent composition of local Hironaka-permissible blowing ups (5.2) such that $\iota(x_r) < \iota(x)$, i.e. x is good.

Corollary 5.2. *Theorems 1.1 and 1.4 hold true.*

Proof. Theorem 1.1 has been reduced to theorem 1.4 for residually algebraic valuations, propositions 4.4 and 4.8. By corollary 4.13, it can be furthermore assumed that condition **(E)** is satisfied. Theorem 1.4 is then an immediate consequence of [29] Main Theorem 1.3 ($m(x) < p$), theorem 2.23 ($(m(x), \omega(x)) = (p, 0)$) and theorem 5.1. □

Remark 5.2. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x and consider an independent sequence of local blowing ups (definition 2.18)

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots$$

along μ. For example, the quadratic sequence along μ is an independent sequence.

Then x is resolved for $(p, \omega(x), a)$ if for every μ, there exists some $r = r(\mu) \geq 0$ such that x_r is resolved for $(p, \omega(x), a)$ (the converse follows from definition 5.2 with $r(\mu) = 0$ for every μ). This fact is used all along the next chapters, *vid.* chapter 7 for $a = 2$ and chapter 8 for $a = 3$.

Proposition 5.3. *With assumptions as in theorem 5.1, assume furthermore that* $\mathrm{Max}(\mathrm{in}h) \neq \mathrm{Dir}(\mathrm{in}h)$*, where* $\mathrm{in}h \in k(x)[U_1, U_2, U_3, Z]_p$ *is the initial form of* h *(proposition 2.15). Then* $\kappa(x) \geq 2$ *and* x *is resolved for* $(p, \omega(x), 2)$*.*

Proof. By remark 2.4, the assumption holds only if $p = 2$ and

$$\mathrm{in}h = Z^2 + F, \ F := \lambda_2 U_1^2 + \lambda_1 U_2^2 + \lambda_1 \lambda_2 U_3^2$$

with $[k(x)^2(\lambda_1, \lambda_2) : k(x)^2] = 4$ up to a linear change of variables. We have $H(x) = (1)$, $\omega(x) = \epsilon(x) = 2$ and $\kappa(x) = 4$ (resp. $\kappa(x) = 2$) if $E = \mathrm{div}(u_1 u_2 u_3)$ (resp. otherwise). Since

$$J(F, E, x) =< \frac{\partial F}{\partial \lambda_2}, \frac{\partial F}{\partial \lambda_1} >=< U_1^2 + \lambda_1 U_3^2, U_2^2 + \lambda_2 U_3^2 >,$$

we have $\tau'(x) = 3$. Let $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along x and $x' \in \pi^{-1}(x)$. Since $\tau'(x) = 3$, we have $\iota(x') \leq (2, 2, 1)$ by theorem 3.6. □

6 Maximal contact, resolution of $\kappa(x) = 1$.

We assume in the whole section that (S, h, E) satisfies conditions **(G)** and **(E)**. We consider here any refinement $\mathcal{C}$ of the function $x \mapsto (m(x), \omega(x))$ on $\mathcal{X}$.

Fix an irreducible component $\mathrm{div}(u_1) \subseteq E$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. We consider in this chapter finite sequences (5.2) of local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{6.1}$$

with *permissible centers of the first kind* $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$, where x_i, $0 \leq i \leq r$, denotes the center of μ. It is furthermore assumed that

(1) $\eta_i(\mathcal{Y}_i)$ belongs to the strict transform of $\mathrm{div}(u_1)$ in $\mathrm{Spec} S_i$, where

$$\eta_i : \ (\mathcal{X}_i, x_i) \longrightarrow \mathrm{Spec} S_i$$

is the projection, *vid.* proposition 2.7, and

(2) $\mathcal{C}$ is not increasing along (6.1), i.e. $\mathcal{C}(x_i) \leq \mathcal{C}(x_{i-1})$, $1 \leq i \leq r$.

Definition 6.1. We say that $\mathrm{div}(u_1) \subseteq E \subset \mathcal{X}$ has "maximal contact" (resp. "weak maximal contact") for some refinement $\mathcal{C}$ if for every μ, any sequence (6.1) (resp. the quadratic sequence (6.1) with $\mathcal{Y}_i := \{x_i\}$) satisfies the following:

$$\mathcal{C}(x_r) = \mathcal{C}(x) \Longrightarrow x_r \text{ maps to the strict transform of } \mathrm{div}(u_1). \tag{6.2}$$

Remark 6.1. Take $\mathcal{C} = \iota$, where $\kappa(x) = 1$. Then $\mathrm{div}(u_1) \subseteq E$ has maximal contact for $\mathcal{C}$ if U_1 divides $H^{-1}G^p$, with notations as in definition 2.16. This follows from theorem 3.6.

The purpose of this section is to prove theorem 6.1 below: the value $\mathcal{C}(x)$ of any such refinement can be lowered by permissible blowing ups of the first kind. A direct application proves theorem 5.1 for $\kappa(x) = 1$. Further applications are given in chapter 8. The proof of this theorem uses a secondary invariant $\gamma(x) \in \mathbb{N}$ which is defined and studied afterwards, *viz.* (6.7) and (6.9).

Theorem 6.1. *Assume that div(u_1) has maximal contact for $\mathcal{C}$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x, where $m(x) = p$ and $\omega(x) > 0$. There exists a finite and independent composition of local permissible blowing ups of the first kind:*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{6.3}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that $\mathcal{C}(x_r) < \mathcal{C}(x)$ or x_r is resolved for $m(x) = p$.

Proof. By proposition 3.13, the set

$$\Omega_+(\mathcal{X}) := \{y \in \mathcal{X} : (m(y), \omega(y)) > (p, 0)\} \subseteq \mathcal{X}$$

is Zariski closed and of dimension at most one. By performing the quadratic sequence (6.1), it can be assumed that there exist well adapted coordinates $(u_1, u_2, u_3; Z)$ at x such that any one dimensional irreducible component $\mathcal{Y}$ of $\Omega_+(\mathcal{X})$, with $\eta(\mathcal{Y})$ contained in $\mathrm{div}(u_1)$ either:

(a) maps to an intersection of components of E, i.e.

$$\eta(\mathcal{Y}) = V(Z, u_1, u_j),\ \mathrm{div}(u_j) \subseteq E,\ j \geq 2,\ \text{or}$$

(b) $\eta(\mathcal{Y}) = V(Z, u_1, u_3)$, $E \subseteq \mathrm{div}(u_1 u_2)$.

Furthermore, there exists at most one $\mathcal{Y}$ satisfying (b) and such $\mathcal{Y}$ is permissible of the first kind by proposition 3.8(1). Let $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along such $\mathcal{Y}$. Replacing $(\mathcal{X}, x)$ by $(\mathcal{X}', x')$, where x' is the center of μ, we may therefore assume that any one dimensional irreducible component $\mathcal{Y}$ of $\Omega_+(\mathcal{X})$, with $\eta(\mathcal{Y})$ contained in $\mathrm{div}(u_1)$, satisfies (a) above.

Consider now the quadratic sequence (6.1) and apply proposition 6.8 below. If alternative (ii) of that proposition holds, the theorem follows from proposition 3.8(2), since the conclusion of proposition 3.8(1) does not hold by the above preparation of $\Omega_+(\mathcal{X})$. Assume then that alternative (i) of proposition 6.8 holds. Then the conclusion follows from proposition 6.9 below. □

Corollary 6.2. *Projection Theorem 5.1 holds when $\kappa(x) = 1$.*

The arguments are quite similar to [27] chapter 4 pages 1957 and following and we sketch the argument below. This section may serve as an introduction to the more involved material in the next chapter.

Notation 6.1. By definition of maximal contact or weak maximal contact, we may assume that $\mathrm{div}(u_1 u_2) \subseteq E$.

Cases 1 and 2: $\epsilon(x) = \omega(x)$ and ($E = \mathrm{div}(u_1 u_2)$ or $E = \mathrm{div}(u_1 u_2 u_3)$ respectively). Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates. Consider the characteristic polyhedron

$$\Delta_S(h; u_1, u_2, u_3; Z) \subset \mathbb{R}^3_{\geq 0}$$

in the affine space with origin $\mathbf{v}_0 := (d_1 + \omega(x)/p, d_2, d_3))$. Perform the stereographic projection $\mathbf{p}'_2$ from $\mathbf{v}_0$ on the plane $x_1 = 0$, followed by the

homothety of center $(0,0)$ and ratio $\frac{p}{\omega(x)}$. Let $\mathbf{p}_2$ be the resulting map. Analytically, we have:

$$\mathbf{p}_2 : \ (x_1, x_2, x_3) \mapsto (y_2, y_3) := \frac{1}{\frac{\omega(x)}{p} - (x_1 - d_1)}(x_2 - d_2, x_3 - d_3). \tag{6.4}$$

We denote for simplicity

$$\Delta_2(x) := \mathbf{p}_2'(\Delta(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_1 - d_1 < \omega(x)/p\}). \tag{6.5}$$

There are associated invariants:

$$\begin{cases} A_j(x) & := & \inf \{y_j \mid (y_2, y_3) \in \Delta_2(x)\} \\ B(x) & := & \inf \{y_2 + y_3 \mid (x_2, x_3) \in \Delta_2(x)\} \\ C(x) & := & B(x) - A_2(x) - A_3(x) \geq 0 \\ \beta(x) & := & \inf \{y_3 \mid (A_2(x), y_3) \in \Delta_2(x)\} \\ \beta_2(x) & := & \sup \{y_3 \mid (y_2, y_3) \in \Delta_2(x), y_2 + y_3 = B(x)\} \end{cases} . \tag{6.6}$$

When $E = \mathrm{div}(u_1 u_2)$, we take as a convention in these formulæ that $A_3(x) = 0$. The main secondary invariant is:

$$\gamma(x) := \begin{cases} \max\{1, \lceil \beta(x) \rceil\} & \text{if} \quad E = \mathrm{div}(u_1 u_2) \\ \\ 1 + \lfloor C(x) \rfloor & \text{if} \quad E = \mathrm{div}(u_1 u_2 u_3) \end{cases} . \tag{6.7}$$

Note that $\Delta_2(x) \neq \emptyset$: this follows from (6.4) and the definition of d_1. Therefore

$$A_2(x), A_3(x), B(x) < +\infty.$$

It is easily seen that $\Delta_2(x) \subseteq \mathbb{R}^2_{\geq 0}$ is a polygon. Since all vertices of $\Delta_S(h; u_1, u_2, u_3; X) - (d_1, d_2, d_3)$ have module at least $\frac{\epsilon(x)}{p}$, we have $B(x) \geq 1$.

Case 3: $\epsilon(x) = 1 + \omega(x)$, $E = \mathrm{div}(u_1 u_2)$. The definition is the same as in cases 1 and 2 except that $\mathbf{v}_0$ is replaced by $\mathbf{v}_0' := (d_1 + \omega(x)/p, d_2, 1/p))$. Analytically, we have:

$$\mathbf{p}_2 : \ (x_1, x_2, x_3) \mapsto (y_2, y_3) := \frac{1}{\frac{\omega(x)}{p} - (x_1 - d_1)}(x_2 - d_2, x_3 - 1/p). \tag{6.8}$$

Note that the image set $\Delta_2(x)$ defined by (6.5) may contain points with negative third coordinate. The invariants $A_2(x)$, $B(x)$, $C(x) := B(x) - A_2(x)$ and $\beta(x)$ are defined as in cases 1 and 2. We let:

$$\gamma(x) := \max\{1 + \lfloor \beta(x) \rfloor, 1\}. \tag{6.9}$$

These definitions depend in principle on (u_1, u_2, u_3), but certainly not on Z such that $(u_1, u_2, u_3; Z)$ are well adapted coordinates. Indeed, the above definition are given in terms of $\Delta(h; u_1, u_2, u_3; Z)$. It can be proved that the numbers $A_j(x)$, $B(x)$, $C(x)$, $\beta(x)$ and $\gamma(x)$ are actually independent of $(u_1, u_2, u_3; Z)$ once the numbering of the components of E is fixed. We skip this fact here and refer to the next chapter (theorem 7.12 and definition 7.4 in particular) for similar issues.

Remark 6.2. The numbers $B(x), A_j(x)$ can be computed directly from the equation h.

In cases 1-2, let (a, b) be positive real numbers such that

$$a(d_1 + \frac{\omega(x)}{p}) + b(d_2 + d_3) = 1$$

with the convention $d_3 = 0$ when $\mathrm{div}(u_3) \not\subseteq E$. Define a monomial valuation $v_{(a,b,b)}$ on $S[Z]$ by setting weights:

$$v_{(a,b,b)}(u_1) = a,\ v_{(a,b,b)}(u_2) = v_{(a,b,b)}(u_3) = b,\ v_{(a,b,b)}(Z) = 1.$$

Then

$$B(x) = \sup\{\frac{a}{b} | v_{(a,b,b)}(h) = p\}.$$

The pair (a, b) giving the sup above is said to "define $B(x)$" (*viz.* [27] theorem **I.4**, equation (3) page 1962). As $B(x) \geq 1$, we have $a \geq b$. We denote:

$$H_B := \mathrm{in}_{v_{(a,b,b)}}(h) = Z^p + \sum_{1 \leq i \leq p} \Phi_i Z^{p-i}, \Phi_i \in k(x)[U_1, U_2, U_3], \tag{6.10}$$

where (a, b) "defines $B(x)$". By theorem 2.14, we have $\Phi_i = 0$, $1 \leq i \leq p-2$ and $-\Phi_{p-1} = G^{p-1}$ where G is a constant times a monomial in $U_1, \ldots, U_e$. We expand the corresponding initial form as in (6.10) and let

$$U_1^{-pd1} U_2^{-pd_2} U_3^{-pd_3} \Phi_p = \lambda U_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3),\ \lambda \in k(x), \tag{6.11}$$

where $F_i \in k(x)[U_2, U_3]$ is homogeneous of degree $iB(x)$.

More generally, let σ_2 be a compact face of $\Delta_2(x)$. The topological closure of the set

$$\sigma := \Delta_S(h; u_1, u_2, u_3; Z) \cap \mathbf{p}_2^{-1}(\sigma_2)$$

is a compact face of $\Delta_S(h; u_1, u_2, u_3; Z)$ defined by a weight vector $\alpha := \alpha_{\sigma_2}$. The corresponding initial form polynomial is written

$$H_\alpha = Z^p + \sum_{1 \leq i \leq p} \Phi_{i,\alpha} Z^{p-i}, \Phi_{i,\alpha} \in \mathrm{gr}_\alpha(S), \tag{6.12}$$

In case 3, there exists a unique compact face $\sigma \subset \Delta_S(h; u_1, u_2, u_3; Z)$ whose image by $\mathbf{p}_2$ is the face $y_1 + y_2 = B(x)$, maximal for this property. For $B(x) = 1$,

$$\sigma_{\mathrm{in}} := \{\mathbf{x} \in \mathbb{R}^3_{\geq 0} : x_1 + x_2 + x_3 = \delta(x)\}$$

obviously has this property. For $B(x) > 1$, we expand the corresponding initial form as in (6.10) and let

$$U_1^{-pd1} U_2^{-pd_2} \Phi_p = U_1^{\omega(x)} (\lambda_3 U_3 + \lambda_2 U_2) + \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3), \tag{6.13}$$

with $\lambda_2, \lambda_3 \in k(x)$, $F_i \in k(x)[U_2, U_3]$ homogeneous of degree $1 + iB(x)$.

In cases 1-2-3, let (a, b) be positive real numbers such that

$$a(d_1 + \frac{\omega(x)}{p}) + bd_2 = 1.$$

We have similarly:

$$A_2(x) = \sup\{\frac{a}{b} | v_{(a,b,0)}(h) = p\},$$

this suitable pair (a, b) is also said to "define $A_2(x)$". We denote:

$$H_2 = \mathrm{in}_{v_{(a,b,0)}}(h) = Z^p + \sum_{1 \leq i \leq p} \phi_i Z^{p-i}, \phi_i \in \frac{S}{(u_1, u_2)}[U_1, U_2], \tag{6.14}$$

where (a, b) "defines $A_2(x)$"([27] theorem **I.4**, valuation μ_1 on page 1962). We expand the ϕ_i, $1 \leq i \leq p$:

$$\phi_i = \sum_{j=0}^{\omega(x)} U_1^j U_2^{b(i,j)} \phi_{i,j}, \ b(i,j) = \frac{i}{b} - jA_2(x), \ \phi_{i,j} \in \frac{S}{(u_1, u_2)},$$

where $\frac{1}{b} = 1 + (d_1 + \omega(x))A_2(x)$.

All proofs are based on the following elementary lemma:

Lemma 6.3. *Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension two, $\mathfrak{m} = (v_1, v_2)$, $\mathrm{char} k = p > 0$. Let $f \in R$ with initial form*

$$\mathrm{in}_{\mathfrak{m}} f = V_1^{a_1} V_2^{a_2} F(V_1, V_2) \in G(\mathfrak{m}),\ \mathrm{in}_{\mathfrak{m}} f \notin G(\mathfrak{m})^p.$$

Let furthermore $P(t) \in R[t]$ be monic of degree $d \geq 1$ with irreducible residue $\overline{P}(t) \in k[t]$,

$$R' := R\left[\frac{v_2}{v_1}\right]_{(v_1, v_2')},\ v_2' := P\left(\frac{v_2}{v_1}\right)$$

and define:

$$a' := \max_{g' \in R'}\{\mathrm{ord}_{v_1}(f - g'^p)\},\ e' := \max_{g' \in R'}\{\mathrm{ord}_{\overline{v}_2'}(v_1^{-a'}(f - g'^p) : \mathrm{ord}_{v_1}(f - g'^p) = a'\}.$$

The following hold:

(1) $a' = a_1 + a_2$, $e' \leq 1 + \lfloor \frac{\deg F}{d} \rfloor$; if equality holds, then $\deg F/d \in \mathbb{N}$, $a'/p \in \mathbb{N}$, $e'/p \notin \mathbb{N}$, and

$$J(\mathrm{in}_{\mathfrak{m}} f, \mathrm{div}(v_1 v_2), \mathfrak{m}) =< \left(V_1^d P\left(\frac{V_2}{V_1}\right)\right)^{\frac{\deg F}{d}} >;$$

(2) if $a_2 = 0$, then $e' \leq \max\{\deg F, 1\}$. Equality holds only if $\deg F \leq 1$ or $d = 1$.

Proof. Identical to [27] **II.5.3.2** on p. 1862. Note that it is not necessary to assume R excellent. □

Now we follow [27] chapter 4. Consider the blowing up $\pi :\ \mathcal{X}' \to (\mathcal{X}, x)$ at x and let $x' \in \pi^{-1}(x)$ be a closed point, with $d := [k(x') : k(x)]$. Following [27] Theorem **I.4** on p.1962, we have:

Proposition 6.4. *With hypotheses and notations as above, assume that x is in case 1-2. Let $(u_1, u_2; u_3, Z)$ be well adapted coordinates at x and assume furthermore that*

$$\eta'(x') \in \mathrm{Spec}(S[\frac{u_1}{u_2}, \frac{u_3}{u_2}][Z']/(h')),\ h' := u_2^{-p} h,\ Z' := \frac{Z}{u_2}.$$

If $\mathcal{C}(x') = \mathcal{C}(x)$, we have:

$$A_2(x') = B(x) - 1,\ \gamma(x') \leq \gamma(x), \tag{6.15}$$

and there exist well adapted coordinates $(u_1' := u_1/u_2, u_2, u_3'; Z')$ at x' such that the following holds:

(1) if $x' = (Z/u_2, u_1', u_2, u_3/u_2)$, then x' is again in case 1-2 and

$$C(x') \leq C(x),\ \beta(x') \leq \beta(x);$$

(2) if $x' \neq (Z/u_2, u_1', u_2, u_3/u_2)$, then x' is in case 1 or 3. We have

$$\beta(x') \leq \begin{cases} 1 + \lfloor \frac{C(x)}{d} \rfloor & \text{if } x' \text{ is in case 1} \\ \\ \frac{C(x)}{d} & \text{if } x' \text{ is in case 3} \end{cases}, \tag{6.16}$$

and $\Phi_{p-1} \neq 0$ implies $\beta(x') = 0$ (resp. $\beta(x') < 0$) if x' is in case 1 (resp. in case 3).

If moreover x is in case 1 and $\beta(x) > 0$, we have

$$\begin{cases} \beta(x') \leq \beta(x) & \text{if } x' \text{ is in case 1} \\ \\ \beta(x') < \beta(x) & \text{if } x' \text{ is in case 3} \end{cases}. \tag{6.17}$$

Furthermore, x' is in case 3 only if $k(x')$ is inseparable over $k(x)$ (in particular p divides d).

Proof. Statement (1) is an easy application of proposition 2.6. For (2), we apply lemma 6.3 to the initial form polynomial H_B in (6.10), where $\Phi_i = 0$, $1 \leq i \leq p-2$ and Φ_p is given by (6.13). The initial form polynomial H_2 in (6.14) at x' has $A_1(x') = B(x) - 1$. The upper bounds (6.16) and (6.17) follow from lemma 6.3 and the self evident

$$1 + \left\lfloor \frac{jC(x)}{d} \right\rfloor \leq j(1 + \left\lfloor \frac{C(x)}{d} \right\rfloor),\ j \geq 1.$$

The inequality in (6.15) then follows from definitions (6.7) and (6.9).

If $\Phi_{p-1} \neq 0$, it is a monomial in U_1, U_2 (case 1) or in in U_1, U_2, U_3 (case 2) by theorem 2.14 and the conclusion follows.

Finally, assume that x is in case 1 and x' in case 3. By theorem 2.20, we may furthermore assume that $k(x') = k(x)$ if $k(x')$ is separable over $k(x)$. But then x' is in case 1 by (1) and the conclusion follows. □

Following now [27] Theorem **I.5** on page 1964:

Proposition 6.5. *With hypotheses and notations as above, assume that x is in case 3. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x and assume furthermore that*

$$\eta'(x') \in \mathrm{Spec}(S[\frac{u_1}{u_2}, \frac{u_3}{u_2}][Z']/(h')),\ h' := u_2^{-p}h,\ Z' := \frac{Z}{u_2}.$$

If $\mathcal{C}(x') = \mathcal{C}(x)$, we have

$$A_2(x') = B(x) - 1,\ \gamma(x') \leq \gamma(x),$$

and there exist well adapted coordinates $(u_1' := u_1/u_2, u_2, u_3'; Z')$ at x' such that the following holds:

(1) if x' is in case 1, then

$$\beta(x') \leq \frac{\gamma(x)}{d} + 1;$$

(2) if x' is in case 3, then

$$\beta(x') \leq \max\{\beta(x), 0\}$$

and $\beta(x') < \beta(x)$ if ($k(x') \neq k(x)$ and $\beta(x) > 0$);

Moreover, $\Phi_{p-1} \neq 0$ implies $\beta(x') = 0$ (resp. $\beta(x') < 0$) if x' is in case 1 (resp. in case 3).

Proof. We apply lemma 6.3 to the initial form polynomial H_2 in (6.10). The initial form polynomial H_2 in (6.14) at x' has $A_1(x') = B(x) - 1$ and the upper bounds for $\beta(x')$ follow from lemma 6.3.

By (6.9), note that

$$\deg F_i(U_2, U_3) - jA_2(x) \leq i\gamma(x)$$

in (6.13) whenever $F_i(U_2, U_3) \neq 0$. One deduces the upper bounds $\gamma(x') \leq \gamma(x)$ if $\beta(x) \geq 0$ as well as the sharper bound in (1) for $\gamma(x) \geq 2$, $d \geq 2$.

If $\Phi_{p-1} \neq 0$, it is a monomial in U_1, U_2 by theorem 2.14 and the conclusion follows. □

Following [27] lemma **I.5.3** on page 1966:

Proposition 6.6. *With hypotheses and notations as above, let* $(u_1, u_2, u_3; Z)$ *be well adapted coordinates at* x *and assume furthermore that*

$$x' = (Z' := Z/u_3, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3).$$

If $\mathcal{C}(x') = \mathcal{C}(x)$*, then* x' *is in case 2,* $(u_1', u_2', u_3; Z')$ *are well adapted coordinates at* x'*,*

$$A_3(x') = B(x) - 1,\ \beta(x') = A_2(x) + \beta(x) - 1,\ \gamma(x') \leq \gamma(x),$$

and the following holds:

(1) if x *is in case 1, then* $C(x') \leq \min\{\beta(x) - C(x), C(x)\}$*;*

(2) if x *is in case 2, we have* $C(x') \leq \min\{\beta(x) - A_2(x) - C(x), C(x)\}$*.*

(3) if x *is in case 3, we have* $C(x') \leq \min\{\beta(x) - C(x), C(x) - \beta_2(x)\}$*.*

Proof. This relies on the characteristic free proposition 2.6. The argument in [27] lemma **I.5.3** on page 1966 gives all statements except "$\gamma(x') \leq \gamma(x)$".

Finally, $\gamma(x') \leq \gamma(x)$ is a trivial consequence of the definitions (6.7) and (6.9) except if (x is in case 3 and $C(x) < 0$). But then $\beta_2(x) = -1/i$ for some i, $1 \leq i \leq \omega(x)$ and (3) gives

$$C(x') \leq C(x) - \beta_2(x) < 1,$$

so $\gamma(x') \leq 1$ as required. □

We now go ahead to prove theorem 6.1. The key lemma goes as follows:

Lemma 6.7. *Assume that div*(u_1) *has weak contact maximal for* $\mathcal{C}$*. Let* μ *be valuation of* $L = k(\mathcal{X})$ *centered at* x *and consider the quadratic sequence (6.1) along* μ*, i.e. with* $\mathcal{Y}_i = \{x_i\}$ *for every* $i \geq 0$*.*

Assume that one of the following holds:

(1) x *is in case 1 with:* $\beta(x) = 2$ *and*

$$\Phi_{p,\alpha} = \sum_{i=0}^{\omega(x)} U_1^{\omega(x)-i} \Phi_{p,\alpha,i}(U_2, U_3)$$

has $\Phi_{p,\alpha,1} \neq 0$ *with notations as in (6.12), where* $\sigma_2 := \{(A_1(x), 2)\}$*;*

(2) x is in case 3 with $\beta(x) = 1$.

Assume furthermore that $x_1 = (Z' := Z/u_3, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3)$, $\mathcal{C}(x_1) = \mathcal{C}(x)$ and $\gamma(x_1) = 2$. Then $\mathcal{C}(x_2) < \mathcal{C}(x)$ or $\gamma(x_2) = 1$.

Proof. Note that x_1 is in case 2 with $\gamma(x_1) = 2$ by assumption. By proposition 6.6, we get $A_2(x_1) = A_2(x)$ and respectively:
(1) $C(x) = C(x_1) = 1$, $A_3(x_1) = A_2(x)$, $\beta(x_1) = A_2(x) + 1$;
(2) $C(x) = 0$, $\beta_2(x) = -1$, $C(x_1) = 1$, $A_3(x_1) = A_2(x) - 1$, $\beta(x_1) = A_2(x)$.

These facts furthermore imply that

$$\Delta_2(x_1) = \{(y_1, y_2) \in \mathbb{R}^2_{\geq 0} : y_1 + y_2 \geq 1\}.$$

We are done by proposition 6.6 if x_2 is again in case 2. Otherwise, we may assume that $\mathcal{C}(x_2) = \mathcal{C}(x)$ and apply proposition 6.4 to estimate $\gamma(x_2)$. We get $\gamma(x_2) = 1$ if $k(x_2) \neq k(x)$ by (1) of this proposition.

Assume that $k(x_2) = k(x)$. We claim that the following sharper bound holds, which concludes the proof:

$$\beta(x_2) \leq 1 \text{ (resp. } \beta(x_2) \leq 0) \tag{6.18}$$

if x_2 is in case 1 (resp. in case 3).

There are associated $d_1', d_2', d_3' \in 1/p\mathbb{N}$ at x_1 with $d_1' = d_1$, $d_2' = d_2$ and

$$d_3' = d_1 + d_2 - 1 + \frac{\omega(x)}{p} \text{ (resp. } d_3' = d_1 + d_2 - 1 + \frac{1 + \omega(x)}{p})$$

if x is in case 1 (resp. in case 3).

Under assumption (1), the initial form (6.11) at x_1 has $F_1(U_2', U_3) \neq 0$ and is of the form

$$F_1(U_2', U_3) = U_2'^{a_2} U_3^{a_3} F(U_2', U_3),$$

where $a_2 \geq A_2(x)$, $a_3 \geq A_3(x)$, and either $F \in k(x)$ or

$$a_2 = a_3 = A_2(x_1) \in \mathbb{N} \text{ and } F = \lambda_2 U_2' + \lambda_3 U_3,\ \lambda_3 \neq 0. \tag{6.19}$$

By lemma 6.3(1) with $d = 1$, we get (6.18) provided

$$d_1' + (\omega(x) - 1)/p \notin \mathbb{N} \text{ or } d_2' + d_3' + \frac{A_2(x_1) + A_3(x_1) + 1}{p} \notin \mathbb{N}.$$

When this fails to hold, we have (6.19) with $\lambda_2 \neq 0$ and

$$d_1 + (\omega(x) - 1)/p \in \mathbb{N},\ 2(d_2 + \frac{A_2(x)+1}{p}) \in \mathbb{N} \tag{6.20}$$

by the above calculations. Furthermore, $p \geq 3$ (statement $e'/p \notin \mathbb{N}$ in lemma 6.3(1)). We deduce that $d_2 + \frac{A_2(x)+1}{p} \in \mathbb{N}$, which in turn implies that

$$U_2' \in J(U_2'^{pd_2+A_2(x_1)} U_3^{pd_3'+A_2(x_1)} F(U_2', U_3), \mathrm{div}(u_2' u_3), \mathfrak{m})$$

with notations as in lemma 6.3(1), applying $U_3 \frac{\partial}{\partial U_3}$. Then equality is strict in lemma 6.3(1) and the conclusion follows.

Under assumption (2), note that since $\beta_2(x) = -1$ we necessarily have $F_1(U_2', U_3) \neq 0$ or

$$H^{-1}G^p =< U_1{}^{\omega(x)-1} U_2^{1+A_2(x)} > .$$

In the former case, the proof is parallel to that under assumption (1), exchanging the roles of U_2', U_3; in the latter case, we conclude from proposition 6.4 with $\Phi_{p-1} \neq 0$. □

Proposition 6.8. *Assume that div(u_1) has weak maximal contact for $\mathcal{C}$. Let μ be valuation of $L = k(\mathcal{X})$ centered at x and consider the quadratic sequence (6.1) along μ, i.e. with $\mathcal{Y}_i = \{x_i\}$ for every $i \geq 0$.*

If $\mathcal{C}(x_i) = \mathcal{C}(x)$ for every $i \geq 0$, one of the following is true:

(i) $\gamma(x_i) = 1$ for every $i >> 0$, or

(ii) there exists a formal arc $\varphi :$ Spec$\mathcal{O} \to (\mathcal{X}, x)$ with $l|k(x)$ finite algebraic, support $Z := Z(\varphi)$ with

$$\eta(Z) \subseteq \mathrm{div}(u_1),$$

$\eta(Z)$ not an intersection of components of E, whose strict transform passes through x_i for every $i \geq 0$.

Proof. Note that (ii) fails to hold if and only if: for every $i \geq 0$, there exists $i' > i$ such that either $k(x_{i'}) \neq k(x_i)$ (i.e. some of proposition 6.4, 6.5 applies to $x_{i'}$ with $d \geq 2$) or $x_{i'}$ is in case 2.

Assume therefore that (ii) does not hold. By propositions 6.4, 6.5 and 6.6, we have $\gamma(x_{i+1}) \leq \gamma(x_i)$ for every $i \geq 0$ and inequality is strict for i'

as above if $\gamma(x_{i'}) \geq 3$. W.l.o.g. it can be assumed that $\gamma(x_i) = 2$ for every $i \geq 0$. We now derive a contradiction by studying different cases.

(a) if x is in case 1 with $\beta(x) < 2$, we are done by propositions 6.4 and 6.6.

Assume that x is in case 1 with $\beta(x) = 2$. If proposition 6.4 applies, we obtain $\beta(x_1) \leq 2$ ($\beta(x_1) < 2$ if $k(x_1) \neq k(x)$) if x_1 is again in case 1. If x_1 is in case 3, we get $\beta(x_1) < 1$ (strict inequalities follow from lemma 6.3(1) in case $p = 2$).

Assume that x is in case 3. If proposition 6.5 applies, we obtain $\beta(x_1) \leq \beta(x)$ (with strict equality if $k(x_1) \neq k(x)$) if x_1 is again in case 3. If x_1 is in case 1, we get $\beta(x_1) \leq 2$; if furthermore $\beta(x) = 1$, inequality is strict unless x_1 satisfies the assumptions of lemma 6.7(1). We deduce:

(b) if x is in case 3 with $\beta(x) = 1$, we are done: this follows from lemma 6.7 and the previous comments.

(c) if x is in case 1 with $\beta(x) = 2$, we are done: we may assume that proposition 6.6 applies by the previous comments; we reach (a)(b) or the assumptions of lemma 6.7(1) at x_2 since it is assumed that $\gamma(x_2) = 2$.

(d) the remaining cases: x is in case 2 (resp. in case 3 with $\beta(x) > 1$). The result is trivial if x_i is in case 2 for every $i >> 0$. Otherwise, note that: $C(x_1) \leq C(x)$ if x_1 is in case 2; $\beta(x_1) \leq C(x)$ if x_1 is in case 3 (resp. $C(x_1) < \beta(x)$ if x_1 is in case 2; $\beta(x_1) \leq \beta(x)$ if x_1 is in case 3), applying propositions 6.5 and 6.6. The conclusion follows easily.

□

Proposition 6.9. *Assume that div(u_1) has maximal contact for $\mathcal{C}$ and that $\gamma(x) = 1$. Let μ be valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent composition of local permissible blowing ups of the first kind:*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r),$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that $\mathcal{C}(x_r) < \mathcal{C}(x)$ or x_r is resolved for $m(x) = p$.

Proof. We may assume that $\mathcal{C}(x_i) = \mathcal{C}(x)$ for every $i \geq 1$ for the resolution process to be defined below; we will either derive a contradiction or prove that x_r is resolved for $m(x) = p$ for some $r \geq 0$. Suppose that $i \geq 1$ and that

$$A_2(x_{i-1}) < 1 \text{ and } (x_{i-1} \text{ is in case } 2 \Longrightarrow \beta(x_{i-1}) < 1). \tag{6.21}$$

Then we consider the quadratic sequence (6.1) along μ. In every case, we have

$$A_2(x_i) \leq A_2(x_{i-1}),$$

where inequality is strict except if either proposition 6.6 applies, or (x_{i-1} is in case 1 with $\beta(x_{i-1}) = 1$). If proposition 6.6 applies, we have

$$\beta(x_i) = A_2(x_{i-1}) + \beta(x_{i-1}) - 1 < 1.$$

This proves in particular that (6.21) holds at $x_{i'}$ for every $i' \geq i$. W.l.o.g. it can be assumed that $i = 0$.

If x is in case 1 with $\beta(x) = 1$ and $k(x_1) \neq k(x)$, then $\beta(x_1) < 1$ by proposition 6.5; if proposition 6.6 applies to x, then $\beta(x_1) < \beta(x)$. In other terms, we have

$$(A_2(x_1), \beta(x_1)) < (A_2(x), \beta(x))$$

for the lexicographical ordering except possibly if x is in case 1 with $\beta(x) = 1$ and $k(x_1) = k(x)$. So in the sequence (6.1), we may assume that x_i is in case 1 with

$$A_2(x_i) = A_2(x) < 1,\ \beta(x_i) = \beta(x) = 1,\ k(x_i) = k(x)$$

for every $i \geq 0$. Applying proposition 3.8, we are done if alternative (2) of this proposition holds; if alternative (1) holds, it can be assumed that there exists a permissible curve of the first kind $\mathcal{Y} = V(Z, u_1, u_3) \subseteq (\mathcal{X}, x)$. Then x is resolved by blowing up $\mathcal{Y}$: in view of definition 6.1, we need only consider the point $x' := (Z/u_3, u_1/u_3, u_2, u_3)$ and get $\omega(x') < \omega(x)$ from proposition 2.6. This proves the proposition under the extra assumption (6.21).

We now consider several cases which are proved consecutively:

(a) *x is in case 1.* We have $A_2(x) \geq 1$ if the extra assumption (6.21) does not hold. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x and note that $\mathcal{Y} := V(Z, u_1, u_2)$ is a permissible curve of first kind. Blowing up along $\mathcal{Y}$, we are done except possibly if $x_1 = (Z/u_2, u_1/u_2, u_2, u_3)$, in which case x_1 is again in case 1 with

$$(A_2(x_1), \beta(x_1)) = (A_2(x) - 1, \beta(x)).$$

The proof concludes by induction on $A_2(x)$. Before going along with the proof in cases 2 and 3, we make the following remark:

Remark 6.3. Assume that x is in case 2 with $A_2(x) \geq 1$. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x and denote $\mathcal{Y} := V(Z, u_1, u_2)$ with generic point y. Since $\epsilon(y) = \epsilon(x)$, $\mathcal{Y}$ is permissible of the first kind if and only if it is Hironaka-permissible w.r.t. E, i.e. if $m(y) = m(x) = p$. Thus:

$$\mathcal{Y} \text{ is permissible of the first kind} \Leftrightarrow d_1 + d_2 + \frac{\omega(x)}{p} \geq 1. \tag{6.22}$$

Suppose that $\mathcal{Y}$ is Hironaka-permissible. Blowing up along $\mathcal{Y}$ and arguing as in (a), we achieve:

$$x_1 \text{ in case 2, } A_2(x_1) = A_2(x) - 1, \ A_3(x_1) = A_3(x). \tag{6.23}$$

This proves that it can be assumed to begin with that

$$A_j(x) < 1 \text{ or } d_1 + d_j + \frac{\omega(x)}{p} < 1 \tag{6.24}$$

for each of $j = 2, 3$.

Assume that x is in case 2 with $d_1 + \omega(x)/p < 1$ and x is blown up. If $x' := x_1$ is in case 3, we have:

$$F_{p,Z} \in k(x)[U_2, U_3], \ d_1 = 0 \text{ and } d_2 + d_3 + \frac{\omega(x)}{p} \in \mathbb{N}$$

by theorem 3.6(1). Let $(u_1' := u_1/u_2, u_2, v'; Z')$ be well adapted coordinates at x', so we have

$$E' = \operatorname{div}(u_1' u_2), \epsilon(x') = 1 + \omega(x) < p, \ d_1' = 0 \text{ and } d_2' \in \mathbb{N}.$$

Therefore x' is resolved for $m(x) = p$ by blowing up codimension one centers of the form $\mathcal{Y}' := V(Z', u_2)$.

Algorithm: if x is in case 2 and $\mathcal{Y}_j := V(Z, u_1, u_j)$ is permissible for some of $j = 2, 3$, blow up along $\mathcal{Y}_j$; otherwise blow up along x.

We claim that this algorithm succeeds, i.e. produces x_r in case 1, *cf.* (a), or x_r resolved for $m(x) = p$. The proof is different for small values of $\omega(x)$:

(b) *proof when* $d_1 + \omega(x)/p < 1$. Let x be in case 2. We may assume that (6.24) holds.

(b1) if $d_1 + d_j + \omega(x)/p < 1$, $j = 2, 3$, the algorithm blows up along x. By the above remark 6.3, it can be assumed that $x_1 = (Z/u_2, u_1/u_2, u_2, u_3/u_2)$ up to renumbering u_2, u_3. We obtain

$$d_1' = d_1,\ d_2' = d_1 + d_2 + d_3 + \omega(x)/p - 1 < d_2,\ d_3' = d_3.$$

Assumption (b1) is stable by blowing up and can possibly repeat only finitely many times.

(b2) by the above remark 6.3, the algorithm succeeds or produces an infinite sequence of points in case 2. By (6.23), any subsequence of blowing ups along curves is finite, in particular $C(x_r) = 0$ for every $r >> 0$. Take $r = 0$ to begin with and assume w.l.o.g. that x is blown up. The extra assumption (6.21) holds if $0 \leq A_2(x), A_3(x) < 1$. Up to renumbering u_2, u_3, we may furthermore assume by (6.24) that

$$(d_1 + d_2 + \frac{\omega(x)}{p} < 1, A_2(x) \geq 1),\ (d_1 + d_3 + \frac{\omega(x)}{p} \geq 1, A_3(x) < 1). \quad (6.25)$$

Let

$$x_1' := (Z/u_2, u_1/u_2, u_2, u_3/u_2) \text{ and } x_1'' := (Z/u_3, u_1/u_3, u_2/u_3, u_3). \quad (6.26)$$

If $x_1 = x_1'$ (resp. $x_1 = x_1''$), we have $d_1' = d_1$ and

$$d_3' = d_3,\ A_3(x_1) = A_3(x),\ A_2(x_1) = A_2(x) + A_3(x) - 1 < A_2(x)$$

$$(\text{resp. } d_2' = d_2,\ A_2(x_1) = A_2(x),\ A_3(x_1) < A_2(x),\ d_3' < d_3).$$

When $x_1 = x_1''$ and the algorithm blows up again along a curve ($A_3(x_1) \geq 1$), note that

$$d_1' + d_3' + \omega(x)/p - 1 < d_3'$$

since $d_1 + \omega(x)/p < 1$. This proves that any further blowing up at a closed point either satisfies: some of (b1) or (6.21), or satisfies again (6.25) with a smaller value of $(A_2(x), d_3)$ for the lexicographical ordering. Induction on $(A_2(x), d_3)$ completes the proof for x in case 2 (*vid.* the same argument in [27] **1.7.4** on p. 1968).

Let now x be in case 3. We are done unless x_1 is again in case 3. Then

$$A_2(x_1) = A_2(x) + C(x) - 1 < A_2(x).$$

Therefore the algorithm reaches (6.21) after finitely many steps. This completes the proof of (b).

(c) *proof when* $d_1 + \omega(x)/p \geq 1$. By the above remark 6.3, we may assume that $0 \leq A_2(x), A_3(x) < 1$ to begin with if x is in case 2. If x is in case 2 (resp. in case 3), we let

$$c'(x) := \beta(x) \text{ (resp. } c'(x) := A_2(x)).$$

We have $c'(x) \geq 1$ if the extra assumption (6.21) does not hold. Applying propositions 6.4, 6.5 and 6.6, we obtain:

• if x is in case 2 and $x_1 = x_1'$ (resp. $x_1 = x_1''$), notations of (6.26), then

$$A_3(x_1') = A_3(x),\ c'(x_1') \leq A_2(x) + \beta(x) - 1 < c'(x)$$

$$\text{(resp. } A_2(x_1'') = A_2(x),\ c'(x_1'') = A_2(x) + \beta(x) - 1 < c'(x)).$$

Note that blowing up along the curve

$$\mathcal{Y}' := V(Z/u_2, u_1/u_2, u_2) \text{ (resp. } \mathcal{Y}'' := V(Z/u_3, u_1/u_3, u_3))$$

if $A_2(x_1') \geq 1$ (resp. if $A_3(x_1'') \geq 1$) does not change $c'(x_1')$ (resp. does not increase again $c'(x_1'')$). If x is in case 2 and x_1 is in case 3, then

$$c'(x_1) = A_2(x) + A_3(x) + C(x) - 1 \leq A_2(x) + \beta(x) - 1 < c'(x).$$

• if x is in case 3 and x_1 is in case 2 (resp. in case 3), then

$$c'(x_1) = A_2(x) + \beta(x) - 1 < c'(x) \text{ (resp. } c'(x_1) = A_2(x) + C(x) - 1 < c'(x)).$$

Induction on $c'(x)$ completes the proof. □

7 Projection theorem: very transverse case, resolution of $\kappa(x) = 2$.

In this chapter, we prove theorem 5.1 when $\kappa(x) = 2$ (definition 5.1). This is restated as theorem 7.18 at the end of this chapter.

Assume that a valuation μ of $L = k(\mathcal{X})$ centered at x is given. We consider finite sequences of local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{7.1}$$

with Hironaka-permissible centers $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$, where x_i, $0 \leq i \leq r$, denotes the center of μ, see (5.2) and following comments. Also recall the definition of "resolved" and "good" (definition 5.2) and remark 5.2 about the logical scheme of the proof of theorem 5.1.

Up to the end of this chapter, "resolved" stands for "resolved for $(p, \omega(x), 2)$*".*

7.1 Preliminaries.

In this section, we study points x' obtained by performing a permissible blowing up and such that $(m(x'), \omega(x')) = (m(x), \omega(x))$ and $\kappa(x') > \kappa(x) = 2$.

Lemma 7.1. *Let* $(u_1, u_2, u_3; Z)$ *be well adapted coordinates at* x*. Assume that* $\epsilon(x) = \omega(x) \geq 2$*,* $\kappa(x) \geq 2$ *and* $\mathrm{div}(u_1) \subseteq E$*.*

Assume furthermore $(d_1, d_2 + 1/p, d_3 + \omega(x)/p)$ *is the only vertex* $\mathbf{v} = (v_1, v_2, v_3)$ *of* $\Delta_S(h; u_1, u_2, u_3; Z)$ *in the region* $v_1 = d_1$*.*

Then x *is resolved.*

Proof. Since $\kappa(x) \geq 2$, there is an expansion

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z},\ H^{-1} F_{p,Z} \subseteq k(x)[U_1, U_2, U_3]_{\omega(x)}.$$

Any vertex of $\Delta_S(h; u_1, u_2, u_3; Z) \cap \{\mathbf{v} : v_1 + v_2 + v_3 = \delta(x)\}$ lies in the region $v_1 > d_1$ by assumption and we deduce that $U_1 \in \mathrm{Vdir}(x)$. Let

$$\mathrm{in}_{\mathbf{v}} h = Z^p + \sum_{i=1}^{p} F_{i,\mathbf{v}} Z^{p-i} \in k(x)[U_1, U_2, U_3][Z]$$

be the initial form polynomial with respect to $\mathbf{v}$. By theorem 2.14, we have $F_{i,\mathbf{v}} = 0$, $1 \leq i \leq p-2$, and $F_{p-1,\mathbf{v}} = -G_{\mathbf{v}}^{p-1}$ since $\epsilon(x) > 0$. Moreover $G_{\mathbf{v}}^{p-1} \neq 0$ implies that

$$\mathbf{v} \in \mathbb{N}^3,\ E = \mathrm{div}(u_1 u_2 u_3) \text{ and } (\mathrm{Disc}_Z(h)) = (u_1^{pd_1} u_2^{pd_2+1} u_3^{pd_3+\epsilon(x)})^{p-1}. \quad (7.2)$$

Let $\mathcal{Y} := V(Z, u_1, u_3) \subset \mathcal{X}$ and $y \in \mathcal{X}$ be its generic point. If $\mathcal{Y}$ is permissible of the first kind, i.e. $m(y) = p$ and $\epsilon(y) = \epsilon(x)$, we take $\mathcal{Y}_0 := \mathcal{Y}$ in (7.1). By theorem 3.6, we have $\iota(x_1) \leq (p, \omega(x), 1)$ unless

$$\mathrm{Vdir}(x) =< U_1 > \text{ and } x_1 = x' := (Z' := Z/u_3, u_1' := u_1/u_3, u_2, u_3).$$

By proposition 2.6, $\Delta_{S'}(h'; u_1', u_2, u_3; Z')$ is minimal, and we deduce that $H(x') = ({u_1'}^{pd_1} u_2^{pd_2+1} u_3^{p(d_1+d_3-1)+\epsilon(x)})$ and $\mathbf{v}' := (d_1, d_2 + 1/p, d_1 + d_3 - 1 + \epsilon(x)/p)$ is a vertex of $\Delta_{S'}(h'; u_1', u_2, u_3; Z')$. Therefore $\omega(x') \leq \epsilon(x') = 1$ and the lemma holds.

Assume now that $\mathcal{Y}$ is not permissible of the first kind. We take $\mathcal{Y}_0 := \{x\}$ in (7.1). If $\iota(x_1) \geq (p, \omega(x), 2)$, x_1 belongs to the strict transform of $\mathrm{div}(u_1)$ by theorem 3.6.

If $x_1 = x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2)$ is the point on the strict transform of $\mathcal{Y}$, then $\Delta_{S'}(h'; u_1', u_2, u_3'; Z')$ is minimal by proposition 2.6 and we deduce as above that $H(x') = ({u_1'}^{pd_1} u_2^{p(d_1+d_2+d_3-1)+\epsilon(x)} {u_3'}^{pd_3})$ and $\mathbf{v}' := (d_1, d_1 + d_2 + d_3 - 1 + (1 + \epsilon(x))/p, d_3 + \epsilon(x)/p)$ is the only vertex of $\Delta_{S'}(h'; u_1', u_2, u_3; Z')$ in the region $v_1' = d_1$. Since $\epsilon(x') \leq \epsilon(x) = \omega(x)$, we deduce that x_1 satisfies again the assumptions of the lemma if $\iota(x_1) \geq (p, \omega(x), 2)$.

The conclusion of proposition 3.8(2.b) is not satisfied by the formal arc $\hat{\mathcal{Y}} \to \mathcal{X}$. Iterating, we deduce from proposition 3.8(1) that one of the following three properties is satisfied for some $r \geq 1$:

(1) $\iota(x_r) \leq (p, \omega(x), 1)$;

(2) x_r belongs to the strict transform $\mathcal{Y}_r$ of $\mathcal{Y}$ in $\mathcal{X}_r$ and $\mathcal{Y}_r$ is permissible of the first kind at x_r, or

(3) x_r does not belong to $\mathcal{Y}_r$.

The lemma holds when (1) is satisfied; it has been proved above that the lemma also holds when (2) is satisfied. If (3) is satisfied, it can be assumed w.l.o.g. that $r = 1$. We claim that x_1 satisfies the conclusion of the lemma if $x_1 \neq (Z/u_3, u_1/u_3, u_2/u_3, u_3)$.

To prove the claim, first note that there exists a unitary polynomial $P(t) \in S[t]$, whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible, $\overline{P}(t) \neq t$, and $x_1 = (X' := Z/u_3, u_1' := u_1/u_3, u_2' := u_3, u_3' := P(u_2/u_3))$. We then denote $S' := S[u_1/u_3, u_2/u_3]_{(u_1', u_2', u_3')}$ and

$$h' = X'^p + \sum_{i=1}^{p} f_{i,X'} X'^{p-i} \in S'[X'], \;\; E' = \mathrm{div}(u_1' u_2').$$

We have $H(x') = ({u'_1}^{pd_1} {u'_2}^{p(d_1+d_2+d_3-1)+\epsilon(x)})$ by proposition 3.5(iv) and

$$\mathbf{v}' := (d_1, d'_2 := d_1 + d_2 + d_3 - 1 + (1 + \epsilon(x))/p, 0)$$

is a vertex of $\Delta_{S'}(h'; u'_1, u'_2, u'_3; X')$.

If $\mathbf{v}'$ is not solvable (in particular if $G_{\mathbf{v}} \neq 0$, see (7.2) above), we deduce that $\omega(x') \leq \epsilon(x') = 1$ and the lemma holds.

If $\mathbf{v}'$ is solvable, we had

$$\mathrm{in}_{\mathbf{v}} h = Z^p + \lambda U_1^{pd_1} U_2^{pd_2+1} U_3^{pd_3+\epsilon(x)},\ \lambda \in k(x),\ \lambda \neq 0$$

to begin with. Let $\sigma' \subset \Delta_{S'}(h'; u'_1, u'_2, u'_3; X')$ be the (noncompact) face with equations $v'_1 = d_1$, $v'_2 = d'_2$. The initial form polynomial corresponding to σ' is

$$\mathrm{in}_{\sigma'} h' = X'^p + \lambda \overline{\left(\frac{u_2}{u_3}\right)^{pd_2+1}} U_1'^{pd_1} U_2'^{pd'_2},$$

where $\overline{\theta}$ denotes the image in $S'/(u'_1, u'_2)$ of $\theta \in S'$. Let $\mu \in k(x')$ be the residue of u_2/u_3. Since $\mathbf{v}'$ is solvable, we have:

$$(d_1, d'_2) \in \mathbb{N}^2,\ \lambda\mu^{pd_2+1} \in k(x')^p. \tag{7.3}$$

Take $Z' := X' - \varphi'$, $\varphi' \in S'$ such that $\Delta_{S'}(h'; u'_1, u'_2, u'_3; Z')$ is minimal. We have $\varphi' = \gamma' {u'_1}^{d_1} {u'_2}^{d'_2}$, where $\gamma' \in S'$ is a preimage of $(\lambda\mu^{pd_2+1})^{1/p} \in k(x')$. By (7.3), $\lambda U_2^{pd_2+1}$ is not a pth-power, since $\mathbf{v}$ was not a solvable vertex. We deduce that

$$\lambda \overline{\left(\frac{u_2}{u_3}\right)^{pd_2+1}} + \overline{\gamma'}^p \in S'/(u'_1, u'_2)$$

is a regular parameter. Therefore $\mathbf{v}'_1 := (d_1, d'_2 + 1/p, 1/p)$ is a vertex of $\Delta_{S'}(h'; u'_1, u'_2, u'_3; Z')$ and this proves that $\omega(x') \leq 1$. This concludes the proof of the claim.

To conclude, take $x_1 = x' := (Z' := Z/u_3, u'_1 := u_1/u_3, u'_2 := u_2/u_3, u'_3 := u_3)$. Since $\Delta_{S'}(h'; u'_1, u'_2, u'_3; Z')$ is minimal by proposition 2.6, we deduce as before that $H(x') = ({u'_1}^{pd_1} {u'_2}^{pd_2} {u'_3}^{pd'_3})$ and $\mathbf{v}' := (d_1, d_2 + 1/p, d'_3 + 1/p)$ is the only vertex of $\Delta_{S'}(h'; u'_1, u'_2, u'_3; Z')$ in the region $v'_1 = d_1$, where $d'_3 := d_1 + d_2 + d_3 - 1 + \epsilon(x)/p$. Therefore $\epsilon(x') \leq 2$ and we are done unless $\omega(x) = 2$, $\iota(x') \geq (p, 2, 2)$ and $E = \mathrm{div}(u_1 u_2 u_3)$, which we assume from now on.

We have $E' = \mathrm{div}(u_1'u_2'u_3')$ and the initial form polynomial has an expansion

$$\mathrm{in}_{m_{S'}} h' = Z'^p + F_{p,Z'}.$$

with $H'^{-1}F_{p,Z'} = U_1'(\lambda_1 U_1' + \lambda_2 U_2' + \lambda_3 U_3') + \mu U_2' U_3'$. The assumptions imply that $\mu H' U_2' U_3' \notin G(m_{S'})^p$ and $(\lambda_1, \lambda_2) \neq (0,0)$. Moreover, we have

$$\lambda_j \neq 0 \Longrightarrow \lambda_j H' U_1' U_j' \notin G(m_{S'})^p,\ 1 \leq j \leq 3.$$

If $\tau(x_1) = 3$, we take $\mathcal{Y}_1 := \{x_1\}$ in (7.1) and obtain $\iota(x_2) \leq (p,2,1)$. We conclude by analyzing the cases $\tau(x_1) \leq 2$. By [27] **II.1.5** p.1888, this implies that $\lambda_1 = 0$. Therefore $\lambda_2 \neq 0$, since $(\lambda_1, \lambda_2) \neq (0,0)$. It is then obvious that $\tau(x_1) \leq 2$ implies that $\lambda_3 = 0$ and we get

$$H'^{-1}F_{p,Z'} = U_2'(\lambda_2 U_1' + \mu U_3'),\ \lambda_2 \mu \neq 0. \tag{7.4}$$

By lemma 7.3 below with $(i,\omega) = (1,2)$, we have $p \geq 3$ and

$$d_2 + 1/p \in \mathbb{N},\ d_1, d_3' \notin \mathbb{N} \text{ and } \widehat{pd_1} + \widehat{pd_3'} + 1 = p. \tag{7.5}$$

Let $\mathcal{Y}_1 := (Z', u_1', u_3')$, $y_1 \in \mathcal{X}_1$ be its generic point and $W_1 := (u_1', u_3')$. For i, $1 \leq i \leq p-1$, consider a finite monomial expansion (2.4):

$$f_{i,Z} = \sum_{\mathbf{a} \in \mathbf{S}(f_{i,Z})} \gamma(\mathbf{a}) u_1^{ia_1} u_2^{ia_2} u_3^{ia_3} \in S,\ \mathbf{S}(f_{i,Z}) \subset \Delta_S(h; u_1, u_2, u_3; Z).$$

The polyhedron assumption on h gives

$$\mathbf{a} \in \mathbf{S}(f_{i,Z}) \Longrightarrow (a_1 \geq d_1,\ a_2 + a_3 \geq d_2 + d_3 + \frac{3}{p})$$

and that at least one of these inequalities is strict. Now $f_{i,Z'} = u_3^{-i} f_{i,Z}$ and one deduces that

$$\frac{\mathrm{ord}_{W_1} f_{i,Z'}}{i} = \min_{\mathbf{a} \in \mathbf{S}(f_{i,Z})} \{2a_1 + a_2 + a_3 - 1\} > d_1 + d_3'. \tag{7.6}$$

By (7.5), we have $i(d_1 + d_3' + 1) \in \mathbb{N}$, so (7.6) actually implies that

$$\frac{\mathrm{ord}_{W_1} f_{i,Z'}}{i} \geq d_1 + d_3' + 1,$$

since $1 \leq i \leq p-1$ and $\mathrm{ord}_{W_1} f_{i,Z'} \in \mathbb{N}$. On the other hand, $\mathrm{ord}_{W_1} f_{p,Z'} = p(d_1 + d_3') + 1 \geq p$ and we deduce that $\mathcal{Y}_1$ is Hironaka-permissible w.r.t. E' with $\epsilon(y_1) = 1$. Arguing as before, on gets $\omega(x_2) \leq \epsilon(x_2) \leq 1$ (resp. $\iota(x_2) \leq (p,2,1)$) if the residue $\mu' \in k(x_2)$ of u_3'/u_1' does not satisfy (resp. satisfies) $\lambda_2 + \mu\mu' = 0$. This concludes the proof. $\square$

The following lemma extends the previous result when $\omega(x) = 1$.

Lemma 7.2. *Lemma 7.1 remains valid when* $\epsilon(x) = \omega(x) = 1$, $\kappa(x) \geq 2$ *and* $\mathrm{div}(u_1) \subseteq E \subseteq \mathrm{div}(u_1 u_2)$, *all other assumptions being otherwise unchanged.*

Proof. Let $\mathcal{Y} := V(Z, u_1, u_2) \subset \mathcal{X}$ and y be its generic point. Arguing as in (7.2) above, any vertex of $\Delta_S(h; u_1, u_2, u_3; Z)$ is induced by $f_{p,Z}$. By proposition 2.4, we have $\delta(y) = d_1 + d_2 + \frac{1}{p} = \delta(x)$, since $H^{-1}F_{p,Z} =< U_1 >$. Then proposition 2.3(ii) implies that

$$(m(y'), \epsilon(y')) = (m(x'), \epsilon(x')) = (p, 1).$$

Therefore $\mathcal{Y}$ is permissible of the first kind and we take $\mathcal{Y}_0 := \mathcal{Y}$ in (7.1). By theorem 3.6, we have $\iota(x_1) \leq (p, \omega(x), 1)$ unless

$$x_1 = x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3),\ E' := \mathrm{div}(u_1' u_2).$$

By proposition 2.6, $\Delta_{S'}(h'; u_1', u_2, u_3; Z')$ is minimal. We deduce that $H(x') = ({u_1'}^{pd_1} u_2^{p(d_1+d_2-1)+1})$ and $\mathbf{v}' := (d_1, d_1 + d_2 - 1 + 1/p, 1/p)$ is a vertex of $\Delta_{S'}(h'; u_1', u_2, u_3; Z')$. Therefore $(m(x'), \omega(x')) \leq (p, 0)$ and the lemma holds. $\square$

Given an integer $\alpha \in \mathbb{N}$, we denote by $\widehat{\alpha} \in \{0, \ldots, p-1\}$ the remainder of α modulo p. The following elementary lemma is useful.

Lemma 7.3. *Let* $(i, \omega) \in \mathbb{N}^2$ *satisfy* $0 < i < \omega$ *and* $F_0 \in k(x)[U_1, U_2]_i$, $F_0 \neq 0$. *Take* $E := \mathrm{div}(u_1 u_2 u_3)$ *and let*

$$(a(1), a(2), a(3)) \in \mathbb{N}^3,\ H := U_1^{a(1)} U_2^{a(2)} U_3^{a(3)} \in G(m_S) = k(x)[U_1, U_2, U_3].$$

We define $F := HU_3^{\omega - i} F_0$; *assume that* $F \notin G(m_S)^p$ *and that*

$$< U_3, U_j > \not\subseteq \mathrm{Vdir}(J(F, E, m_S)) \text{ for } j = 1 \text{ and } j = 2.$$

Then

$$\mathrm{Vdir}(J(F, E, m_S)) =< U_3, U_1 + \lambda U_2 >,\ \lambda \neq 0, \tag{7.7}$$

and the following holds:

(i) if $i \equiv 0 \bmod p$, *there exists* $0 \neq c \in k(x)$ *such that*

$$F - cHU_3^{\omega - i}(U_1 + \lambda U_2)^i \in G(m_S)^p;$$

(ii) if $i \not\equiv 0 \bmod p$, let $a_j := \widehat{a(j)}$, $1 \leq j \leq 3$, and $a := \widehat{i} \neq 0$. Then:

$$a_3 + \omega - a \equiv 0 \bmod p,\ a_1 a_2 \neq 0 \text{ and } a_1 + a_2 + a = p. \tag{7.8}$$

In particular $p \geq 3$. There exists $0 \neq c \in k(x)^p$ such that

$$F - cU_3^{a(3)+\omega-i}\Phi_i(U_1, \lambda U_2) \in G(m_S)^p,$$

where

$$\Phi_i(U_1, U_2) := (-1)^{a_2} U_1^{a(1)} U_2^{a(2)} \sum_{k=0}^{a} \begin{pmatrix} a_2 + k - 1 \\ k \end{pmatrix} U_1^{a-k}(U_1 + U_2)^{i-a+k}. \tag{7.9}$$

Proof. [27] **II.5** p.1896 for (i) and (7.8). There remains to prove that there exists $0 \neq c \in k(x)^p$ such that

$$H_0 F_0 - c\Phi_i(U_1, \lambda U_2) \in (k(x)[U_1, U_2])^p,\ H_0 := U_1^{a(1)} U_2^{a(2)}.$$

It is easily checked that (7.7) holds when $F = U_3^{a(3)+\omega(x)-i}\Phi_i(U_1, \lambda U_2)$. Note that

$$H_0^{-1}\Phi_i(U_1, \lambda U_2) = (-1)^{a_2}\lambda^{a(2)} \begin{pmatrix} a_2 + a \\ a \end{pmatrix} U_1^i + \cdots . \tag{7.10}$$

Let $(\lambda_l)_{l \in \Lambda}$ be an absolute p-basis of $k(x)$ and let

$$D_l := \frac{\partial}{\partial \lambda_l}\ D_j := U_j \frac{\partial}{\partial U_j},\ j = 1, 2.$$

We expand

$$F_0 := \alpha U_1^i + \alpha_1 U_1^{i-1} U_2 + \cdots ,\ \alpha, \alpha_1 \in k(x). \tag{7.11}$$

Since $H_0^{-1} D_l \cdot (H_0 F_0) \in < (U_1 + \lambda U_2)^i >$ by (7.7), $l \in \Lambda \cup \{1, 2\}$, it is easily seen that $\alpha \neq 0$.

Suppose that $\alpha \in k(x)^p$. Since $H_0^{-1} D_l \cdot (H_0 F_0) \in < (U_1 + \lambda U_2)^i >$, $l \in \Lambda$, and this polynomial is divisible by U_2, we have $D_l \cdot H_0 F_0 = 0$ for $l \in \Lambda$ by (7.7). We deduce that $H_0 F_0 \in k(x)^p[U_1, U_2]$ and in particular that $\lambda \in k(x)^p$. Let

$$F' := H_0 F_0 - c\Phi_i(U_1, \lambda U_2),\ c := \alpha(-1)^{a_2}\lambda^{-a(2)} \begin{pmatrix} a_2 + a \\ a \end{pmatrix}^{-1} \in k(x)^p.$$

By construction, we have $H^{-1}D_l \cdot F' = 0$, $l \in \Lambda \cup \{1,2\}$, and (ii) is proved.

Suppose that $\alpha \notin k(x)^p$. Without loss of generality, it can be assumed that $\alpha = \lambda_l$ for some $l \in \Lambda$. For $l' \neq l$, U_2 divides $H_0^{-1}D_{l'} \cdot (H_0F_0)$, so $D_{l'} \cdot (H_0F_0) = 0$ by (7.7). This proves that $F_0 \in k(x)^p(\alpha)[U_1, U_2]$. We have

$$\begin{cases} H_0^{-1}D_l \cdot (H_0F_0) &= U_1^i + (D_l \cdot \alpha_1)U_1^{i-1}U_2 + \cdots \\ \\ H_0^{-1}D_1 \cdot (H_0F_0) &= (a_1 + a)\alpha U_1^i + (a_1 + a - 1)\alpha_1 U_1^{i-1}U_2 + \cdots \end{cases}$$

from which we deduce the identity

$$\begin{cases} a\lambda &= D_l \cdot \alpha_1 \\ \\ a(a_1 + a)\alpha\lambda &= (a_1 + a - 1)\alpha_1 \end{cases} . \tag{7.12}$$

Therefore $(a_1 + a - 1)\alpha_1 = (a_1 + a)(D_l \cdot \alpha_1)$. Expanding $\alpha_1 =: \sum_{j=0}^{p-1} c_j^p \alpha^j$, we then deduce that

$$\alpha_1 = c_j^p \alpha^j, \text{ where } (a_1 + a)j \equiv a_1 + a - 1 \bmod p. \tag{7.13}$$

Since $a_1 + a + a_2 = p$ in this case (ii), we get $a_2(j-1) \equiv 1 \bmod p$ from (7.13). One deduces from (7.12)-(7.13) that $\alpha = d\lambda^{a(2)}$ for some $d \in k(x)^p$, $d \neq 0$. The proof now concludes as in the above case $\alpha \in k(x)^p$. □

Lemma 7.4. *Assume that $E = \mathrm{div}(u_1)$. If $(u_1, u_2, u_3; Z)$ are well adapted coordinates at x, then*

$$\mathrm{in}_E h = Z^p + U_1^{pd_1}\overline{F} \in S/(u_1)[U_1][Z],\ \overline{F} \neq 0.$$

Proof. This is obvious if $\mathrm{char} S = p > 0$ and h is purely inseparable (case (c) of assumption **(G)**). Otherwise, **(E)** implies that $\mathrm{Disc}_Z(h) = \gamma u_1^D$ for some $D \geq p(p-1)d_1$ and $\gamma \in S$ a unit. Let

$$\mathrm{in}_E h = Z^p + \sum_{i=1}^{p} U_1^{id_1} F_i Z^{p-i},\ F_i \in S/(u_1)[U_1]_{id_1},$$

where $F_i = 0$ if $id_1 \notin \mathbb{N}$. Since $\mathrm{char} S/(u_1) = p > 0$, condition **(G)** implies that $\mathrm{in}_E h$ has p distinct roots over an algebraic closure of $k(E)$ if $F_i \neq 0$ for some $i \neq p$. But then $D = p(p-1)d_1$: a contradiction since $\epsilon(x) > 0$. We deduce that $F_i = 0$, $1 \leq i \leq p-1$. We have $F_p \neq 0$ by proposition 2.4.

Proposition 7.5. *Assume that* $\epsilon(x) = \omega(x)$, $\kappa(x) \geq 2$ *and* $E = \mathrm{div}(u_1)$. *Let* $(u_1, u_2, u_3; Z)$ *be well adapted coordinates at* x. *Assume furthermore that* $S/(u_1) \simeq k(x)[\overline{u}_2, \overline{u}_3]_{(\overline{u}_2, \overline{u}_3)}$ *and the following two conditions are satisfied:*

(i) the initial form polynomial $\mathrm{in}_E h$ *of lemma 7.4 is of the form*

$$\mathrm{in}_E h = Z^p + U_1^{pd_1}\overline{F},\ \ \overline{F} \in k(x)[\overline{u}_2, \overline{u}_3]_{1+\omega(x)};$$

(ii) we have

$$\overline{\mathrm{Vdir}(x)} + \mathrm{Vdir}\left(\frac{\partial \overline{F}}{\partial \overline{u}_2}, \frac{\partial \overline{F}}{\partial \overline{u}_3}\right) =< \overline{U}_2, \overline{U}_3 >,$$

where $\overline{\mathrm{Vdir}(x)}$ *denotes the image of* $\mathrm{Vdir}(x)$ *in* $< \overline{U}_2, \overline{U}_3 >$.

Then x *is resolved.*

Proof. The proof is the same as that of [27] **II.3** p.1890 and we only indicate the necessary changes. Since $\kappa(x) \geq 2$, we have

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z},\ \ H^{-1}F_{p,Z} \subseteq k(x)[U_1, U_2, U_3]_{\omega(x)} \tag{7.14}$$

and $U_1 \in \mathrm{Vdir}(x)$ as in the beginning of the proof of lemma 7.1. We discuss according to the value of $\tau'(x)$.

• *Assume that* $\tau'(x) = 3$. The proposition follows from theorem 3.6.

• *Assume that* $\tau'(x) = 2$. Note that $\omega(x) \geq 2$. Since $E = \mathrm{div}(u_1)$ and $U_1 \in \mathrm{Vdir}(x)$, we have $\mathrm{Vdir}(x) =< U_1, \lambda_2 U_2 + \lambda_3 U_3 >$, $(\lambda_2, \lambda_3) \neq (0, 0)$. By symmetry, it can be assumed that $\lambda_2 = 1$. If $\lambda_3 \neq 0$, we let $v_2 := u_2 + \gamma_3 u_3$, where $\gamma_3 \in S$ is a preimage of $\lambda_3 \in S/(u_1) \simeq k(x)[\overline{u}_2, \overline{u}_3]_{(\overline{u}_2, \overline{u}_3)}$.

Let $(u_1, v_2, u_3; Z_1)$ be well adapted coordinates at x, $Z_1 = Z - \phi$, $\phi \in S$. By lemma 7.4, we have $\mathrm{ord}_{u_1}\phi > d_1$. Therefore

$$\mathrm{in}_E h = Z_1^p + U_1^{pd_1}(\overline{F} + \overline{\phi}^p),$$

where $\overline{\phi} = 0$ (resp. $\overline{\phi} = cl_0(u_1^{-d_1}\phi) \in S/(u_1)$) if $d_1 \notin \mathbb{N}$ (resp $d_1 \in \mathbb{N}$). Without loss of generality, it can be assumed that $(1+\omega(x) \equiv 0 \bmod p$ and $\overline{\phi} \in k(x)[\overline{u}_2, \overline{u}_3]_{(1+\omega(x))/p})$ if $\overline{\phi} \neq 0$. Assumptions (i) and (ii) are then unchanged, so it can be assumed w.l.o.g. that $\mathrm{Vdir}(x) =< U_1, U_2 >$. Assumption (ii) now implies

$$\overline{F}(\overline{u}_2, \overline{u}_3) \notin < \overline{u}_2^{1+\omega(x)} > \ \ (\text{resp. } \overline{F}(U_2, U_3) \notin < \overline{u}_2^{1+\omega(x)}, \overline{u}_3\overline{u}_2^{\omega(x)} >)$$

if $\omega(x) \not\equiv 0 \bmod p$ (resp. if $\omega(x) \equiv 0 \bmod p$).

Let $\mathcal{X}' \longrightarrow (\mathcal{X}, x)$ be the blowing along x and $x' \in \mathcal{X}'$ be the center of μ. By theorem 3.6, we have $\iota(x') \leq (p, \omega(x), 1)$ except possible if $x' = (Z' := Z/u_1, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3)$, since $\mathrm{Vdir}(x) =< U_1, U_2 >$. If then $\iota(x') \geq (p, \omega(x), 2)$, there are two cases to be considered as in [27] end of p.1891:

Case 1: $\overline{F}(\overline{u}_2, \overline{u}_3) = \lambda_0 \overline{u}_2^{1+\omega(x)} + \lambda_1 \overline{u}_3 \overline{u}_2^{\omega(x)}$, $\lambda_1 \neq 0$. Then $(\mathcal{X}', x')$ satisfies the assumption of lemma 7.1 (instead of *ibid.* **II.1** on p.1885) whose conclusion proves the proposition.

Case 2: $\overline{F}(\overline{u}_2, \overline{u}_3) = \lambda_0 \overline{u}_2^{1+\omega(x)} + \lambda_1 \overline{u}_3 \overline{u}_2^{\omega(x)} + \lambda_2 \overline{u}_3^2 \overline{u}_2^{\omega(x)-1}$, $\lambda_2 \neq 0$ and $\omega(x) - 1 \equiv 0 \bmod p$. Then $\tau(x') = 3$ by the characteristic free *ibid.* lemma **II.3.3** on p.1892. Blowing up again x' then gives $\iota(x'') \leq (p, \omega(x), 1)$ by theorem 3.6, where x'' is the center of μ.

• *Assume that* $\tau'(x) = 1$. We have $\mathrm{Vdir}(x) = k(x)U_1$ and $F_{p,Z} = \lambda U_1^{pd_1+\omega(x)}$ in (7.14). Assumption (ii) now reads

$$\mathrm{Vdir}\left(\frac{\partial \overline{F}}{\partial \overline{u}_2}, \frac{\partial \overline{F}}{\partial \overline{u}_3}\right) =< \overline{U}_2, \overline{U}_3 > . \tag{7.15}$$

Let $\mathcal{X}' \longrightarrow (\mathcal{X}, x)$ be the blowing along x and $x' \in \mathcal{X}'$ be the center of μ. By theorem 3.6, we have $\iota(x') \leq (p, \omega(x), 1)$ except possible if $\eta'(x')$ lies on the strict transform of $\mathrm{div}(u_1)$. By symmetry on $\overline{u}_2, \overline{u}_3$, it can be assumed that $x' = (Z' := Z/u_1, u_1' := u_1/u_2, u_2, u_3' := P(u_3/u_2))$, where $P(t) \in S[t]$ is a unitary polynomial whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible. We have $E' = \mathrm{div}(u_1' u_2)$. Let

$$\tilde{P}(\overline{U}_2, \overline{U}_3) := \overline{U}_2^{\deg \overline{P}} \overline{P}(\overline{U}_3/\overline{U}_2) \in k(x)[\overline{U}_2, \overline{U}_3]_{\deg \overline{P}}.$$

By (7.15), we have

$$a := \mathrm{ord}_{\tilde{P}}(\frac{\partial \overline{F}}{\partial \overline{u}_2}, \frac{\partial \overline{F}}{\partial \overline{u}_3}) \leq \omega(x) - 1.$$

If $\iota(x') \geq (p, \omega(x), 2)$, there are two cases to be considered as in [27] p.1894:

Case 1: $a = \omega(x) - 1$. If $\omega(x) \geq 2$, this implies that $k(x') = k(x)$. Arguing as in the above case $\tau'(x) = 2$, it can be assumed that $P(t) = t$. Then $(\mathcal{X}', x')$ satisfies the assumption of lemma 7.1, whose conclusion proves the proposition.

If $\omega(x) = 1$, then $(\mathcal{X}', x')$ satisfies the assumption of lemma 7.2 or there is an expansion

$$\mathrm{in}_{m_{S'}} h' = Z'^p + U_1'^{pd_1} U_2'^{p(d_1-1)+1}(\lambda_1' U_1' + \lambda_2' U_2') \in G(m_{S'})[Z'] \tag{7.16}$$

with $\lambda_1'\lambda_2' \neq 0$, where $(u_1', u_2', u_3'; Z')$ are well adapted coordinates at x'. With notations as in lemma 7.3, we let $a_1 := \widehat{pd_1}$.

Assume in addition that $p = 2$, or that $a_1 \neq (p-1)/2$. We have $\mathrm{Vdir}(x') =< U_1', U_2' >$ by lemma 7.3. Let $\mathcal{Y}' := V(Z', u_1', u_2') \subset \mathcal{X}'$ with generic point y'. By (7.16), any vertex of $\Delta_{S'}(h'; u_1', u_2', u_3'; Z')$ is induced by $f_{p,Z'}$ and we have $\delta(y') = 2d_1 - 1 + 2/p = \delta(x')$, so $\mathcal{Y}'$ is permissible of the first kind at x'. Blowing up $\mathcal{Y}'$ then gives $\iota(x'') \leq (p, \omega(x), 1)$ by theorem 3.6, where x'' is the center of μ.

Assume now that $p \geq 3$ and $a_1 = (p-1)/2$. If $d_1 \geq 1$, the centers $\mathcal{Y}_j' := V(Z', u_j')$, $j = 1, 2$ are Hironaka permissible w.r.t. E'. Blowing up consecutively $\mathcal{Y}_1'$, then $\mathcal{Y}_2'$, and iterating, we reduce to the case $d_1 = a_1/p < 1$. Blowing up again $\mathcal{Y}'$ as above then gives $m(x'') \leq 2 < p$, where x'' is the center of μ and the conclusion follows.

Case 2: $a = \omega(x) - 2$. Then $\omega(x) - 1 \equiv 0 \bmod p$ and there is an expansion

$$\mathrm{in}_{m_{S'}} h' = Z'^p + U_1'^{pd_1} U_2'^{pd_1'}(U_1'\Phi(U_1', U_2, U_3') + \lambda' U_2' U_3'^{\omega(x)-1}) \in G(m_{S'})[Z']$$

with $d_1' := d_1 - 1 + \omega(x)/p$, $\lambda' \neq 0$, $\Phi \neq 0$, where $(u_1', u_2', u_3'; Z')$ are well adapted coordinates at x'. Then $\tau(x') = 3$ by the characteristic free *ibid.* lemma **II.3.3** on p.1892. Blowing up again x' then gives $\iota(x'') \leq (p, \omega(x), 1)$ by theorem 3.6, where x'' is the center of μ. □

Proposition 7.6. *Assume that $\epsilon(x) = \omega(x) \geq 2$, $\kappa(x) \geq 2$ and $E = \mathrm{div}(u_1)$. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x. Assume furthermore that the initial form polynomial $\mathrm{in}_E h = Z^p + U_1^{pd_1}\overline{F}$, $\overline{F} \in S/(u_1)$, of lemma 7.4 satisfies the following two conditions:*

(i) $\mathrm{ord}_{(\overline{u}_2, \overline{u}_3)} \overline{F} = \omega(x) + 1$;

(ii) the form $\Phi := \mathrm{cl}_{\omega(x)+1}\overline{F} \in k(x)[\overline{U}_2, \overline{U}_3]_{\omega(x)+1}$ *is such that*

$$\frac{\partial \Phi}{\partial \overline{U}_3} = 0 \text{ and } \mathrm{Vdir}(\frac{\partial \Phi}{\partial \overline{U}_2}) =< \overline{U}_2, \overline{U}_3 > .$$

Then x is resolved.

Proof. This is a simpler variation of proposition 7.5 and we build upon its proof. To begin with, let $(u_1, u_2, u_3'; Z')$ be another set of well adapted coordinates at x. There is an equality

$$U_3' = \lambda_3 U_3 + \lambda_2 U_2 + \lambda_1 U_1 \in G(m_S)_1 =< U_1, U_2, U_3 >, \ \lambda_3 \neq 0.$$

The corresponding initial form polynomial $\mathrm{in}_E h = Z'^p + U_1^{pd_1}\overline{F}'$ satisfies

$$\Phi' := \mathrm{cl}_{\omega(x)+1}\overline{F}' = \overline{F}(\overline{U}_2, \lambda_3^{-1}(\overline{U}_3' - \lambda_2\overline{U}_2)) + \Theta^p \in k(x)[\overline{U}_2, \overline{U}_3']_{\omega(x)+1},$$

where $\Theta \in k(x)[\overline{U}_2, \overline{U}_3']_{(\omega(x)+1)/p}$, $\Theta = 0$ if $d_1 \notin \mathbb{N}$ or if $\omega(x) + 1 \not\equiv 0 \bmod p$. We deduce that

$$\frac{\partial \Phi'}{\partial \overline{U}_3'} = 0 \text{ and } \mathrm{Vdir}(\frac{\partial \Phi}{\partial \overline{U}_2}) =< \overline{U}_2, \overline{U}_3' > . \tag{7.17}$$

In other terms, (i) and (ii) remains valid for the well adapted coordinates $(u_1, u_2, u_3'; Z')$.

Also note that no Φ satisfies (ii) when $\omega(x) + 1 \equiv 0 \bmod p$, since then

$$\frac{\partial \Phi}{\partial \overline{U}_3} = 0 \Longrightarrow \Phi \in k(x)[\overline{U}_2^p, \overline{U}_3^p] \Longrightarrow \frac{\partial \Phi}{\partial \overline{U}_2} = 0. \tag{7.18}$$

Let $\mathcal{X}' \longrightarrow (\mathcal{X}, x)$ be the blowing along x, $x' \in \mathcal{X}'$ be the center of μ and suppose that $\iota(x') \geq (p, \omega(x), 2)$. We discuss according to the values of $\tau'(x)$ as in the proof of proposition 7.5.

• *Assume that* $\tau'(x) = 3$. The proposition follows from theorem 3.6.

• *Assume that* $\tau'(x) = 2$. By (7.17), it can be assumed that $\mathrm{Vdir}(x) =< U_1, U_2 >$ or $\mathrm{Vdir}(x) =< U_1, U_3 >$. The polynomial assumption proposition 7.5 (i) on $\overline{F}$ is used only in cases 1 and 2 of the corresponding proof. Therefore under the assumptions of this proposition, it is sufficient to prove that

$$\begin{cases} \Phi \notin < \overline{U}_2^{1+\omega(x)}, \overline{U}_3\overline{U}_2^{\omega(x)} > & \text{if} \quad \mathrm{Vdir}(x) =< U_1, U_2 > \\ \\ \Phi \notin < \overline{U}_3^{1+\omega(x)}, \overline{U}_2\overline{U}_3^{\omega(x)} > & \text{if} \quad \mathrm{Vdir}(x) =< U_1, U_3 > \end{cases} \tag{7.19}$$

and that

$$\begin{cases} \Phi \notin < \overline{U}_2^{1+\omega(x)}, \overline{U}_3\overline{U}_2^{\omega(x)}, \overline{U}_3^2\overline{U}_2^{\omega(x)-1} > & \text{if} \quad \mathrm{Vdir}(x) =< U_1, U_2 > \\ \\ \Phi \notin < \overline{U}_3^{1+\omega(x)}, \overline{U}_2\overline{U}_3^{\omega(x)}, \overline{U}_2^2\overline{U}_3^{\omega(x)-1} > & \text{if} \quad \mathrm{Vdir}(x) =< U_1, U_3 > \end{cases} \tag{7.20}$$

if furthermore $\omega(x) - 1 \equiv 0 \bmod p$. By (ii), we have

$$\Phi \in k(x)[\overline{U}_2, \overline{U}_3^p] \backslash k(x)[\overline{U}_2] \text{ and } \frac{\partial \Phi}{\partial \overline{U}_2} \notin < \overline{U}_3^{\omega(x)} >$$

and (7.19) follows easily. Furthermore, (7.20) reduces to (7.19) except possibly if $p = 2$; but assumption (ii) then implies that $\omega(x) \equiv 0 \bmod 2$ by (7.18).

• *Assume that* $\tau'(x) = 1$. We have $\mathrm{Vdir}(x) =< U_1 >$. The polynomial assumption proposition 7.5 (i) on $\overline{F}$ is also used only in cases 1 and 2 of the corresponding proof.

If $k(x') = k(x)$, one is then reduced to proving (7.19)-(7.20) and the proof is identical as in (b).

If $[k(x') : k(x)] \geq 2$, the argument in [27] proof of **II.3** (cases 1 and 2 on p.1894) shows that $p = 2$, $\omega(x) = 3$ and $[k(x') : k(x)] = 2$; but assumption (ii) then implies $\omega(x) \equiv 0 \bmod 2$ by (7.18) and the conclusion follows. □

Proposition 7.7. *Assume that* $E = \mathrm{div}(u_1)$, $\epsilon(x) = \omega(x)$, $\kappa(x) = 2$ *and*

$$\mathrm{Vdir}(x) + k(x)U_1 =< U_1, U_2, U_3 > .$$

Then x *is good.*

Proof. This follows from theorem 3.6 if $\mathrm{Vdir}(x) =< U_1, U_2, U_3 >$, i.e. $\tau'(x) = 3$.

Assume that $\tau'(x) = 2$. Since $\mathrm{Vdir}(x)$ and $\iota(x)$ do not depend on the choice of well adapted coordinates, it can be assumed that $\mathrm{Vdir}(x) =< U_2, U_3 >$. Since $\kappa(x) = 2$, there is an expansion

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z},\ H^{-1} F_{p,Z} \subseteq k(x)[U_2, U_3]_{\omega(x)}.$$

Let μ be a valuation of $L = k(\mathcal{X})$ centered at x, $\mathcal{X}_1 \longrightarrow \mathcal{X}$ be the blowing up along x and $x_1 \in \mathcal{X}_1$ be the center of μ. By theorem 3.6, $\iota(x_1) < \iota(x)$ except possibly if $x_1 = x' := (Z' := Z/u_1, u_1, u_2' := u_2/u_1, u_3' := u_3/u_1)$, so $E' = \mathrm{div}(u_1)$ and $k(x') = k(x)$.

By proposition 2.6, $\Delta_{S'}(h'; u_1, u_2', u_3'; Z')$ is minimal. We deduce that $\epsilon(x') \leq \epsilon(x)$; if x_1 is very near x, we have $\epsilon(x_1) = \epsilon(x) = \omega(x_1)$ and

$$\mathrm{in}_{m_S} h = Z'^p - G'^{p-1} Z' + F_{p,Z'},\ H'^{-1} F_{p,Z'} \subseteq k(x)[U_1, U_2', U_3']_{\omega(x)}.$$

Moreover proposition 3.5(v) implies that

$$J(F_{p,Z'}, E', m_{S'}) \equiv U_1^{-\epsilon(x)} J(F_{p,Z}, E, m_S) \bmod U_1.$$

We deduce that $\kappa(x_1) = 1$ (so $\iota(x_1) < \iota(x)$) if $G' \neq 0$. Otherwise we have $\mathrm{Vdir}(x_1) \equiv < U_2', U_3' > \bmod U_1$, so x_1 satisfies again the assumptions of the proposition. The proposition then follows from corollary 3.9. □

Remark 7.1. All local blowing ups considered in this section are permissible of the first kind except when $p \geq 3$ and $\omega(x) \leq 2$ (proof of lemma 7.1 for $\omega(x) = 2$, proof of lemma 7.5 for $\omega(x) = 1$).

7.2 Reduction to monic expansions.

In this section, we further reduce the proof of the projection theorem to those points with $\kappa(x) = 2$ satisfying condition (*) below. To begin with, let $(u_1, u_2, u_3; Z)$ be well adapted coordinates and

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z] \tag{7.21}$$

be the corresponding initial form. If $\kappa(x) = 2$, we have $\mathrm{div}(u_1) \subseteq E \subseteq \mathrm{div}(u_1 u_2)$, $E = \mathrm{div}(u_1)$ if $\omega(x) = \epsilon(x) - 1$. We recall from definition 5.1 that $G = 0$ if $\omega(x) = \epsilon(x)$.

Definition 7.1. Assume that $\kappa(x) = 2$. We say that x satisfies condition (*) if there exists well adapted coordinates $(u_1, u_2, u_3; Z)$ such that one of the following properties are satisfied:

(i) $\omega(x) = \epsilon(x)$, $U_3 \in \mathrm{Vdir}(x)$ and $J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)}$ contains a unitary polynomial in U_3;

(ii) $\omega(x) = \epsilon(x) - 1$, $U_3 \in \mathrm{Vdir}(x)$ and $H^{-1} \frac{\partial F_{p,Z}}{\partial U_2}$ is (generated by) a unitary polynomial in U_3.

Condition (*) is labeled (*1) (resp. (*2)) if $E = \mathrm{div}(u_1)$ (resp. if $E = \mathrm{div}(u_1 u_2)$) when condition (i) holds. Condition (*) is labeled (*3) when condition (ii) holds.

Proposition 7.8. *Assume that $\kappa(x) = 2$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x and consider the quadratic sequence*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots$$

along μ. The following holds:

(i) there exists $r \geq 0$ such that x_r is resolved or ($\iota(x_r) = \iota(x)$ and x_r satisfies condition (*));

(ii) if x satisfies condition (*), then x_1 is resolved or ($\iota(x_1) = \iota(x)$ and x_1 satisfies again condition (*));

(iii) if $\omega(x) \not\equiv 0 \bmod p$, then x is good.

Proof. We first prove together (i) and (ii) by a casuistic analysis. The discussion goes according to the value of $\tau'(x)$ and subdivides in the different situations $\omega(x) = \epsilon(x)$ and $\omega(x) = \epsilon(x) - 1$.

• *Assume that $\tau'(x) = 3$.* Then $\iota(x_1) < \iota(x)$ by theorem 3.6, so x is good and there is nothing more to be proved.

• *Assume that $\tau'(x) = 1$ and $\omega(x) = \epsilon(x)$.* We may pick well adapted coordinates $(u_1, u_2, u_3; Z)$ such that $U_3 \in \mathrm{Vdir}(x)$, so

$$J(F_{p,Z}, E, m_S) =< U_3^{\omega(x)} > .$$

We deduce that $\omega(x) \equiv 0 \bmod p$ and x satisfies condition (*1) or (*2). This proves that (i) holds with $r = 0$.

To prove (ii), we may assume that $\iota(x_1) \geq \iota(x)$ (in particular $\omega(x_1) = \omega(x)$). There is an expansion (7.21) with

$$G = 0 \text{ and } U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = \lambda U_3^{\omega(x)},\ \lambda \neq 0. \tag{7.22}$$

By theorem 3.6, x_1 lies on the strict transform of $\mathrm{div}(u_3)$. Let

$$x' := (Z' := \frac{Z}{u_2}, u_1' := \frac{u_1}{u_2}, u_2, u_3' := \frac{u_3}{u_2}),\ E' = \mathrm{div}(u_1' u_2).$$

If $x_1 = x'$, then $\Delta_{S'}(h'; u_1', u_2, u_3'; Z')$ is minimal by proposition 2.6. One deduces from (7.22) that

$$\epsilon(x') = \omega(x) \text{ and } J(F_{p,Z'}, E', m_{S'}) \equiv< U_3^{\omega(x)} > \ \bmod(U_2) \cap G(m_{S'})_{\epsilon(x')}.$$

This proves that ($\iota(x') = \iota(x)$ and x' satisfies condition (*2)), so (ii) holds.

If $x_1 \neq x'$, there exists a unitary polynomial $P(t) \in S[t]$, whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible, such that

$$x_1 = (X' := \frac{Z}{u_1}, u_1, v_2 := P(u_2'), u_3' := \frac{u_3}{u_1}),\ u_2' := \frac{u_2}{u_1},\ E_1 = \mathrm{div}(u_1). \quad (7.23)$$

We have $S_1/(u_1) \simeq k(x)[\overline{u}_2', \overline{u}_3']_{(\overline{v}_2, \overline{u}_3')}$. Let $(u_1, v_2, u_3'; Z_1)$ be well adapted coordinates at x_1, where $Z_1 = X' - \phi_1$, $\phi_1 \in S_1$. Let $d_1' := d_1 + d_2 - 1 + \omega(x)/p$ and $c \in k(x_1)$ be the residue of u_2'. Note that we may furthermore assume that $P(t) \neq t$ if $E = \mathrm{div}(u_1 u_2)$ by symmetry on u_1 and u_2, i.e. $c^{pd_2} \neq 0$ (and $c^{pd_2} = 1$ if $c = 0$).

Case 1: $d_1' \notin \mathbb{N}$ or $\lambda c^{pd_2} \notin k(x_1)^p$. By (7.22), it can be assumed w.l.o.g. that $\mathrm{ord}_{(u_1)}\phi_1 > d_1'$. The initial form $\mathrm{in}_{E_1} h_1$ of lemma 7.4 is then of the form:

$$\mathrm{in}_{E_1} h_1 = {Z_1}^p + \lambda U_1^{pd_1' } {\overline{u}_2'}^{pd_2} {\overline{u}_3'}^{\omega(x)} \in S_1/(u_1)[U_1][Z_1].$$

We have $\epsilon(x_1) = \omega(x)$ and

$$J(F_{p,Z_1}, E_1, m_{S_1}) \equiv < U_3'^{\omega(x)} > \ \mathrm{mod}(U_1) \cap G(m_{S_1})_{\epsilon(x_1)}.$$

Therefore $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*1), so (ii) holds.

Case 2: $d_1' \in \mathbb{N}$ and $\lambda c^{pd_2} \in k(x_1)^p$. It can be assumed w.l.o.g. that

$$u_1^{-d_1'} \phi_1 \equiv \gamma_1 {u_3'}^{\frac{\omega(x)}{p}} \ \mathrm{mod}(u_1),$$

where $\gamma_1 \in S_1$ is a preimage of $-(\lambda c^{pd_2})^{1/p} \in k(x_1)$. Since $\Delta_S(h; u_1, u_2, u_3; Z)$ is minimal, we have

$$0 \neq d(\lambda U_1^{pd_1} U_2^{pd_2}) \in \Omega^1_{G(m_S)/\mathbb{F}_p}.$$

We deduce that $(u_1, v_2' := \gamma {u_2'}^{pd_2} + \gamma_1^p, u_3')$ is a r.s.p. of S_1, where $\gamma \in S$ is a preimage of λ. Let $(u_1, v_2', u_3'; Z_1')$ be well adapted coordinates at x_1, so the initial form $\mathrm{in}_{E_1} h_1$ of lemma 7.4 is now of the form:

$$\mathrm{in}_{E_1} h_1 = {Z_1'}^p + U_1^{pd_1'} \overline{v}_2' {\overline{u}_3'}^{\omega(x)} \in S_1/(u_1)[U_1][Z_1'].$$

If $\epsilon(x_1) = \omega(x)$, then x_1 satisfies the assumptions of lemma 7.1, so x_1 is resolved. Otherwise we have $\epsilon(x_1) = 1 + \omega(x)$ and

$$H'^{-1} \frac{\partial F_{p,Z_1'}}{\partial V_2} \equiv < U_3'^{\omega(x)} > \ \mathrm{mod}(U_1) \cap G(m_{S_1})_{\omega(x)}.$$

Then there exist well adapted coordinates of the form $(u_1, v_2', v_3; Z')$ at x_1 satisfying definition 7.1, so $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*3).

• *Assume that* $\tau'(x) = 1$ *and* $\omega(x) = \epsilon(x) - 1$. By definition 5.1, we then have $H^{-1}\frac{\partial F_{p,Z}}{\partial U_2} \neq (0)$, therefore

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_2} =< U_3^{\omega(x)} >, \tag{7.24}$$

so x satisfies condition (*3). This proves that (i) holds.

To prove (ii), we may assume that $\iota(x_1) \geq \iota(x)$. By (7.24), there is an expansion (7.21) with

$$G = 0,\ U_1^{-pd_1}F_{p,Z} = \lambda U_2 U_3^{\omega(x)} + \Phi_0(U_2^p, U_3^p) + U_1\Phi(U_1, U_2^p, U_3^p), \lambda \neq 0. \tag{7.25}$$

This furthermore implies that $\omega(x) \equiv 0 \bmod p$, so $\Phi_0 = 0$. By theorem 3.6, x_1 lies on the strict transform of $\mathrm{div}(u_3)$. Note that we may furthermore assume that

$$\lambda = 1 \text{ and } \deg_{U_3}\Phi(U_1, U_2^p, U_3^p) < \omega(x) \tag{7.26}$$

in (7.25): this is achieved by possibly changing u_2 to $\gamma_0 u_2 + \gamma u_1$, $\gamma_0\gamma \in S$ a unit, then picking again well prepared coordinates. Let

$$x' := (Z' := \frac{Z}{u_2}, u_1' := \frac{u_1}{u_2}, u_2, u_3' := \frac{u_3}{u_2}),\ E' = \mathrm{div}(u_1' u_2).$$

If $x_1 = x'$, the proof is identical as when $\omega(x) = \epsilon(x)$: one gets ($\iota(x') = \iota(x)$ and x' satisfies condition (*2)), so (ii) holds.

If $x_1 \neq x'$, we let $d_1' := d_1 - 1 + (1 + \omega(x))/p$ and use notations as when $\omega(x) = \epsilon(x)$. We have $E_1 = \mathrm{div}(u_1)$ and consider three cases.

Case 1: $d_1' \notin \mathbb{N}$. By (7.25), $\mathrm{ord}_{(u_1)}\phi_1 > d_1'$.

If $x_1 = x_1' := (Z_1' := Z/u_1, u_1, u_2' := u_2/u_1, u_3' := u_3/u_1)$, we have $\Phi \in k(x)[U_2^p, U_3^p]$ and the initial form $\mathrm{in}_{E_1}h_1$ of lemma 7.4 is of the form:

$$\mathrm{in}_{E_1}h_1 = Z_1'^p + U_1^{pd_1'}(\overline{u}_2'\overline{u}_3'^{\omega(x)} + \Phi(\overline{u}_2'^p, \overline{u}_3'^p)) \in S_1/(u_1)[U_1][Z_1].$$

If $\Phi = 0$, we either have $\epsilon(x_1') = \omega(x)$, so x_1' satisfies the assumptions of lemma 7.1 and x_1 is resolved; or $\epsilon(x_1') = 1 + \omega(x)$ and

$$H'^{-1}\frac{\partial F_{p,Z_1'}}{\partial U_2'} \equiv< U_3'^{\omega(x)} > \ \bmod (U_1) \cap G(m_{S_1})_{\omega(x)}.$$

Then $\iota(x_1') = \iota(x)$ and x_1' satisfies condition (*3).

If $\Phi \neq 0$, we have $\epsilon(x_1') = \omega(x)$ and

$$H'^{-1} F_{p,Z_1'} \equiv < \Phi(U_2'^p, U_3'^p) > \ \mathrm{mod}(U_1) \cap G(m_{S_1})_{\omega(x)}.$$

If U_2U_3 divides Φ, then x_1 is good by proposition 7.7; otherwise Φ is monic in U_2 or in U_3, so $\iota(x_1') = \iota(x)$ and x_1' satisfies condition (*1).

If $x_1 \neq x_1'$, then $\epsilon(x_1) = \omega(x)$, $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*1).

Case 2: $d_1' \in \mathbb{N}$ and $c \notin k(x_1)^p$. With notations as in (7.23) *sqq.*, we get $\epsilon(x') = \omega(x)$ and

$$H'^{-1} F_{p,Z_1} \equiv < cU_3'^{\omega(x)} + \Phi_1(U_2'^p, U_3'^p) > \ \mathrm{mod}(U_1) \cap G(m_{S_1})_{\omega(x)},$$

where $\deg_{U_3'} \Phi_1(U_2'^p, U_3'^p) < \omega(x)$ by (7.26). Therefore $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*1).

Case 3: $d_1' \in \mathbb{N}$ and $c \in k(x_1)^p$. It can be assumed w.l.o.g. that

$$u_1^{-d_1'} \phi_1 \equiv \gamma_1 u_3'^{\frac{\omega(x)}{p}} + \sum_{i=1}^{\frac{\omega(x)}{p}} \psi_i u_3'^{\frac{\omega(x)}{p} - i} \ \mathrm{mod}(u_1),$$

where $\gamma_1 \in S_1$ is a preimage of $c^{1/p} \in k(x_1)$ and

$$\overline{\psi}_i \in k(x)[\overline{u}_2']_{(\overline{v}_2)} \subset S_1/(u_1), \ 1 \leq i \leq \frac{\omega(x)}{p}.$$

Then $(u_1, v_2' := u_2' + \gamma_1^p, u_3')$ is a r.s.p. of S_1 (*viz.* above $\omega(x) = \epsilon(x)$, case 2). Let $(u_1, v_2', u_3'; Z_1')$ be well adapted coordinates. We have

$$\mathrm{in}_{E_1} h_1 = Z_1'^p + U_1^{pd_1'} (\overline{v}_2' \overline{u}_3'^{\omega(x)} + \Psi(\overline{u}_2', \overline{u}_3')) \in S_1/(u_1)[U_1][Z_1'],$$

where $\Psi(\overline{u}_2', \overline{u}_3') \in k(x)[\overline{u}_2']_{(\overline{v}_2)}[\overline{u}_3']$, $\mathrm{ord}_{(\overline{v}_2', \overline{u}_3')} \Psi \geq \omega(x)$. Since $\omega(x') = \omega(x)$, we have

$$\overline{\Psi} := \mathrm{cl}_{\omega(x)} \Psi(\overline{u}_2', \overline{u}_3') \in (\overline{V}_2'^p k(x')[\overline{V}_2'^p, \overline{U}_3'^p])_{\omega(x)}. \tag{7.27}$$

If ($\epsilon(x_1) = \omega(x)$ and $\overline{\Psi} = 0$), then x_1 satisfies the assumptions of lemma 7.1, so x_1 is resolved.

If ($\epsilon(x_1) = \omega(x)$ and $0 \neq \overline{\Psi} \in < \overline{V}_2'^{\omega(x)} >$), we have

$$J(F_{p,Z_1'}, E_1, m_{S_1}) \equiv < V_2'^{\omega(x)} > \ \mathrm{mod}(U_1) \cap G(m_{S_1})_{\omega(x)}.$$

Therefore $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*1).

If ($\epsilon(x_1) = \omega(x)$ and $\overline{\Psi} \notin < \overline{V}_2'^{\omega(x)} >$), we have

$$\kappa(x') = 2 \text{ and } \mathrm{Vdir}(x') + k(x')U_1 =< U_1, V_2', U_3' >,$$

so x_1 is good by lemma 7.7.

If $\epsilon(x_1) = 1 + \omega(x)$, we have

$$H'^{-1}\frac{\partial F_{p,Z_1'}}{\partial V_2'} \equiv < U_3'^{\omega(x)} > \ \mathrm{mod}(U_1, V_2') \cap G(m_{S_1})_{\omega(x_1)}.$$

Then there exist well adapted coordinates of the form $(u_1, v_2', v_3; Z')$ at x_1 satisfying definition 7.1, so $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*3). This concludes the proof of (ii) when $\tau'(x) = 1$.

• *Assume that* $\tau'(x) = 2$. Up to a change of well adapted coordinates, it is easily seen that x belongs to one of the following types:

(T0) $\omega(x) = \epsilon(x)$, $E = \mathrm{div}(u_1)$ and $\mathrm{Vdir}(x) =< U_3, U_2 >$;
(T1) $\omega(x) = \epsilon(x) - 1$ and $\mathrm{Vdir}(x) =< U_3, U_2 >$;
(T2) $\omega(x) = \epsilon(x)$, $E = \mathrm{div}(u_1u_2)$ and $\mathrm{Vdir}(x) =< U_3, U_1 + \lambda U_2 >$ with $\lambda \neq 0$;
(T3) $\omega(x) = \epsilon(x)$ and $\mathrm{Vdir}(x) =< U_3, U_1 >$;
(T4) $\omega(x) = \epsilon(x) - 1$ and $\mathrm{Vdir}(x) =< U_3, U_1 >$.

Claim: suppose x is of type (Tk), $0 \leq j \leq 4$. Then x_1 is resolved or one of the following properties hold:

(a) $\iota(x_1) = \iota(x)$ and x_1 satisfies condition (*);

(b) $\iota(x_1) = \iota(x)$, $\tau'(x_1) = 2$ and x_1 is of type (Tl) with $l \leq k$.

If moreover x satisfies condition (*), then x_1 is resolved or (a) holds.

To prove the claim, we do a case by case analysis. If $k = 0$, then x is good by proposition 7.7.

Assume that $k = 1$. There is an expansion (7.21) with

$$G = 0 \text{ and } U_1^{-pd_1}F_{p,Z} = F_{1+\omega(x)}(U_2, U_3) + \sum_{i=1}^{1+\omega(x)} F_{1+\omega(x)-i}(U_2, U_3)U_1^i.$$

Since $\mathrm{Vdir}(x) =< U_2, U_3 >$, we have

$$\begin{cases} \mathrm{Vdir}\left(\frac{\partial F_{1+\omega(x)}}{\partial U_2}, \frac{\partial F_{1+\omega(x)}}{\partial U_3}\right) =< U_2, U_3 > \\ \\ F_{1+\omega(x)-i}(U_2, U_3) \in k(x)[U_2^p, U_3^p],\ 1 \leq i \leq 1+\omega(x) \end{cases}. \tag{7.28}$$

Assume that $\iota(x') \geq \iota(x)$. By theorem 3.6, $x_1 = x'$, where

$$x' := (Z' := Z/u_1, u_1, u_2' := u_2/u_1, u_3' := u_3/u_1).$$

We have

$$E' = \mathrm{div}(u_1), S'/(u_1) \simeq k(x)[\overline{u}_2', \overline{u}_3']_{(\overline{u}_2', \overline{u}_3')} \text{ and } H(x') = (u_1^{p(d_1-1)+1+\omega(x)}).$$

Assume that $\iota(x') \geq \iota(x)$. By proposition 2.6, $\Delta_{S'}(h'; u_1, u_2', u_3'; Z')$ is minimal. The initial form $\mathrm{in}_{E'}h'$ of lemma 7.4 is of the form:

$$\mathrm{in}_{E'}h' = Z'^p + U_1^{p(d_1-1)+1+\omega(x)} \left(F_{1+\omega(x)}(\overline{u}_2', \overline{u}_3') + \sum_{i=1}^{1+\omega(x)} F_{1+\omega(x)-i}(\overline{u}_2', \overline{u}_3') \right).$$

This proves that $F_i(U_2, U_3) = 0$, $2 \leq i \leq 1+\omega(x)$. We consider two cases:
Case 1: $F_{\omega(x)}(U_2, U_3) = 0$. If $\epsilon(x') = \omega(x)$, then x' satisfies all assumptions of proposition 7.5 by (7.28), so x is good.

If $\epsilon(x') = \epsilon(x)$, then $\iota(x') = \iota(x)$ and

$$H'^{-1}\frac{\partial F_{p,Z'}}{\partial U_j'} \equiv < \frac{\partial F_{1+\omega(x)}}{\partial U_j}(U_2', U_3') > \ \mathrm{mod}(U_1) \cap G(m_{S'})_{\omega(x)},$$

for $j = 2, 3$ again by (7.28). We conclude that $\tau'(x') = 3$ (so x is good) or x' is again of type (T1) as required. If x satisfies condition (*), so does x'.
Case 2: $F_{\omega(x)}(U_2, U_3) \neq 0$. We have $\epsilon(x') = \omega(x)$ and

$$\mathrm{in}_{m_{S'}}h' = Z'^p + U_1^{p(d_1-1)+1+\omega(x)}(F_{\omega(x)}(U_2', U_3') + U_1\Phi'),\ \Phi' \in k(x')[U_1, U_2'^p, U_3'^p].$$

Therefore $\iota(x') = \iota(x)$. If $F_{\omega(x)}(U_2, U_3)$ is monic in U_2 or in U_3, then x' satisfies condition (*1). Otherwise x' is of type (T0) and the conclusion follows.

Note that if $\omega(x) = 1$, x is of type (T1) and satisfies condition (*3). So we may assume from this point on that $\omega(x) \geq 2$.

Assume that $k = 2$. There is an expansion (7.21) with $G = 0$ and

$$F_{p,Z} = U_1^{pd_1} U_2^{pd_2} \sum_{i=0}^{\omega(x)} F_i(U_1, U_2) U_3^{\omega(x)-i}.$$

Note that $F_i(U_1, U_2) = 0$ whenever $\omega(x) - i \not\equiv 0 \bmod p$, since $\omega(x) = \epsilon(x)$; we have $F_i \neq 0$ for some i, $0 \leq i \leq \omega(x) - 1$ since $\kappa(x) = 2$; moreover $F_0 \neq 0$ iff x satisfies condition (*).

Assume that $\iota(x') \geq \iota(x)$. By theorem 3.6, we have

$$x_1 = x' := (X' := Z/u_1, u_1, u_2' := 1 + \gamma u_2/u_1, u_3' := u_3/u_1),$$

$\gamma \in S$ being a preimage of λ. We have

$$E' = \mathrm{div}(u_1),\ k(x') = k(x) \text{ and } H(x') = (u_1^{p(d_1+d_2-1)+\omega(x)}).$$

Assume that $\iota(x') \geq \iota(x)$. Since $\mathrm{Vdir}(x) =< U_3, U_1 + \lambda U_2 >$, we consider two cases deduced from lemma 7.3:

Case 1: $\omega(x) \equiv 0 \bmod p$. By lemma 7.3(i), it can be assumed w.l.o.g that

$$F_{pi}(U_1, U_2) = c_{pi}(U_1 + \lambda U_2)^{pi},\ c_{pi} \in k(x), 1 \leq i \leq \frac{\omega(x)}{p}. \tag{7.29}$$

After blowing up, there is an expansion $\mathrm{in}_{m_{S'}} h' = X'^p + F_{p,X'}$, where

$$U_1^{-pd_1'} F_{p,X'} = (-\lambda)^{-pd_2} \sum_{i=0}^{\omega(x)/p} c_{pi} U_2'^{pi} U_3'^{\omega(x)-pi} + U_1 \Phi', \tag{7.30}$$

for some $\Phi' \in k(x)[U_1, U_2'^p, U_3'^p]$, $d_1' := d_1 + d_2 - 1 + \omega(x)/p$.

If $d_1' \notin \mathbb{N}$, then $\epsilon(x') = \omega(x)$ and $\iota(x') = \iota(x)$. Moreover

$$k(x')U_1 + \mathrm{Vdir}(x') =< U_1, U_2', U_3' >,$$

so $\tau'(x') = 3$ or x' is of type (T0). In both cases, x is good.

If $(d_1, d_2) \in \mathbb{N}^2$, it can be assumed furthermore that $c_{pi} = 0$ or $c_{pi} \notin k(x)^p$ in (7.29). We have $d_1' \in \mathbb{N}$ and we also get $\epsilon(x') = \omega(x)$ and $\iota(x') = \iota(x)$. Since

$$J(F_{p,Z}, E, x) = H^{-1} < \frac{\partial F_{p,Z}}{\partial \lambda_l})_{l \in \Lambda_0} >$$

with notations as in (2.44), we get in any case since $k(x') = k(x)$:

$$k(x')U_1 + \mathrm{Vdir}(x') =< U_1, U_2', U_3' > .$$

Therefore $\tau'(x') = 3$ or x' is of type (T0), so x is good.

If $d_1' \in \mathbb{N}$, $d_2 \notin \mathbb{N}$, we define

$$I := \{i : (-\lambda)^{-pd_2} c_{pi} \notin k(x)^p\}.$$

If $I \neq \emptyset$, we also get $\epsilon(x') = \omega(x)$ and $\iota(x') = \iota(x)$. If $\omega(x) \in I$, x' satisfies condition (*1); otherwise x' is good.

If $I = \emptyset$, let $(u_1, u_2', u_3'; Z')$ be well adapted coordinates at x'. We denote by $a \in \mathbb{F}_p$ the residue of pd_2. Since $d_2 \notin \mathbb{N}$, we have $a \neq 0$. The initial form $\mathrm{in}_{E'} h'$ of lemma 7.4 is of the form:

$$\mathrm{in}_{E'} h' = Z'^p + U_1^{pd_1'} \overline{F}'(\overline{u}_2', \overline{u}_3') \in S'/(u_1)[U_1, Z'],$$

where $S'/(u_1) \simeq k(x)[\overline{u}_2', \overline{u}_3']_{(\overline{u}_2', \overline{u}_3')}$. The form $\Phi' := \mathrm{cl}_{\omega(x)+1} \overline{F}'$ is given by

$$\Phi' = -a(-\lambda)^{-pd_2} \sum_{i=0}^{\omega(x)/p} c_{pi} \overline{U}_2'^{\,pi+1} \overline{U}_3'^{\,\omega(x)-pi} \in k(x)[\overline{U}_2', \overline{U}_3']_{\omega(x)+1}.$$

If $\epsilon(x') = \omega(x)$, x' thus satisfies all assumptions of proposition 7.6, so x is good. Otherwise, we have $\epsilon(x') = 1 + \omega(x)$ and

$$k(x')U_1 + \mathrm{Vdir}(x') =< U_1, U_2', U_3' > .$$

Therefore $\iota(x') = \iota(x)$ and x is good (if $\tau(x') = 3$) or x' is of type (T1). If x satisfies condition (*2), i.e. $c_0 \neq 0$, then x' satisfies condition (*3).

Case 2: $\omega(x) \not\equiv 0 \bmod p$. Recall that $F_i(U_1, U_2) = 0$ whenever $\omega(x) - i \not\equiv 0 \bmod p$. Therefore $a := \widehat{\omega(x)} = \widehat{i}$ whenever $F_i \neq 0$. Let $a_j := \widehat{pd_j}$, $j = 1, 2$. By lemma 7.3(ii), we have $a_1 a_2 \neq 0$, $a_1 + a_2 + a = p$. Moreover, it can be assumed w.l.o.g. that

$$U_1^{a(1)} U_2^{a(2)} F_i(U_1, U_2) = c_i \Phi_i(U_1, \lambda U_2),\ c_i \in k(x)^p, 1 \leq i \leq \omega(x), \qquad (7.31)$$

with notations as in (7.9). After blowing up, the initial form $\mathrm{in}_{E'}h'$ of lemma 7.4 is of the form:

$$\mathrm{in}_{E'}h' = Z'^p + U_1^{pd_1'}\overline{F}'(\overline{u}_2', \overline{u}_3') \in S'/(u_1)[U_1, Z'],$$

where $S'/(u_1) \simeq k(x)[\overline{u}_2', \overline{u}_3']_{(\overline{u}_2', \overline{u}_3')}$. The form $\Phi' := \mathrm{cl}_{\omega(x)+1}\overline{F}'$ is given explicitly by

$$\Phi' = \begin{pmatrix} a_2 + a \\ a+1 \end{pmatrix} \sum_{i=0}^{\lfloor \omega(x)/p \rfloor} c_{pi+a} \overline{U}_2'^{\,a+pi+1} \overline{U}_3'^{\,\omega(x)-a-pi} \in k(x)[\overline{U}_2', \overline{U}_3']_{\omega(x)+1}.$$

If $\epsilon(x') = \omega(x)$, x' thus satisfies all assumptions of proposition 7.6, so x is good. Otherwise, we have $\epsilon(x') = 1 + \omega(x)$ and

$$k(x')U_1 + \mathrm{Vdir}(x') =< U_1, U_2', U_3' > .$$

Therefore $\iota(x') = \iota(x)$ and x is good (if $\tau(x') = 3$) or is of type (T1). Note that x did not satisfy condition (*2): since $J(F_{p,Z}, E, m_S) \subset k[U_1, U_2, U_3^p]_{\omega(x)}$ and $\omega(x) \not\equiv 0 \bmod p$, $J(F_{p,Z}, E, m_S)$ contains no monic polynomial in U_3.

Assume that $k = 3$. There is an expansion (7.21) with $G = 0$ and

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = \sum_{i=0}^{\omega(x)} \lambda_i U_3^{\omega(x)-i} U_1^i.$$

Assume that $\iota(x') \geq \iota(x)$. By theorem 3.6, we have $x_1 = x'$, where

$$x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2).$$

By proposition 2.6, $\Delta_{S'}(h'; u_1', u_2, u_3'; Z')$ is minimal and we have

$$\mathrm{in}_{m_{S'}}h' = Z'^p + U_1'^{pd_1} U_2^{pd_1'} (\sum_{i=0}^{\omega(x)} \lambda_i U_3'^{\omega(x)-i} U_1'^{\,i} + U_2\Phi'),$$

where $d_1' := d_1 + d_2 - 1 + \omega(x)/p$, $\Phi' \in k(x')[U_1', U_2, U_3'^p]$. since it is assumed that $\iota(x') \geq \iota(x)$. Then

$$\iota(x') = \iota(x) \text{ and } k(x')U_2 + \mathrm{Vdir}(x') =< U_1', U_2, U_3' > .$$

We conclude that $\tau'(x') = 3$ (so x is good) or x' is of either type (T2) or (T3). If moreover x satisfies condition (*), i.e. $\lambda_0 \neq 0$, then x' satisfies condition (*2).

Assume that $k = 4$. We have $H^{-1}G^p \subseteq k(x)U_1^{\omega(x)+1}$ and there is an expansion (7.21) with

$$U_1^{-pd_1}F_{p,Z} = F_{1+\omega(x)}(U_1, U_3) + \sum_{i=1}^{1+\omega(x)} F_{1+\omega(x)-i}(U_1, U_3)U_2^i. \tag{7.32}$$

Assume that $\iota(x') \geq \iota(x)$. By theorem 3.6, we have $x_1 = x'$, where

$$x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2),\ E' = \mathrm{div}(u_1'u_2).$$

By proposition 2.6, $\Delta_{S'}(h'; u_1, u_2', u_3'; Z')$ is minimal. We deduce from (7.32) that

$$F_{1+\omega(x)-i}(U_1, U_3) = 0,\ 2 \leq i \leq 1 + \omega(x),$$

since it is assumed that $\iota(x') \geq \iota(x)$. Since $\kappa(x) = 2$, we deduce from definition 5.1 that

$$F_{\omega(x)}(U_1, U_3) \notin < U_1^{\omega(x)} > . \tag{7.33}$$

In particular, we get from (7.32):

$$\epsilon(x') = \omega(x) \text{ and } \mathrm{Vdir}(x') \nsubseteq < U_1', U_2 > .$$

The initial form polynomial $\mathrm{in}_{m_{S'}}h'$ is therefore given by

$$\mathrm{in}_{m_{S'}}h' = Z'^p + U_1'^{pd_1}U_2^{pd_2'}(F_{\omega(x)}(U_1', U_3') + U_2\Phi') \tag{7.34}$$

where $d_2' := d_1 + d_2 - 1 + (1 + \omega(x))/p$, $\Phi' \in k(x')[U_1', U_2, U_3']$. This proves that $\iota(x') = \iota(x)$.

Suppose that x satisfies condition (*3), i.e. $F_{\omega(x)}(U_1, U_3)$ is unitary in U_3. We deduce from (7.34) that x' satisfies condition (*2). Otherwise, U_1 divides $F_{\omega(x)}(U_1, U_3)$ and we deduce from (7.33) that

$$k(x')U_2 + \mathrm{Vdir}(x') =< U_1', U_2, U_3' > .$$

Then x is good (if $\tau'(x') = 3$), or ($\tau'(x') = 2$ and x' is of type (T2) or (T3)). This concludes the proof of the claim. In particular, we have proved (ii).

We now prove (i). Suppose on the contrary that for every $r \geq 0$, x_r does not satisfy condition (*). The above proof shows that x_r is resolved for some $r \geq 0$ or there exists $r_0 \geq 0$ such that for every $r \geq r_0$, we have

$$\tau'(x_r) = 2 \text{ and } x_r \text{ is of type } (Tk)$$

where $k \in \{1, 3\}$ is independent of r. If $k = 1$, we derive a contradiction from corollary 3.9.

If $k = 3$, there exists

$$\hat{u}_3 := u_3 - \sum_{i=1}^{\infty} \gamma_{i,3} u_2^i \in \hat{S};\ \hat{\phi} := \sum_{i=1}^{\infty} \gamma_i u_2^i \in \hat{S}$$

with the following property: for every $i \geq 0$, we have $\iota(x_i) = \iota(x)$ and the strict transform in $(\mathcal{X}_i, x_i)$ of the formal curve $\hat{\mathcal{Y}} = (Z - \hat{\phi}, u_1, \hat{u}_3) \subset \hat{\mathcal{X}}$ is nonempty.

Note that the conclusion of proposition 3.8(2) applied to the formal arc $\varphi :\ \hat{\mathcal{Y}} \to \mathcal{X}$ does not hold. To see this, note that *ibid.*(2.b) implies that $Z_{r_0(\varphi)}$ is an irreducible component of E_{r_0}; by *ibid.*(2.c) we have $\epsilon(x_{r_0}) = 1$: a contradiction, since it is assumed (from the beginning of this proof) that $\omega(x) \geq 2$.

Therefore the conclusion of proposition 3.8(1) holds. Let $(u_1', u_2', u_3'; Z')$ be well adapted coordinates at x_{r_0}, where $\mathcal{Y} := (Z', u_1', u_3') \subset (\mathcal{X}_{r_0}, x_{r_0})$ is permissible of the first kind at x_0. Since $\mathrm{Vdir}(x_{r_0}) =< U_1', U_3' >$, x_{r_0} is good by theorem 3.6, hence x is good.

To prove (iii), it can be assumed by (i) that x satisfies condition (*). Suppose that $\epsilon(x) = \omega(x)$. Then $J(F_{p,Z}, E, x)$ contains no monic polynomial in U_3, since $\omega(x) \not\equiv 0 \bmod p$. So $\epsilon(x) = \omega(x) + 1$. It has been proved above that

$$\tau'(x) = 1 \Longrightarrow \omega(x) \equiv 0 \bmod p.$$

We deduce that $\tau'(x) \geq 2$. Therefore x_r is resolved for some $r \geq 0$ or

$$\iota(x_i) = \iota(x),\ \epsilon(x_i) = \omega(x) + 1 \text{ and } \tau'(x_i) = 2$$

for every $i \geq 0$. The above claim shows that x_r is of type (T1) for every $r >> 0$. We get x_r resolved for some $r \geq 0$ arguing as in the above proof of (i), so x is good. $\square$

A direct consequence of proposition 7.8(iii) and remark 7.1 is:

Corollary 7.9. *Projection Theorem 5.1 holds when* $\kappa(x) = 2$ *and* $\omega(x) \not\equiv 0 \bmod p$. *One may take all local blowing ups in (5.2) permissible of the first kind if* $p = 2$ *or if* $\omega(x) \geq 3$.

Remark 7.2. Assume that $\kappa(x) = 2$, $\omega(x) \equiv 0 \bmod p$ and use notations as in proposition 7.8.

Suppose that x satisfies condition (*1) or (*2) and x_1 satisfies condition (*3). It follows from the above proof that x_1 is resolved or there exist well adapted coordinates $(u_1, u_2, u_3; Z)$ at x_1 such that

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_2} \equiv < \Phi(U_2, U_3) > \ \bmod(U_1) \cap G(m_S)_{\omega(x)}, \tag{7.35}$$

where $\Phi(U_2, U_3) \in k(x_1)[U_2, {U_3}^p]$. This is precisely the definition used by the authors for $\kappa(x) = 2$ when $\epsilon(x) = 1 + \omega(x)$ in [27] **I.1**(ii) on p.1899.

Suppose now that $\kappa(x) = 2$, x satisfies condition (*3) and $(u_1, u_2, u_3; Z)$ are well adapted coordinates satisfying the requirements in definition 7.1. It also follows from the above proof that x is good or

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_2} = \begin{cases} < U_3^{\omega(x)} > & \text{if} \quad \tau'(x) = 1 \\ \\ < F_{\omega(x)}(U_1, U_3) > & \text{if} \quad \mathrm{Vdir}(x) =< U_1, U_3 > \end{cases}.$$

In particular, (7.35) holds in both cases with $< \Phi >=< U_3^{\omega(x)} >$. We deduce the following: there exists $r \geq 0$ such that x_r is resolved or for every $r >> 0$, we have ($\iota(x_r) = \iota(x)$, x_r satisfies condition (*) and

$$x_r \text{ satisfies condition } (*3) \Longrightarrow (7.35) \text{ holds at } x_r).$$

Namely, otherwise we would have ($\iota(x_r) = \iota(x)$, $\tau'(x_r) = 2$ and x_r is of type (T1)) for every $r >> 0$ by the above. But this implies that x_r is resolved for some $r \geq 0$ (*viz.* proof of proposition 7.8(i) for $\tau'(x) = 2$).

This matches the present definition of $\kappa(x) = 2$ with that used in [27], and reduces the proof to the same situation (7.35).

7.3 Monic expansions: secondary invariants.

Proposition 7.8(i) has reduced the proof of the projection theorem to those points with $\kappa(x) = 2$ satisfying condition (*). Moreover, we may assume that $\omega(x) \equiv 0 \bmod p$ by corollary 7.9. For such points, we introduce a new invariant $\gamma(x) \in \mathbb{N}$ in definition 7.4.

We assume in this section and in the following one that $\omega(x) \equiv 0 \bmod p$ and x satisfies condition (*).

Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates satisfying the condition in definition 7.1. If x satisfies condition (*1) or (*2) (resp. condition (*3)), then

$$\mathbf{v}_0 := (\mathbf{b}_0, \frac{\omega(x)}{p}),\ \mathbf{b}_0 := (d_1, d_2)\ (\text{resp. } \mathbf{b}_0 := (d_1, \frac{1}{p})) \tag{7.36}$$

is a vertex of $\Delta_S(h; u_1, u_2, u_3; Z)$. Consider the projection from the point $\mathbf{v}_0$:

$$\mathbf{p}_2' : \mathbb{R}^3 \backslash \{x_3 = \omega(x)/p\} \longrightarrow \mathbb{A} := \mathbf{b}_0 + \{(x_1, x_2, 0),\ x_1, x_2 \in \mathbb{R}^2\}.$$

We view here $\mathbb{A}$ as an *affine* plane with origin $\mathbf{b}_0$ and coordinates (x_1, x_2). Of course $\mathbb{A}$ as a set is independent of our choice of $\mathbf{b}_0$. Let $\mathbf{p}_2 := \tau \circ \mathbf{p}_2'$, where

$$\tau : \mathbb{A} \longrightarrow \mathbb{A},\ \mathbf{b}_0 + (y_1, y_2) \mapsto \mathbf{b}_0 + \frac{1}{\frac{\omega(x)}{p}}(y_1, y_2).$$

Analytically, we have:

$$\mathbf{p}_2 :\ (x_1, x_2, x_3) \mapsto \mathbf{b}_0 + \frac{(x_1, x_2) - \mathbf{b}_0}{\frac{\omega(x)}{p} - x_3}. \tag{7.37}$$

From now on, we will use affine coordinates in $\mathbb{A}$, i.e. $(y_1, y_2) \in \mathbb{R}^2$ represents the point $\mathbf{b}_0 + (y_1, y_2) \in \mathbb{A}$.

Definition 7.2. With notations as above, we define a convex set:

$$\Delta_2(h; u_1, u_2; u_3; Z) := \mathbf{p}_2 \left(\Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \le x_3 < \frac{\omega(x)}{p}\} \right) \subseteq \mathbb{A}.$$

Let furthermore

$$\begin{cases} B(h; u_1, u_2; u_3; Z) & := & \inf_{\mathbf{y} \in \Delta_2(h; u_1, u_2; u_3; Z)} \{y_1 + y_2\} \ge 1 \\ \\ \beta_2(h; u_1, u_2; u_3; Z) & := & \sup \begin{cases} \mathbf{y} \in \Delta_2(h; u_1, u_2; u_3; Z) \\ y_1 + y_2 = B(h; u_1, u_2; u_3; Z) \end{cases} \{y_2\} \end{cases}. \tag{7.38}$$

Indeed, $\Delta_2(h; u_1, u_2; u_3; Z)$ is a convex set because the set

$$\Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\}$$

is convex. We now prove some basic properties of $\Delta_2(h; u_1, u_2; u_3; Z)$. The situation is different and somewhat simpler when (*1) or (*2) holds.

Lemma 7.10. *With notations as above, the following holds:*

(1) there exists $\mathbf{a} = (a_1, a_2, a_3) \in \Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\}$ *such that* $\mathbf{p}_2(\mathbf{a}) =: (\alpha_2, \beta_2) \in \Delta_2(h; u_1, u_2; u_3; Z)$ *satisfies*

$$\beta_2 = \beta_2(h; u_1, u_2; u_3; Z), \;\; \alpha_2 + \beta_2 = B(h; u_1, u_2; u_3; Z).$$

*(2) if x satisfies condition (*1) or (*2), then* $\Delta_2(h; u_1, u_2; u_3; Z)$ *is a (nonempty) rational polygon.*

*(3) if x satisfies condition (*3), then* $\Delta_2(h; u_1, u_2; u_3; Z) \cap \{y_2 \geq \beta_2\}$ *is a (nonempty) rational polygon.*

*(4) assume that x satisfies condition (*1) or (*2) (resp. condition (*3)). Let*

$$\sigma_2 \subset \Delta_2(h; u_1, u_2; u_3; Z) \text{ (resp.} \sigma_2 \subset \Delta_2(h; u_1, u_2; u_3; Z) \cap \{y_2 \geq \beta_2\})$$

be a compact face. The topological closure σ of

$$\sigma^\circ := \mathbf{p}_2^{-1}(\sigma_2) \cap \Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\} \tag{7.39}$$

is a compact face of the polyhedron $\Delta_S(h; u_1, u_2, u_3; Z)$ *(so $\sigma = \sigma_\alpha$ for some weight vector $\alpha \in \mathbb{R}^3_{>0}$, viz. definition 2.2). Moreover* $\mathbf{p}_2(\sigma^\circ) = \sigma_2$ *and*

$$\sigma = \sigma^\circ \cup \{\mathbf{v}_0\}. \tag{7.40}$$

*(5) assume that x satisfies condition (*3) and let*

$$\sigma_{2,\text{in}} := \Delta_2(h; u_1, u_2; u_3; Z) \cap \{y_1 + y_2 = B(h; u_1, u_2; u_3; Z)\}.$$

If $B(h;u_1,u_2;u_3;Z) > 1$, statement (4) extends to $\sigma_2 = \sigma_{2,\mathrm{in}}$, with (7.40) possibly replaced by

$$\sigma = \mathrm{Conv}\left(\sigma^\circ \cup \{\mathbf{v}_0\} \cup \{(\frac{1}{p}, 0, \frac{\omega(x)}{p})\}\right).$$

If $B(h;u_1,u_2;u_3;Z) = 1$, then

$$\sigma_{\mathrm{in}} := \{\mathbf{x} \in \Delta_S(h;u_1,u_2,u_3;Z) : x_1 + x_2 + x_3 = \delta(x)\}$$

is the unique compact face σ of $\Delta_S(h;u_1,u_2,u_3;Z)$ such that

$$\mathbf{p}_2\left(\sigma \cap \{0 \le x_3 < \frac{\omega(x)}{p}\}\right) = \sigma_2.$$

Proof. Let $\mathbf{V}$ be the set of all vertices of $\Delta_S(h;u_1,u_2,u_3;Z)$ and

$$\mathbf{V}_- := \mathbf{V} \cap \{0 \le x_3 < \frac{\omega(x)}{p}\}.$$

We claim that $\mathbf{V}_- \neq \emptyset$. Namely, suppose that $\mathbf{V}_- = \emptyset$. By definition, this means that

$$\mathrm{ord}_{(u_3)} f_{i,Z} \ge i\frac{\omega(x)}{p},\ 1 \le i \le p.$$

Since $\omega(x)/p \ge 1$, we deduce that $\mathcal{Y} := V(Z,u_3) \subset \mathrm{Sing}_p\mathcal{X}$ by proposition 2.3: a contradiction with assumption **(E)**.

In order to prove the lemma, we must understand the limit points $\mathbf{p}_2(\mathbf{x}) \in \Delta_2(h;u_1,u_2;u_3;Z)$ when $\mathbf{x} \in \Delta_S(h;u_1,u_2,u_3;Z)$ tends to the hyperplane $\{x_3 = \omega(x)/p\}$. By convexity, we have

$$\mathbf{x} \in \mathrm{Conv}\left(\bigcup_{\mathbf{v}\in\mathbf{V}} \{\mathbf{v} + \mathbb{R}^3_{\ge 0}\}\right).$$

• *Assume that x satisfies condition (*1) or (*2).* Let $\mathbf{v} \in \mathbf{V}\backslash\mathbf{V}_-$. Since $v_j \ge d_j$, $j = 1,2$, and $v_3 \ge \omega(x)/p$, we have $\mathbf{v} = \mathbf{v}_0$. One deduces immediately that

$$\Delta_2(h;u_1,u_2;u_3;Z) = \mathrm{Conv}\left(\{\mathbf{p}_2(\mathbf{v}) + \mathbb{R}^2_{\ge 0},\ \mathbf{v} \in \mathbf{V}_-\}\right).$$

All statements in the lemma follow easily.

• *Assume that x satisfies condition (*3).* Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbf{V}_-$ be chosen in such a way that

$$(\alpha_2 + \beta_2, -\beta_2) := \left(\frac{a_1 + a_2 - d_1 - \frac{1}{p}}{\frac{\omega(x)}{p} - a_3}, \frac{-a_2 + \frac{1}{p}}{\frac{\omega(x)}{p} - a_3} \right) \tag{7.41}$$

is minimal for the lexicographical ordering, *viz.* (7.37). We now prove (1). Let $\mathbf{v} \in \mathbf{V} \backslash \mathbf{V}_-$. Since $v_3 > 0$, theorem 2.14 implies that

$$\mathrm{in}_{\mathbf{v}} h = Z^p + \lambda U^{p\mathbf{v}}, \ \lambda \neq 0.$$

If $\mathbf{v} \neq \mathbf{v}_0$, we therefore have

$$v_3 \geq \frac{1 + \omega(x)}{p} \text{ or } \mathbf{v} = \mathbf{v}_k := (d_1 + \frac{k}{p}, 0, \frac{\omega(x)}{p}) \text{ for some } k \geq 1. \tag{7.42}$$

Let

$$\alpha := (\frac{\omega(x)}{p}, \frac{\omega(x)}{p}, \alpha_2 + \beta_2) \in \mathbb{R}^3_{>0}, \ L_\alpha(x_1, x_2, x_3) := x_1 + x_2 + (\alpha_2 + \beta_2) x_3.$$

By (7.41)-(7.42), we have

$$\begin{cases} L_\alpha(\mathbf{v}_0) = L_\alpha(\mathbf{b}_0) + \frac{\omega(x)}{p}(\alpha_2 + \beta_2) = L_\alpha(\mathbf{a}) \\ L_\alpha(\mathbf{v}) \geq L_\alpha(\mathbf{v}_0) \text{ if } \mathbf{v} \notin \{\mathbf{v}_0, \mathbf{a}\} \end{cases} . \tag{7.43}$$

This shows that $\mathbf{v}_0, \mathbf{a} \in \sigma_\alpha$, where σ_α is the compact face of the polyhedron $\Delta_S(h; u_1, u_2, u_3; Z)$ defined by α. In particular we have proved that

$$\alpha_2 + \beta_2 = B(h; u_1, u_2; u_3; Z).$$

Similarly, let

$$\alpha' := (\frac{\omega(x)}{p} \alpha'_1, \frac{\omega(x)}{p}, \alpha'_1 \alpha_2 + \beta_2) \in \mathbb{R}^3_{>0},$$

where $\alpha'_1 > 1$ is chosen in such a way that $L_{\alpha'}(\mathbf{v}) > L_{\alpha'}(\mathbf{a})$ for every $\mathbf{v} \in \mathbf{V}_-$. Such $\alpha'_1 > 1$ exists thanks to the minimal property in (7.41). We now have

$$\begin{cases} L_{\alpha'}(\mathbf{v}_0) = L_{\alpha'}(\mathbf{b}_0) + \frac{\omega(x)}{p}(\alpha'_1 \alpha_2 + \beta_2) = L_{\alpha'}(\mathbf{a}) \\ L_{\alpha'}(\mathbf{v}) > L_{\alpha'}(\mathbf{v}_0) \text{ if } \mathbf{v} \notin \{\mathbf{v}_0, \mathbf{a}\} \end{cases}$$

and this proves that the line $(\mathbf{v}_0\mathbf{a})$ meets $\Delta_S(h;u_1,u_2,u_3;Z)$ along an edge. This completes the proof of (1), and of (4) when $\sigma_2=\{(\alpha_2,\beta_2)\}$.

Statement (4) is proved along the same lines for arbitrary

$$\sigma_2 \subseteq \Delta_2(h;u_1,u_2;u_3;Z)\cap\{y_2\geq\beta_2\}$$

and we omit the proof. Then (3) is a consequence of (4) because $\mathbf{V}_-$ is a finite set.

To prove (5) when $B(h;u_1,u_2;u_3;Z)>1$, note that equality possibly holds in (7.43) only if $\mathbf{v}=\mathbf{v}_1$ and the conclusion follows.

If $B(h;u_1,u_2;u_3;Z)=1$, we have $\alpha=(1,1,1)$ with notations as above and σ_{in} is the compact face of $\Delta_S(h;u_1,u_2,u_3;Z)$ generated by σ°. $\square$

Corollary 7.11. *With notations as above, let:*

$$\Delta_2^+(h;u_1,u_2;u_3;Z):=\Delta_2(h;u_1,u_2;u_3;Z)\cap\{y_2\geq\beta_2(h;u_1,u_2;u_3;Z)\}.$$

Then $\Delta_2^+(h;u_1,u_2;u_3;Z)=\operatorname{Conv}\big(\{\mathbf{p}_2(\mathbf{x})+\mathbb{R}^2_{\geq 0},\ \mathbf{x}\in\mathbf{S}\}\big)$, *where* $\mathbf{S}$ *is the set of vertices* $\mathbf{x}\in\Delta_S(h;u_1,u_2,u_3;Z)$ *with*

$$0\leq x_3<\frac{\omega(x)}{p}\ \text{and}\ y_2:=(\mathbf{p}_2(\mathbf{x}))_2\geq\beta_2(h;u_1,u_2;u_3;Z).$$

Taking $\sigma=\sigma_\alpha$ as in lemma 7.10(4) or (5), we deduce from theorem 2.14 that:

$$\operatorname{in}_\alpha h=Z^p+F_{p-1,Z,\alpha}Z+F_{p,Z,\alpha}\in\operatorname{gr}_\alpha S[Z].$$

Moreover, $F_{p-1,Z,\alpha}\neq 0$ implies that $F_{p-1,Z,\alpha}=-G_\alpha^{p-1}$ and

$$\operatorname{cl}_{p(p-1)\delta_\alpha}(\operatorname{Disc}_Z(h))=<G^{p(p-1)}>.$$

In order to associate relevant combinatorial data to $\Delta_2(h;u_1,u_2;u_3;Z)$, some minimizing process on the u_3 coordinate is required. This process is similar to that used in definition 2.3 and proposition 2.2.

Definition 7.3. Let x satisfy condition (*), $(u_1,u_2,u_3;Z)$ be well adapted coordinates at x satisfying definition 7.1 and $\mathbf{y}=(y_1,y_2)\in\mathbb{R}^2$ be a vertex of $\Delta_2(h;u_1,u_2;u_3;Z)$ (of $\Delta_2^+(h;u_1,u_2;u_3;Z)$ in case (*3)).

With notations as in lemma 7.10(4) with $\sigma_2 = \{\mathbf{y}\}$, we say that $\mathbf{y}$ is 2-solvable if $\mathbf{y} \in \mathbb{N}^2$ and

$$\begin{cases} \mathrm{in}_\alpha h = Z^p + \lambda U_1^{pd_1} U_2^{pd_2} (U_3 - cU_1^{y_1} U_2^{y_2})^{\omega(x)} + \Phi^p & \text{in cases } (*1) \text{ or } (*2) \\ \\ \mathrm{in}_\alpha h = Z^p + \lambda U_1^{pd_1} U_2 (U_3 - cU_1^{y_1} U_2^{y_2})^{\omega(x)} + \Phi^p & \text{in case } (*3) \end{cases}$$

where $\Phi \in \mathrm{gr}_\alpha S$ and $\lambda, c \in k(x)$.

We say that $(u_1, u_2; u_3; Z)$ are well 2-adapted if furthermore the polygon $\Delta_2(h; u_1, u_2; u_3; Z)$ ($\Delta_2^+(h; u_1, u_2; u_3; Z)$ in case (*3)) has no 2-solvable vertex.

Theorem 7.12. *With notations as above, there exists well 2-adapted coordinates. Furthermore, the polygon $\Delta_2^+(h; u_1, u_2; u_3; Z)$ is independent of the well 2-adapted coordinates $(u_1, u_2; u_3; Z)$. For such $(u_1, u_2; u_3; Z)$, let*

$$A_1(x) := \min_{\mathbf{y} \in \Delta_2^+(h; u_1, u_2; u_3; Z)} \{y_1\} \geq 0;$$

the curve $\mathcal{Y} := V(Z, u_1, u_3) \subset \mathcal{X}$ satisfies the equivalence:

$$A_1(x) \geq 1 \Leftrightarrow \mathcal{Y} \text{ is permissible (of the first or second kind).}$$

Proof. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates and assume on the contrary that $(u_1, u_2; u_3; Z)$ are not well 2-adapted. Let $\mathbf{y} \in \mathbb{N}^2$ be a 2-solvable vertex of $\Delta_2(h; u_1, u_2; u_3; Z)$ with $y_1 + y_2$ minimal (and $y_2 \geq \beta_2(h; u_1, u_2; u_3; Z)$ if x satisfies condition (*3)). Let $\gamma \in S$ be a preimage of $c \in k(x)$ given by definition 7.3. Since $\mathbf{y}$ is a vertex of $\Delta_2(h; u_1, u_2; u_3; Z)$, we have $c \neq 0$, so γ is a unit. We let $u_3' := u_3 - \gamma u_1^{y_1} u_2^{y_2}$. Let $\alpha \in \mathbb{R}^3_{>0}$ define the edge

$$\sigma := \mathbf{p}_2^{-1}(\mathbf{y}) \cap \Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\} \cup \{\mathbf{v}_0\}$$

of $\Delta_S(h; u_1, u_2, u_3; Z)$. Computing now initial forms for the polyhedron $\Delta_S(h; u_1, u_2, u_3'; Z)$, we obtain

$$\begin{cases} \mathrm{in}_\alpha h = Z^p + \lambda U_1^{pd_1} U_2^{pd_2} U_3'^{\omega(x)} + \Phi^p & \text{in cases } (*1) \text{ or } (*2) \\ \\ \mathrm{in}_\alpha h = Z^p + \lambda U_1^{pd_1} U_2 U_3'^{\omega(x)} + \Phi^p & \text{in cases } (*3) \end{cases} \tag{7.44}$$

with notations as in definition 7.3.

Let now $\mathbf{y}' \neq \mathbf{y}$ be a vertex of $\Delta_2(h; u_1, u_2; u_3; Z)$ (of $\Delta_2^+(h; u_1, u_2; u_3; Z)$ if x satisfies condition (*3)). Let $\alpha' \in \mathbb{R}^3_{>0}$ define the corresponding edge

$$\sigma' := \mathbf{p}_2^{-1}(\mathbf{y}') \cap \Delta_S(h; u_1, u_2, u_3; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\} \cup \{\mathbf{v}_0\}$$

given by lemma 7.10(4). In particular we have

$$\mu_{\alpha'}(u_1^{y_1} u_2^{y_2}) > \mu_{\alpha'}(u_3).$$

This implies that $\mathrm{in}_{\alpha'} h$ is unchanged when computed in $\Delta_S(h; u_1, u_2, u_3; Z)$ or in $\Delta_S(h; u_1, u_2, u_3'; Z)$, i.e. obtained by substituting the variable U_3 by the variable U_3'. Therefore σ' is again an edge of $\Delta_S(h; u_1, u_2, u_3'; Z)$.

If x satisfies condition (*1) or (*2), we deduce that

$$\mathbf{p}_2(\Delta_S(h; u_1, u_2, u_3'; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\}) \subseteq \Delta_2(h; u_1, u_2; u_3; Z).$$

If x satisfies condition (*3), we obtain

$$\mathbf{p}_2(\Delta_S(h; u_1, u_2, u_3'; Z) \cap \{0 \leq x_3 < \frac{\omega(x)}{p}\}) \cap \{y_2 \geq \beta_2\} \subseteq \Delta_2^+(h; u_1, u_2; u_3; Z).$$

Let $(u_1, u_2, u_3'; Z')$ be well adapted coordinates, $Z' := Z - \phi$, $\phi \in S$. We first check that $(u_1, u_2, u_3'; Z')$ satisfies definition 7.1, i.e. that $U_3' \in \mathrm{Vdir}(x)$. This is obvious if $y_1 + y_2 > 1$, since $\mathrm{in}_{m_S} h$ is then unchanged. If $y_1 + y_2 = 1$, then $\mathbf{y} \in \{(1, 0), (0, 1)\}$ because 2-solvable vertices have integer coordinates. By definition 7.3 and definition 7.1, we have

$$U_3 - cU_1 \in \mathrm{Vdir}(x) + < U_2 > \ \ (\text{resp. } U_3 - cU_2 \in \mathrm{Vdir}(x) + < U_1 >)$$

if $\mathbf{y} = (1, 0)$ (resp. if $\mathbf{y} = (0, 1)$). Therefore $\tau'(x) = 3$ or

$$\mathrm{Vdir}(x) =< U_3, U_1 + dU_2 > \ \ (\text{resp. } \mathrm{Vdir}(x) =< U_3, U_2 + dU_1 >)$$

for some $d \in k(x)$. In all cases, $U_3' \in \mathrm{Vdir}(x)$ follows from the invariance of $\mathrm{Vdir}(x)$ (definition 2.17) if $\tau'(x) = 3$ or if $d = 0$, or if ($d \neq 0$ and x satisfies condition (*2)). Otherwise, it can be assumed w.l.o.g. that $d = 0$ by substituting u_2 by $u_2' = u_2 + \delta u_1$, where $\delta \in S$ is a preimage of $d \in k(x)$. Note that this substitution does not change the requirements in definition 7.1 and we thus get $U_3' \in \mathrm{Vdir}(x)$ as required.

By (7.44), we now have

$$\begin{cases} \mathbf{v}_0 \in \Delta_S(h; u_1, u_2, u_3'; Z') \subset \Delta_S(h; u_1, u_2, u_3; Z) \\ \\ \mathbf{y} \notin \Delta_2(h; u_1, u_2; u_3'; Z') \end{cases}.$$

Iterating this construction, we deduce that there exists a sequence (finite or infinite) of 2-solvable vertices $(\mathbf{y}^{(i)})_{i\geq 0}$, $\mathbf{y}^{(0)} := \mathbf{y}$ and corresponding well adapted coordinates $(u_1, u_2, u_3^{(i)}; Z^{(i)})$, $Z^{(i)} := Z^{(i-1)} - \phi^{(i-1)}$, $\phi^{(i-1)} \in S$ such that

$$\begin{cases} \mathbf{v}_0 \in \Delta_S(h; u_1, u_2, u_3^{(i)}; Z^{(i)}) \subset \Delta_S(h; u_1, u_2, u_3^{(i-1)}; Z^{(i-1)}) \\ \\ \mathbf{y}^{(i)} \notin \Delta_2(h; u_1, u_2; u_3^{(i-1)}; Z^{(i-1)}) \end{cases} \tag{7.45}$$

for $i \geq 1$. Since $y_1^{(i)} + y_2^{(i)}$ is chosen to be minimal at each step, we have $y_1^{(i)} + y_2^{(i)} \to +\infty$ as $i \to +\infty$ if the process is infinite. Therefore

$$\hat{u}_3 = \lim_i u_3^{(i)} \in \hat{S},\ \hat{Z} := Z - \hat{\phi},\ \hat{\phi} := \sum_i \phi^{(i-1)} \in \hat{S}$$

exist and $(u_1, u_2; \hat{u}_3; \hat{Z})$ are well 2-adapted coordinates of $\hat{\mathcal{X}} = \mathrm{Spec}(\hat{S}[X]/(h))$. This proves the existence of well 2-adapted coordinates when $S = \hat{S}$.

Let now $(u_1, u_2; u_3; Z)$ and $(u_1', u_2'; u_3'; Z')$ be two sets of well 2-adapted coordinates. To prove that $\Delta_2^+(h; u_1', u_2'; u_3'; Z') = \Delta_2^+(h; u_1, u_2; u_3; Z)$, let first $\mathbf{y} \in \Delta_2^+(h; u_1, u_2; u_3; Z)$ and let $\alpha \in \mathbb{R}^3_{>0}$ be given by lemma 7.10(4) w.r.t. the face $\sigma_2 := \mathbf{y}$. Since $\mathbf{y} \in \Delta_2^+(h; u_1, u_2; u_3; Z)$, we have

$$\mu_\alpha(u_2) < \min\{\mu_\alpha(u_1), \mu_\alpha(u_3)\}.$$

Therefore $\mu_\alpha(u_2') = \mu_\alpha(u_2)$. We deduce that $\mathrm{in}_\alpha h$ is unchanged when computed w.r.t. the coordinates $(u_1', u_2'; u_3; Z)$. This implies furthermore that $\mathbf{y}$ is not 2-solvable in $\Delta_2(h; u_1', u_2'; u_3'; Z')$ provided $\mu_\alpha(u_3') = \mu_\alpha(u_3)$ for every $\alpha = \alpha(\mathbf{y})$. Otherwise, there is an expansion

$$u_3' = \delta u_3 + \sum_{\mathbf{x}\in\Sigma} \gamma(\mathbf{x}) u_1^{x_1} u_2^{x_2},$$

with Σ finite, $\delta, \gamma(\mathbf{x}) \in S$ units and $\mu_\alpha(u_1^{x_1} u_2^{x_2}) < \mu_\alpha(u_3)$ for some $\mathbf{x} = \mathbf{x}_0 \in \Sigma$ and α. One deduces that $(\mathbf{v}_0\mathbf{x}_0)$ supports an edge of $\Delta_2(h; u_1', u_2'; u_3'; Z')$ and

that $1/\frac{\omega(x)}{p}\mathbf{x}_0$ is a 2-solvable vertex of $\Delta_2(h;u_1',u_2';u_3';Z')$. Choosing $\mathbf{x}_0$ with x_1 minimal gives

$$\frac{1}{\frac{\omega(x)}{p}}\mathbf{x}_0 \in \Delta_2^+(h;u_1',u_2';u_3';Z').$$

This is a contradiction since $(u_1',u_2';u_3';Z')$ are well 2-adapted coordinates, so we get

$$\Delta_2^+(h;u_1',u_2';u_3';Z') = \Delta_2^+(h;u_1,u_2;u_3;Z)$$

as required.

Let now $(u_1,u_2,u_3;Z)$ be well adapted coordinates at x satisfying definition 7.1. Applying finitely many times the above algorithm and (7.45), it can be assumed w.l.o.g. that

$$\begin{cases} \alpha_2(h;u_1,u_2;u_3;Z) &= \alpha_2(h;u_1,u_2;\hat{u}_3;\hat{Z}) \\ \beta_2(h;u_1,u_2;u_3;Z) &= \beta_2(h;u_1,u_2;\hat{u}_3;\hat{Z}) \end{cases},$$

where $(u_1,u_2;\hat{u}_3;\hat{Z})$ are well 2-adapted coordinates of $\hat{\mathcal{X}} = \mathrm{Spec}(\hat{S}[X]/(h))$, $\hat{Z} = Z - \hat{\phi}$. Moreover,

$$(\alpha_2,\beta_2) := (\alpha_2(h;u_1,u_2;u_3;Z),\beta_2(h;u_1,u_2;u_3;Z))$$

is a vertex of both $\Delta_2^+(h;u_1,u_2;u_3;Z)$ and $\Delta_2^+(h;u_1,u_2;\hat{u}_3;\hat{Z})$. Let $\hat{x}$ be the closed point of $\hat{\mathcal{X}}$ and assume that

$$A_1(\hat{x}) > A_1 := \min_{\mathbf{y}\in\Delta_2^+(h;u_1,u_2;u_3;Z)}\{y_1\}. \tag{7.46}$$

Let $J := \{1,3\}$ and consider the weight vector $\alpha := (\frac{\omega(x)}{p}, A_1) \in \mathbb{R}^J_{>0}$. We consider the initial form polynomial

$$\mathrm{in}_\alpha h = Z^p + \sum_{i=1}^{p} F_{i,Z,\alpha} Z^{p-i} \in (\mathrm{gr}_\alpha S)[Z],$$

where

$$\begin{cases} \mathrm{gr}_\alpha S = S/(u_1)[U_1] \subseteq \mathrm{gr}_\alpha \hat{S} = \hat{S}/(u_1)[U_1] & \text{if } A_1 = 0 \\ \mathrm{gr}_\alpha S = S/(u_1,u_3)[U_1,U_3] \subseteq \mathrm{gr}_\alpha \hat{S} = \hat{S}/(u_1,u_3)[U_1,U_3] & \text{if } A_1 > 0 \end{cases}.$$

Case 1: $A_1 = 0$. One deduces from the above algorithm and (7.45) that there exists some $\hat{c} \in (\overline{u}_2)\hat{S}/(u_1)$ such that

$$F_{i,\hat{Z},\alpha} = \hat{g}_i U_1^{id_1} \overline{u}_2^{d_{2,i}} (\overline{u}_3 - \hat{c})^{i\frac{\omega(x)}{p}}, \ 1 \leq i \leq p-1 \tag{7.47}$$

for some $\hat{g}_i \in \hat{S}/(u_1)$ ($\hat{g}_i = 0$ if $d_1 \notin \mathbb{N}$), $d_{2,i} \geq id_2$, and

$$\begin{cases} F_{p,\hat{Z},\alpha} = \hat{l} U_1^{pd_1} \overline{u}_2^{pd_2} (\overline{u}_3 - \hat{c})^{\omega(x)} & \text{in cases } (*1) \text{ or } (*2) \\ F_{p,\hat{Z},\alpha} = \hat{l} U_1^{pd_1} \overline{u}_2 (\overline{u}_3 - \hat{c})^{\omega(x)} & \text{in case } (*3) \end{cases} \tag{7.48}$$

for some $\hat{l} \in \hat{S}/(u_1)$ a unit.

The regular local ring $T := (\mathrm{gr}_\alpha S)_{(U_1,\overline{u}_2,\overline{u}_3)}$ is excellent and the polynomial $\mathrm{in}_\alpha h \in T[Z]$ satisfies the assumptions of proposition 2.4. Let

$$\Xi := \mathrm{Spec}(T[Z]/(\mathrm{in}_\alpha h)), \ \hat{\Xi} := \mathrm{Spec}(\hat{T}[Z]/(\mathrm{in}_\alpha h)).$$

Since $\mathbf{v}_0$ is a nonsolvable vertex of $\Delta_{\hat{T}}(\mathrm{in}_\alpha h; U_1, \overline{u}_2, \overline{u}_3; Z)$, we deduce from (7.47)-(7.48) that

$$\hat{V} := V(\hat{Z}, \overline{u}_3 - \hat{c}) \subseteq \mathrm{Sing}_p \widehat{\Xi} \subseteq V(\hat{Z}, U_1 \overline{u}_2^{pd_2} (\overline{u}_3 - \hat{c})). \tag{7.49}$$

Since T is excellent, one deduces that the Zariski closure V of $\hat{V}$ in Ξ is contained in $\mathrm{Sing}_p \Xi$. Let

$$P: \ \Xi \longrightarrow \mathrm{Spec} T$$

be the projection. By (7.49), $P(V)$ is an irreducible component of $P(\mathrm{Sing}_p \Xi)$ contained in $\mathrm{div}(U_1 \overline{u}_2^{pd_2} (\overline{u}_3 - \hat{c}))$. Since each of $\mathrm{div}(U_1)$, $\mathrm{div}(\overline{u}_2)$ is Zariski closed, there exist $\hat{\delta}' \in \hat{S}/(u_1)$ a unit such that $\overline{u}_3' := \hat{\delta}'(\overline{u}_3 - \hat{c}) \in S/(u_1)$. Let $u_3' \in S$ be a preimage of $\overline{u}_3'$. Applying again proposition 2.4, there exist well adapted coordinates $(u_1, u_2, u_3'; Z')$ at x satisfying definition 7.1 and such that

$$\min_{\mathbf{y} \in \Delta_2^+(h; u_1, u_2; u_3'; Z')} \{y_1\} > A_1. \tag{7.50}$$

Case 2: $A_1 > 0$. The argument runs along the same lines: we now have some $\hat{c} \in (\overline{u}_2)\hat{S}/(u_1, u_3)$, (7.49) is replaced by

$$V(\hat{Z}, U_3 - \hat{c} U_1^{A_1}) \subseteq \mathrm{Sing}_p \widehat{\Xi} \subseteq V(\hat{Z}, U_1 \overline{u}_2^{pd_2} (U_3 - \hat{c} U_1^{A_1})),$$

with Ξ as above and (7.50) holds.

Applying this procedure and (7.50) finitely many times, it can be assumed w.l.o.g. that $A_1 = A_1(\hat{x})$. When x satisfies condition (*1) or (*2), one introduces similarly

$$A_2 := \min_{\mathbf{y}\in\Delta_2(h;u_1,u_2;u_3;Z)}\{y_2\} \leq \min_{\mathbf{y}\in\Delta_2^+(h;u_1,u_2;u_3;Z)}\{y_2\} = \beta_2(h;u_1,u_2;u_3;Z).$$

The same argument shows that there exists well adapted coordinates $(u_1,u_2,u_3;Z)$ at x satisfying definition 7.1 and well 2-adapted coordinates $(u_1,u_2;\hat{u}_3;\hat{Z})$ of $\hat{\mathcal{X}} = \mathrm{Spec}(\hat{S}[X]/(h))$, $\hat{Z} = Z - \hat{\phi}$, such that

$$A_j := \min_{\mathbf{y}\in\Delta_2(h;u_1,u_2;u_3;Z)}\{y_j\} = \min_{\mathbf{y}\in\Delta_2^+(h;u_1,u_2;\hat{u}_3;\hat{Z})}\{y_j\},\ j=1,2. \tag{7.51}$$

Finally, if x satisfies condition (*1) or (*2) (resp. (*3)), (7.51) (resp. (7.50)) proves that the region

$$\Delta_2(h;u_1,u_2;u_3;Z)\backslash\Delta_2(h;u_1,u_2;\hat{u}_3;\hat{Z}) \subseteq \mathbb{R}^2_{\geq 0}$$

(resp. $\Delta_2^+(h;u_1,u_2;u_3;Z)\backslash\Delta_2^+(h;u_1,u_2;\hat{u}_3;\hat{Z})$) is bounded. Therefore the above algorithm and (7.45) can repeat only finitely many times. This proves the existence of well 2-adapted coordinates for arbitrary S.

Let then $(u_1,u_2;u_3;Z)$ be well 2-adapted coordinates and define the curve $\mathcal{Y} := V(Z,u_1,u_3) \subset \mathcal{X}$. By proposition 2.4, the polyhedron

$$\Delta_{\hat{S}}(h;u_1,u_3;Z) = \mathrm{pr}^{\{1,3\}}\Delta_{\hat{S}}(h;u_1,u_2,u_3;Z)$$

is minimal and we have

$$\epsilon(y) = \omega(x) \times \min\{1, A_1(x)\}. \tag{7.52}$$

By definition 3.1, $\mathcal{Y}$ is permissible of the first kind at x if and only if (x satisfies condition (*1) or (*2)) and $A_1(x) \geq 1$.

By proposition 3.3, $\mathcal{Y}$ is permissible of the second kind at x only if x satisfies condition (*3) and $A_1(x) \geq 1$ by (7.52). Conversely, definition 3.2(i) is satisfied because

$$m(y) \geq \epsilon(y) = \omega(x) \geq p.$$

By (7.52), we have $\epsilon(y) = \epsilon(x) - 1$. Suppose that $i_0(y) = p - 1$. Let $W := \eta(\mathcal{Y})$, so we have

$$\mathrm{in}_W h = Z^p - G_W^{p-1} Z + F_{p,W,Z} \in G(W)[Z]$$

with $\delta(y) \in \mathbb{N}$, $G_W = g_W U_1^{\delta(y)}$ and

$$0 \neq \mathrm{cl}_{p(p-1)\delta(y)} \mathrm{Disc}_Z h =< g_W^{p(p-1)} U_1^{p(p-1)\delta(y)} > \in G(W)_{p(p-1)\delta(y)}$$

by theorem 2.14. Since $E = \mathrm{div}(u_1)$, $g_W \in S/(u_1, u_3)$ is a unit by assumption **(E)**. We then get

$$\epsilon(x) \leq \frac{\mathrm{ord}_{m_S}(H(x)^{-(p-1)} f_{p-1,Z}^p)}{p-1} = \epsilon(y) = \epsilon(x) - 1,$$

a contradiction. Therefore definition 3.2(ii) is satisfied because $i_0(y) = p$. Finally it follows from definition 7.1(ii) that definition 3.2(iii) is satisfied. $\square$

The previous theorem shows that the following invariants are actually independent of the choice of well 2-adapted coordinates.

Definition 7.4. Let x satisfy condition (*) and $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates. We let

$$A_j(x) := \min_{\mathbf{y} \in \Delta_2(h; u_1, u_2; u_3; Z)} \{y_j\} \geq 0 \quad \text{for } \mathrm{div}(u_j) \subseteq E;$$

$$B(x) := B(h; u_1, u_2; u_3; Z); \; C(x) := B(x) - \sum_{\mathrm{div}(u_j) \subseteq E} A_j(x);$$

$$\beta(x) := \min_{(A_1(x), y_2) \in \Delta_2^+(h; u_1, u_2; u_3; Z)} \{y_2\} \geq 0;$$

$$(\alpha_2(x), \beta_2(x)) := (\alpha_2(h; u_1, u_2; u_3; Z), \beta_2(h; u_1, u_2; u_3; Z)).$$

Finally, we define $\gamma(x) \in \mathbb{N}$ by:

$$\gamma(x) := \begin{cases} \lceil \beta(x) \rceil & \text{in case } (*1) \\ 1 + \lfloor C(x) \rfloor & \text{in case } (*2) \\ 1 + \lfloor \beta(x) \rfloor & \text{in case } (*3) \end{cases}.$$

Lemma 7.13. *Assume that $\kappa(x) = 2$ and x satisfies condition (*). Let $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates and assume furthermore that*

$$A_1(x) \geq 1 \text{ (resp. } A_1(x) > 1 \text{ or } (A_1(x) = 1 \text{ and } \beta(x) < 1 - \frac{1}{\omega(x)})$$

*if x satisfies condition (*1) or (*2) (resp. condition (*3)). Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along $\mathcal{Y} := V(Z, u_1, u_3) \subset \mathcal{X}$ and $x' \in \pi^{-1}(x)$. Then x' is resolved or the following holds:*

$$x' = (Z' := Z/u_1, u_1, u_2, u_3' := u_3/u_1)$$

*and x' satisfies again condition (*1) or (*2) (resp. (*3)); the coordinates $(u_1, u_2; u_3'; Z')$ are well 2-adapted at x' and*

$$\begin{cases} \Delta_2(u_1, u_2; u_3'; Z') = \Delta_2(u_1, u_2; u_3; Z) - (1, 0) & \text{in case } (*1) \text{ and } (*2) \\ \Delta_2^+(u_1, u_2; u_3'; Z') = \Delta_2^+(u_1, u_2; u_3; Z) - (1, 0) & \text{in case } (*3) \end{cases};$$

in particular $A_1(x') = A_1(x) - 1$ and we have:

$$A_2(x') = A_2(x),\ C(x') = C(x),\ \beta(x') = \beta(x) \text{ and } \gamma(x') = \gamma(x).$$

Proof. By theorem 7.12, the curve $\mathcal{Y}$ is permissible since $A_1(x) \geq 1$.

Since $U_3 \in \mathrm{Vdir}(x)$ by definition of well 2-prepared coordinates, x is then good except possibly if $\mathrm{Vdir}(x) =< U_3 >$ by theorem 3.6; in this case, we have $x' = (Z/u_1, u_1, u_2, u_3/u_1)$.

Let $h' := u_1^{-p}h$. By proposition 2.6, $\Delta_{\hat{S}'}(h'; u_1, u_2, u_3'; Z')$ is again minimal. With usual notations, we have $d_1' = d_1 + \frac{\omega(x)}{p} - 1$ and $\mathbf{v}_0' := (d_1', 0, \omega(x)/p)$ ($\mathbf{v}_0' := (d_1', 1/p, \omega(x)/p)$ in case (*3)) is a nonsolvable vertex. We may assume that x' is very near x.

If x satisfies condition (*1) (resp. (*2)), then $\kappa(x') = 2$ and x' satisfies again condition (*1) (resp. (*2)).

If x satisfies condition (*3) and $\epsilon(x') = \epsilon(x)$, then $\kappa(x') = 2$ and x' satisfies again condition (*3).

If x satisfies condition (*3) and $\epsilon(x') = \omega(x)$, then x' satisfies the assumptions of lemma 7.1, so x is good if $A_1(x) > 1$; if ($A_1(x) = 1$ and $\beta(x) < 1 - 1/\omega(x)$), then $\Delta_2^+(u_1, u_2; u_3; Z)$ has a vertex of the form

$$\mathbf{y}_0 = (1, (i_0 - 1)/i) = \beta(x),\ 1 \leq i \leq \omega(x).$$

Therefore $i_0 \leq i < \omega(x)$ or $i_0 < i = \omega(x)$, since $\beta(x) < 1 - 1/\omega(x)$. Taking i_0 minimal with this property, $(d_1', i_0/p, (\omega(x) - i)/p)$ is a vertex of $\Delta_{S'}(h'; u_1, u_2, u_3'; Z')$ and therefore

$$\epsilon(x') = \omega(x) \Longrightarrow i_0 + \omega(x) - i = \omega(x).$$

Therefore $i < \omega(x)$ since $(i_0, i) \neq (\omega(x), \omega(x))$; then x' is good by proposition 7.7, so x is good.

Let $\mathbf{y} = \mathbf{p}_2(\mathbf{v})$ be a vertex of $\Delta_2(u_1, u_2; u_3; Z)$ (of $\Delta_2^+(u_1, u_2; u_3; Z)$ in case (*3)). With notations as in lemma 7.10(4) with $\sigma_2 := \{\mathbf{y}\}$, let

$$\mathrm{in}_\alpha h = Z^p + U_1^{pd_1} U_2^{pd_2} (F_0(U_2) U_3^{\omega(x)} + \sum_{i=1}^{\omega(x)} F_i(U_1, U_2) U_3^{\omega(x)-i}),$$

where $0 \neq F_0(U_2) \in k(x)$ ($0 \neq F_0(U_2) \in k(x)[U_2]_1$ in case (*3)). Then $\mathbf{y}' := \mathbf{y} - (1, 0)$ is a vertex of $\Delta_2(u_1, u_2; u_3'; Z')$ (of $\Delta_2^+(u_1, u_2; u_3'; Z')$ in case (*3)); the corresponding initial form in lemma 7.10(4) with $\sigma_2 := \{\mathbf{y}'\}$ is of the form:

$$\mathrm{in}_{\alpha'} h' = Z'^p + U_1^{pd_1'} U_2^{pd_2} (F_0(U_2) U_3'^{\omega(x)} + \sum_{i=1}^{\omega(x)} U_1^{-i} F_i(U_1, U_2) U_3'^{\omega(x)-i}).$$

It follows from definition 7.3 that $\mathbf{y}'$ is not 2-solvable, since $\mathbf{y}$ is not. The lemma follows easily. □

Proposition 7.14. *Assume that $\kappa(x) = 2$ and x satisfies condition (*). If $\gamma(x) = 0$, then x is good.*

Proof. By theorem 7.12, there exist well 2-adapted coordinates $(u_1, u_2; u_3; Z)$ at x. The assumption $\gamma(x) = 0$ means that (x is in case (*1) and $\beta(x) = 0$) or (x is in case (*3) and $\beta(x) < 0$).

*Assume that x is in case (*1).* We have

$$\Delta_2(h; u_1, u_2; u_3; Z) = (A_1(x), 0) + \mathbb{R}^2_{\geq 0}.$$

Since $B(x) \geq 1$ (*viz.* (7.38)), we have $A_1(x) \geq 1$.

*Assume that x is in case (*3).* We have

$$\Delta_2^+(h; u_1, u_2; u_3; Z) = (A_1(x), \beta(x)) + \mathbb{R}^2_{\geq 0}$$

in this case. Note that we have $A_1(x) \geq 1$: namely, $\beta(x) = -1/i$ for some i, $1 \leq i \leq \omega(x)$ such that

$$\epsilon(x) = 1 + \omega(x) \leq iA_1(x) + \omega(x) - i + 1,$$

so $A_1(x) \geq 1$.

Suppose that $1 \leq A_1(x) < 2$. By lemma 7.13, x is good or x' satisfies again the assumption of the proposition with $A_1(x') = A_1(x) - 1 < 1$: a contradiction with the previous remark. Induction on $\lfloor A_1(x) \rfloor$ concludes the proof. □

7.4 Monic expansions: blowing up a closed point.

In this section, we control the behavior of the secondary invariant $\gamma(x)$ (definition 7.4) by blowing up a closed point. By proposition 7.14 we may furthermore assume that $\gamma(x) \geq 1$. At this point, we connect the proof with the equal characteristic proof given in [27] chapter 3. Namely, this control is considered in lemmas **I.8.3** and **I.8.8** (resp. lemmas **I.8.7** and **I.8.9**) [27] chapter 3 when x satisfies condition (*1) or (*2) (resp. condition (*3)). The proof relies on the definition of the form

$$\mathrm{in}_\alpha h = Z^p - G_\alpha^{p-1} Z + F_{p,Z,\alpha} \in (\mathrm{gr}_\alpha S)[Z]$$

in lemma 7.10(4)(5) w.r.t. the initial face $\sigma_{2,\mathrm{in}}$ of $\Delta_2(h; u_1, u_2; u_3; Z)$, where $(u_1, u_2; u_3; Z)$ are well 2-adapted coordinates at x.

Notations used in [27]. The corresponding notation for $F_{p,Z,\alpha}$ is

$$F_{p,Z,\alpha} = U_1^{a(1)} U_2^{a(2)} \left(\overline{\phi}_0 U_3^{\omega(x)} + \sum_{j \in J_0} U_3^{\omega(x)-j} \Phi_j(U_1, U_2) \right) \tag{7.53}$$

when x satisfies condition (*1) or (*2) (definition **I.8.2.1**), with

$$a(j) = pd_j, \ j = 1, 2, \ 0 \neq \overline{\phi}_0 \in k(x) \text{ and } \Phi_j(U_1, U_2) \in k(x)[U_1, U_2].$$

By definition 7.3, we have $\Phi_j(U_1, U_2) \neq 0$ for some $j_0 \neq 0$.

When x satisfies condition (*3), the notation is the same except that $\overline{\phi}_0$ and $\Phi_j(U_1, U_2)$ are replaced respectively by $U_2 \overline{\phi}_0$, $\overline{\phi}_0 \in U_2^{-1} k(x)[U_1, U_2, U_3]_1$,

and by $U_2\Phi_j(U_1,U_2)$ with $\Phi_j(U_1,U_2)\in U_2^{-1}k(x)[U_1,U_2]$ (definition **I.8.6.1**). We have $a(2)=0$ in these formulæ in cases (*1) and (*3).

Similarly, the corresponding notation for G_α is

$$G_\alpha^p = U_1^{a(1)}U_2^{a(2)}\mathrm{cl}_{B(x)\omega(x)}(H(x)^{-1}g^p) \tag{7.54}$$

when x satisfies condition (*1) or (*2). When x satisfies condition (*3), we have

$$G_\alpha^p = U_1^{a(1)}\mathrm{cl}_{1+B(x)\omega(x)}(H(x)^{-1}g^p). \tag{7.55}$$

The numerical invariants $\beta(x)$ and $B(x)$ are denoted respectively by $\beta 3(x)$ and $B3(x)$ in [27] when x satisfies condition (*3). The statement "$\kappa(x)\le 1$" in [27] stands for "x is resolved" in this article. The vector spaces $\mathrm{cl}_{\mu_0,\omega(x)}J$ ([27] definitions **I.8.2.3** and **I.8.6.3**) are determined by the initial form polynomial $\mathrm{in}_\alpha h$. The proofs of the following lemmas are almost entirely based on the numerical lemmas **I.8.2.2** and **I.8.6.2** in [27] which are characteristic free. We simply refer to their counterpart in [27] except when they do not immediately adapt to our characteristic free setting.

Assume that ($\kappa(x)=2$, x satisfies condition (*) and $\gamma(x)\ge 1$). Let $\pi:\mathcal{X}'\longrightarrow\mathcal{X}$ be the blowing up along x and $x'\in\pi^{-1}(x)$. We denote by $d:=[k(x'):k(x)]$.

Lemma 7.15. *With notations as above, assume that x is in case (*1) or (*2). Let $(u_1,u_2;u_3;Z)$ be well 2-adapted coordinates at x and assume furthermore that*

$$\eta'(x')\in\mathrm{Spec}(S[\frac{u_2}{u_1},\frac{u_3}{u_1}][Z']/(h')),\ h':=u_1^{-p}h,\ Z':=\frac{Z}{u_1}.$$

Then x' is resolved or ($\kappa(x')=2$, x' satisfies again condition (*) with

$$A_1(x')=B(x)-1,\ \gamma(x')\le\gamma(x),$$

and there exist well 2-adapted coordinates $(u_1',u_2';u_3';Z')$ at x' such that the following holds:)

*(1) if $x'=(Z/u_1,u_1,u_2/u_1,u_3/u_1)$, then x' is again in case (*1) (resp. in case (*2)) and we have $C(x')\le C(x)$, $\beta(x')\le\beta(x)$;*

*(2) if $x'\ne(Z/u_1,u_1,u_2/u_1,u_3/u_1)$, then x' satisfies condition (*1) or (*3), and either (3') below holds or (3)-(4) below hold;*

*(3') the point x satisfies condition (*2) with*

$$U_1^{-pd_1}U_2^{-pd_2}F_{p,Z} = \mu U_3^p + c_p(U_1 + \lambda U_2)^p,$$

*where $d_1, d_2 \notin \mathbb{N}$, $\lambda, \mu, c_p \in k(x)$, $\lambda\mu c_p \neq 0$ and $\mu^{-1}c_p \notin k(x)^p$ up to change of well 2-adapted coordinates; furthermore, x' satisfies condition (*1), $k(x') = k(x)$ and we have*

$$\mathbf{y}' := (\alpha_2(x'), \beta_2(x')) = (0, p/(p-1)) \in \Delta_2(h'; u_1', u_2'; u_3'; Z')$$

and

$$\mathrm{in}_{\alpha'} h' = Z'^p + U_1'^{pd_1'}(\lambda' U_3'^p + U_3' U_2'^p), \tag{7.56}$$

with $d_1' \in \mathbb{N}$, $\lambda' \notin k(x)^p$, notations as in lemma 7.10(4) with $\sigma_2 = \mathbf{y}'$;

(3) we have

$$\beta(x') \leq \frac{C(x)}{d} + \frac{1}{p};$$

(4) we have

$$\beta(x') < \begin{cases} 1 + \lfloor \frac{C(x)}{d} \rfloor & \text{if } x' \text{ is in case } (*1) \\ 1 + \lfloor \frac{C(x)}{d} \rfloor - \frac{1}{\omega(x)} & \text{if } x' \text{ is in case } (*3) \end{cases}.$$

Proof. We already know from proposition 7.8(ii) that x' is resolved or ($\kappa(x') = 2$ and x' satisfies condition (*)). Note that we have

$$B(x) > 1 \Leftrightarrow \tau'(x) = 1.$$

Namely, we have $< U_3 > \subseteq \mathrm{Vdir}(x)$ by definition 7.1, so

$$\tau'(x) = 1 \Leftrightarrow H^{-1}F_{p,Z} \in < U_3^{\omega(x)} > \Leftrightarrow B(x) > 1,$$

where the left hand side equivalence is true because $\Delta_S(h; u_1, u_2, u_3; Z)$ is minimal.

If $B(x) = 1$, then x is of type (T0), (T2) or (T3) as defined along the proof of proposition 7.8. What follows has been proved along the course of that proof: for type (T0), x is good; for type (T3), x' is resolved by theorem 3.6 since $\mathrm{Vdir}(x) =< U_3, U_1 >$; for type (T2), x is good or ($d_1 + d_2 \in \mathbb{N}$,

$d_2 \notin \mathbb{N}$, $B(x) = C(x) = 1$). In this situation, we have $\kappa(x') = 2$, x' satisfies condition (*) and there exist well 2-adapted coordinates $(u'_1, u'_2; u'_3; Z')$ at x' such that $A_1(x') = 0$ and one of the following holds:

• x' is in case (*1)

$$\beta(x') = \frac{i+1}{i},\ i \equiv 0 \bmod p,\ p \leq i \leq \omega(x); \tag{7.57}$$

• x' is in case (*1) and

$$\beta(x') = \frac{\omega(x)}{\omega(x)-1}; \tag{7.58}$$

• x' is in case (*3) and $\beta(x') = 1$.

See the discussion in the proof of proposition 7.8: these three situations correspond respectively to $I = \{0\}$, $I = \{\omega(x)\}$ and $I = \emptyset$ therein. When (7.58) holds with $\omega(x) = p$, we have (3'); otherwise, we have (3)(4). Note that $\gamma(x') = \gamma(x) = 2$ here.

If $B(x) > 1$, statement (1) is easily deduced from the characteristic free proposition 2.6 as in [27]. The rest of the proof relies on the characteristic free transformation formula [27](4) on p.1918 and numerical lemma **I.8.2.2** and is identical to that of **I.8.3**(1)(2)(ii)(iv)-(vi). If x' satisfies condition (*3), note that (4) is an equivalent formulation of [27] lemma **I.8.3**(1). □

Example 7.1. Let $\omega(x) = \overline{\omega}p^a$, $a \geq 2$, $\overline{\omega}/p \notin \mathbb{N}$. We prove here that the bound in lemma 7.15(3) is sharp when x' satisfies either condition (*1) or (*3).

Let $E = \mathrm{div}(u_1u_2)$, $d_1 \in \frac{1}{p}\mathbb{N}\backslash\mathbb{N}$, $d_2 \in \frac{1}{p}\mathbb{N}$, $C \in \mathbb{N}$. Take

$$G_\alpha = 0,\ U_1^{-pd_1}U_2^{-pd_2}F_{p,Z,\alpha} = \left(U_3^p - U_1^{p-1}U_2(U_2-U_1)^{pC}\right)^{\overline{\omega}p^{a-1}},$$

where $C(x) = C$. Let $S' := S[u_2/u_1, u_3/u_1]_{(u_1,u'_2,u'_3)}$, where

$$u'_2 := u_2/u_1 - 1,\ u'_3 := u_3/u_1 - u_1^C {u'_2}^C.$$

Letting $g' := {u'_3}^p - u_1^{pC}{u'_2}^{pC+1}$, we get

$$h' = Z'^p + \sum_{i=1}^{p-1} f_{i,Z'}Z'^{p-i} + u_1^{pd'_1}(f' + u_1f'_1) \in S'[Z'],$$

where $d_1' = d_1 + d_2 + \omega(x)/p - 1$, $\mathrm{ord}_{u_1} f_{i,Z'} > id_1'$, $f_1' \in S'$ and

$$\begin{cases} f' := \delta' g'^{\overline{\omega} p^{a-1}}, & Z' := Z/u_1 & \text{if} \quad d_1 + d_2 \notin \mathbb{N} \\ f' := \delta' u_2' g'^{\overline{\omega} p^{a-1}}, & Z' := Z/u_1 + u_1^{d_1'} g'^{\overline{\omega} p^{a-2}} & \text{if} \quad d_1 + d_2 \in \mathbb{N} \end{cases},$$

with $\delta' \in S'$ a unit. In both cases we get $\beta(x') = C + 1/p$. Note that the above argument also works for ($a = 1$ and x' satisfies condition (*1)).

We now turn to the (*3)-version of the previous lemma. We point out that the situation $J_0 \subset p\mathbb{N}$ has *not* been correctly analyzed in the proof of [27] lemma **I.8.7**. Namely, the bound (3') (*ibid.* p. 1929) may fail (case 2 on p.1930 when $d = 1$) unlike stated therein; the *same* mistake occurs in **I.8.7.5** case 1.

We review and amend the corresponding statements in lemma 7.16(2) below. Adapting notations of (7.53), there is an expansion

$$U_1^{-pd_1} F_{p,Z,\alpha} = (\mu U_3 + cU_1 + U_2) U_3^{\omega(x)} + \sum_{j \in J_0} U_3^{\omega(x)-j} U_1^{b_j} \Psi_j(U_1, U_2), \quad (7.59)$$

where $\mu, c \in k(x)$, $1 + d_j := \deg_{U_2} \Psi_j(U_1, U_2)$, with $b_j \geq jA_1(x)$, $d_j \leq j\beta(x)$, notations as in lemma 7.10(5) (where $\mu = 0$ if $B(x) > 1$). The subset $J_0 \subseteq \{1, \ldots, \omega(x)\}$ is defined by

$$j \in J_0 \Leftrightarrow \Psi_j(U_1, U_2) \neq 0.$$

Lemma 7.16. *Assume that x satisfies condition (*3). Let $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates at x and assume furthermore that*

$$\eta'(x') \in \mathrm{Spec}(S[\frac{u_2}{u_1}, \frac{u_3}{u_1}][Z']/(h')), \ h' := u_1^{-p} h, \ Z' := \frac{Z}{u_1}.$$

*Then x' is resolved or ($\kappa(x') = 2$, x' satisfies condition (*1) or (*3) with*

$$A_1(x') = B(x) - 1, \ \gamma(x') \leq 1 + \gamma(x)$$

and there exist well 2-adapted coordinates $(u_1', u_2'; u_3'; Z')$ at x' such that either (1') below holds, or (1)-(3) below hold:)

(1') *we have*

$$U_1^{-pd_1} F_{p,Z} = U_2 U_3^p + c_p U_1 (U_2 + \lambda U_1)^p,$$

where $\lambda \neq 0$, *(*$d_1 + 1/p \notin \mathbb{N}$ *or* $c_p \notin k(x)^p$*) up to change of well 2-adapted coordinates; furthermore* x' *satisfies condition (*1) and (7.56) holds at* x' *with* $\lambda' \neq 0$ *and (*$d_1' \notin \mathbb{N}$ *or* $\lambda' \notin k(x)^p$*);*

(1) *we have*

$$\beta(x') \leq \frac{\gamma(x)}{d} + \frac{1}{p}$$

and inequality is strict if x' *satisfies condition (*3);*

(2) *if* $\gamma(x') > \gamma(x)$, *then* $k(x') = k(x)$ *and* x' *is uniquely determined; up to a change of well 2-adapted coordinates,* $x' = (Z/u_1, u_1, u_2/u_1, u_3/u_1)$ *and (7.59) reads*

$$U_1^{-pd_1} F_{p,Z,\alpha} = (\mu U_3 + U_2) U_3^{\omega(x)} + c U_1 (U_3 + \lambda U_1^k U_2^{\gamma(x)})^{\omega(x)} \tag{7.60}$$

with $k \in \mathbb{N}$, $\lambda c \neq 0$, *(*$d_1 + 1/p \notin \mathbb{N}$ *or* $c \notin k(x)^p$*), and* $\mu = 0$ *if* $B(x) = k + \gamma(x) > 1$*; furthermore, we have*

$$A_1(x) = k + \frac{1}{\omega(x)}, \beta(x) = \gamma(x) - \frac{1}{\omega(x)}$$

and x' *satisfies condition (*1) with*

$$\beta(x') = \gamma(x) + \frac{1}{\omega(x)};$$

(3) *if (*$\gamma(x') \leq \gamma(x)$ *and* x' *is in case (*3)), then*

$$\beta(x') \leq \max\{\beta(x), \frac{1}{p}\}$$

and $\beta(x') < \beta(x)$ *if (*$k(x') \neq k(x)$ *and* $\beta(x) > 1/p$*).*

Proof. We already know from proposition 7.8(ii) that x' is resolved or ($\kappa(x') = 2$ and x' satisfies condition (*)). Note that we have

$$B(x) > 1 \Leftrightarrow H^{-1} F_{p,Z} \in < U_1 U_3^{\omega(x)}, U_2 U_3^{\omega(x)}, U_3^{\omega(x)+1} > .$$

If $\tau'(x) \geq 2$, we certainly have $B(x) = 1$ and x is of type (T1) or (T4) as defined along the proof of proposition 7.8. For type (T4), x' is resolved by theorem 3.6 since $\mathrm{Vdir}(x) =< U_3, U_1 >$. For type (T1), note that we have $\beta(x) = 1$, hence $\gamma(x) = 2$. The following holds: x is good or $\kappa(x') = 2$, x' satisfies condition (*) and there exist well 2-adapted coordinates $(u_1', u_2'; u_3'; Z')$ at x' such that $A_1(x') = 0$ and either:

- x' is in case (*3) and $\beta(x') = 1$, or
- x' is in case (*1) and

$$\beta(x') = \frac{1+i}{i},\ i \geq 1.$$

See the discussion along the course of the proof of proposition 7.8: these two situations correspond respectively to case 1 and case 2 therein. This proves that x' is resolved or ($\gamma(x') = \gamma(x) = 2$ and (1)(3) hold) when $\tau'(x) = 2$.

Assume now that ($B(x) = 1$ and $\tau'(x) = 1$). The argument in the proof of proposition 7.8, *viz.* (7.25)-(7.26), gives

$$\mathrm{in}_{m_S} h = Z^p + U_1^{pd_1}\left((\mu U_3 + U_2)U_3^{\omega(x)} + U_1 \sum_{i=1}^{\omega(x)/p} U_3^{\omega(x)-pi}\Phi_i(U_1^p, U_2^p)\right)$$

where $\mu \in k(x)$ and $\Phi_i \in k(x)[T_1, T_2]_i$, $1 \leq i \leq \omega(x)/p$. It is easily seen from this expression that

$$\omega(x') \leq \omega(x) - p \min_{1 \leq i \leq \frac{\omega(x)}{p}} \left\{ i - \frac{\deg_{T_2}\Phi_i}{d} \right\},$$

so $\omega(x') = \omega(x)$ implies $d = 1$, and Φ_i monic in T_2 whenever $\Phi_i \neq 0$. Similarly, we have

$$\sum_{i=1}^{\omega(x)/p} \mathrm{Vdir}\left(\{\frac{\partial \Phi_i(U_1^p, U_2^p)}{\partial \lambda_l}\}_{l \in \Lambda_0}\right) =< U_1, U_2 > \Longrightarrow \omega(x') < \omega(x),$$

with notations as in (2.37). After possibly changing Z with $Z - \phi$, $\phi \in S$, it can thus be assumed that

$$\mathrm{in}_{m_S} h = Z^p + U_1^{pd_1}\left((\mu U_3 + U_2)U_3^{\omega(x)} + U_1 \sum_{i=1}^{\omega(x)/p} c_i U_3^{\omega(x)-pi}(U_2 + \lambda U_1)^{pi}\right), \tag{7.61}$$

where $\mu \in k(x)$, $\lambda \in k(x)$ and $c_i \in k(x)$, $1 \leq i \leq \omega(x)/p$. Furthermore, we have $x' = (Z'/u_1, u_1, u_2/u_1 + \gamma, u_3/u_1)$, where $\gamma \in S$ is a preimage of λ. The proof now goes on along the same lines as that of the case $B(x) = 1$ in the previous lemma: x' is resolved or x' satisfies condition (*1), $A_1(x') = 0$ and one of (7.57)-(7.58) holds (in particular $\gamma(x') = 2$). When (7.58) holds with $\omega(x) = p$, we have (1)'; otherwise, we have (1), (3) being pointless.

For (2), note that x' satisfies the assumptions of proposition 7.7 (so x is good) if $c_i \neq 0$ for some $i < \omega(x)/p$. Otherwise, we have

$$(\alpha_2(x), \beta_2(x)) = (\frac{1}{\omega(x)}, 1 - \frac{1}{\omega(x)}). \tag{7.62}$$

By definition 7.4, we also have $\beta(x) = (i_1 - 1)/i$, $1 \leq i \leq \omega(x)$ and $i_1 \in \mathbb{N}$. By assumption, $\gamma(x) = 1$, so $\beta(x) < 1$ and we get

$$1 - \frac{1}{\omega(x)} = \beta_2(x) \leq \beta(x) \leq 1 - \frac{1}{i}.$$

We deduce that $i_1 = i = \omega(x)$. By (7.62), this implies that

$$(A_1(x), \beta(x)) = (\alpha_2(x), \beta_2(x)) = (\frac{1}{\omega(x)}, 1 - \frac{1}{\omega(x)})$$

and the conclusion follows.

If $B(x) > 1$, the proof is identical to that of [27] lemma **I.8.7**(b)(b')(d)(i)-(iii)(v): this relies on the numerical lemma **I.8.6.2** and characteristic free transformation formula for $\mathrm{cl}_{\mu_0,\omega(x)}J$ (definition **I.8.6.3**). As observed before stating this lemma, a mistake in [27] **I.8.7.8** (case 2, $B(x) \in \mathbb{N}$) has to be amended at this point. Namely, the bounds (3)(4) on p.1929 only hold when $G = \mu_2^{-1}\frac{\partial F}{\partial U_2} \neq 0$ with notations as in there. The correct bounds are thus no better than those given in **I.8.7.8** case 3:

$$\beta(x') \leq \frac{1 + d_{j_1}}{dj_1} + \frac{1}{p},\ \beta 3(x') \leq \frac{1 + d_{j_1}}{dj_1} + \frac{1}{p} - \frac{1}{p^a}, \tag{7.63}$$

where $a := \mathrm{ord}_p\omega(x)$: this gives (1) of the present lemma.

We note however that the bounds (3)(3')(4)(4') on p.1929-1930 are correct if $d \geq 2$ (this relies on lemma 6.3(2), statement "$d = 1$ if equality holds"). This proves that $\gamma(x') \leq \gamma(x)$ if $k(x') \neq k(x)$. There remains to

prove (2) and (3) (resp. (3)) of the present lemma for $d = 1$ (resp. for $d \geq 2$).

First assume that $d \geq 2$, i.e. $k(x') \neq k(x)$. The conclusion follows trivially from (1) if $\beta(x) \geq 1$, so we may assume that $\beta(x) < 1$.

The proof involves picking some element $G \in \mathrm{cl}_{\mu_0,\omega(x)}$, $G \neq 0$ [27] middle of p. 1930 and computing the order of its transform. *This is done after possibly performing the Tschirnhausen transformation described in* [27] **I.8.3.6**. We consider several cases:

Case 1: $J_0 \nsubseteq p\mathbb{N}$. Arguing as in [27] **I.8.7.7**, we get

$$\beta(x') \leq \frac{1 + d_{j_1}}{j_1 d} - \frac{1}{j_1} < \frac{\beta(x)}{d}.$$

Case 2: $J_0 \subseteq p\mathbb{N}$ *and* $B(x) \notin \mathbb{N}$. By [27] (4) on p.1930, we get

$$\beta(x) - \beta(x') \geq \left(1 - \frac{1}{d}\right)\beta(x) - \frac{1}{pd} > \frac{1}{p}\left(1 - \frac{2}{d}\right) \geq 0.$$

Case 3: $J_0 \subseteq p\mathbb{N}$, $B(x) \in \mathbb{N}$ *and* $G = U_1^{-pd_1}\frac{\partial F_{p,Z,\alpha}}{\partial U_2}$. Amending [27] **I.8.7.8** as in (7.63), we obtain the bound $\beta(x') \leq \beta(x)/d$ except possibly if $j_1 = p^a$; in this case, we let

$$a' := \max\{b : U_1^{b_{p^a}}\Psi_{p^a}(U_1, U_2) \in (k(x)[U_1, U_2])^{p^b}\} < a \tag{7.64}$$

and obtain the bound:

$$\beta(x') \leq \max\{p^{a'-a}, \beta(x)\} \text{ (resp. } \beta(x') < \beta(x)) \tag{7.65}$$

from lemma 6.3(2) (resp. *ibid.* with $\deg F \geq 2$ if $\beta(x) > 1/p$).

Case 4: $J_0 \subseteq p\mathbb{N}$, $B(x) \in \mathbb{N}$ *and* $U_1^{-pd_1}\frac{\partial F_{p,Z,\alpha}}{\partial U_2} = U_3^{\omega(x)}$. The bound is:

$$\beta(x') \leq \frac{1 + d_{j_1}}{dj_1}$$

as in case 2 with the same conclusion.

Assume that $k(x') = k(x)$. By the independence statement in theorem 7.12, it can be assumed that x' is the origin of the chart. We build upon (7.59) and connect the proof with [27] **I.8.7.5**. First note that x' satisfies condition (*3)

if and only if $\mu = 0$, since $\Delta_S(h; u_1, u_2, u_3; Z)$ is minimal. In this situation one gets easily $\beta(x') \leq \beta(x)$ from proposition 2.6 as in case 3 of [27] **I.8.7.5**. This completes the proof when x' satisfies condition (*3).

Assume now that x' satisfies condition (*1), so $c \neq 0$ in (7.59). Note to begin with that we have

$$\frac{d_j}{j} \leq \beta(x) \Longrightarrow \frac{1+d_j}{j} \leq \gamma(x) \tag{7.66}$$

for each $j \in J_0$ in (7.59). We again consider the same cases 1 to 4 as for $k(x') \neq k(x)$:

Case 1: $J_0 \nsubseteq p\mathbb{N}$. Arguing as in [27] **I.8.7.7**, we get

$$\beta(x') \leq \frac{1+d_{j_1}}{j_1} \leq \gamma(x).$$

Case 2: $J_0 \subseteq p\mathbb{N}$ *and* $B(x) \notin \mathbb{N}$. Same as in case 1 by [27] (3') on p.1929.

Case 3: $J_0 \subseteq p\mathbb{N}$, $B(x) \in \mathbb{N}$ *and* $G = U_1^{-pd_1} \frac{\partial F_{p,Z,\alpha}}{\partial U_2}$. In this situation, equality in (7.66) implies $1 + d_j \in p\mathbb{N}$. Therefore

$$\deg_{U_2} \frac{\partial \Psi_j}{\partial U_2} \leq d_j - 1$$

in (7.59) and we get the same bound as in case 1.

Case 4: $J_0 \subseteq p\mathbb{N}$, $B(x) \in \mathbb{N}$ *and* $U_1^{-pd_1} \frac{\partial F_{p,Z,\alpha}}{\partial U_2} = U_3^{\omega(x)}$. We now have $\Psi_j(U_1, U_2) = \Phi_j(U_1^p, U_2^p)$ for $j \in J_0$ and must take

$$G := U_1^{-pd_1}(D \cdot F_{p,Z,\alpha}),\ D = \lambda_l \frac{\partial}{\partial \lambda_l} \text{ or } D = U_1 \frac{\partial}{\partial U_1} - (pd_1) U_2 \frac{\partial}{\partial U_2}.$$

Arguing as in the case ($B(x) = 1$ and $\tau'(x) = 1$), we obtain the same bound as in case 1 except possibly if

$$U_1^{-pd_1} F_{p,Z,\alpha} = (cU_1 + U_2) U_3^{\omega(x)} + U_1 \sum_{i=1}^{\omega(x)/p} c_{pi} U_3^{\omega(x)-pi} U_1^{kpi} U_2^{pi\gamma(x)}, \tag{7.67}$$

where $k := B(x) - \gamma(x) \in \mathbb{N}$. Define:

$$P(t) := ct^{\omega(x)} + \sum_{i=1}^{\omega(x)/p} c_{pi} t^{\omega(x)-pi}.$$

If $pd_1 + 1 \notin \mathbb{N}$ (resp. $pd_1 + 1 \in \mathbb{N}$) and $P(t) \neq c(t+\lambda)^{\omega(x)}$ (resp. and $P(t) \neq c(t+\lambda)^{\omega(x)} + Q(t)^p$ with $Q(t) \in k(x)[t]$) for some $\lambda \in k(x)$, then

$$\mathbf{y}' := (B(x) - 1, \gamma(x)) \in \Delta_2(h'; u_1, u_2'; u_3'; Z')$$

is a vertex which is not 2-solvable and we get $\beta(x') \leq \gamma(x)$. Otherwise, we may assume w.l.o.g. that $Q = 0$ after changing Z with $Z - \phi$, $\phi \in S$, which gives (7.60). One concludes as in the case ($B(x) = 1$ and $\tau'(x) = 1$) above. □

We now consider the remaining point "at infinity" for the blowing up $\pi : \mathcal{X}' \longrightarrow \mathcal{X}$ along x.

Lemma 7.17. *With notations as above, assume that x satisfies condition (*). Let $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates at x and assume furthermore that*

$$x' = (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2).$$

*Then x' is resolved or ($\kappa(x') = 2$, x' satisfies condition (*2), $(u_1', u_2; u_3'; Z')$ are well 2-adapted coordinates at x',*

$$A_1(x') = A_1(x),\ A_2(x') = B(x) - 1,\ \beta(x') = A_1(x) + \beta(x) - 1,\ \gamma(x') \leq \gamma(x),$$

and the following holds:)

*(1) if x is in case (*1), then $C(x') \leq \min\{\beta(x) - C(x), C(x)\}$;*

*(2) if x is in case (*2), we have $C(x') \leq \min\{\beta(x) - A_2(x) - C(x), C(x)\}$.*

*(3) if x is in case (*3), we have $C(x') \leq \min\{\beta(x) - C(x), C(x) - \beta_2(x)\}$.*

Proof. This relies on the characteristic free proposition 2.6. The argument in [27] lemmas **I.8.8** and **I.8.9** gives all statements before "$\gamma(x') \leq \gamma(x)$". Moreover equations (2) on p.1933 and (2) on p.1934 give:

$$C(x') \leq \min\{\beta(x) - (B(x) - A_1(x)), \alpha_2(x) - A_1(x)\}. \tag{7.68}$$

Assume that x is in case (*1) or (*3). We have

$$\alpha_2(x) + \beta_2(x) = B(x),\ B(x) - A_1(x) = C(x). \tag{7.69}$$

This proves (3); if x satisfies condition (*1), then $\beta_2(x) \geq 0$ and the conclusion follows from (7.68).

If x satisfies condition (*2), we have $\beta_2(x) \geq A_2(x)$, so (7.69) implies that $\alpha_2(x) - A_1(x) \leq C(x)$ and (2) follows easily. Since $\gamma(x) \geq 1$, $\gamma(x') \leq \gamma(x)$ is a trivial consequence of definition 7.4 except if (x is in case (*3) and $C(x) < 0$). But then we have $\beta_2(x) = -1/i$ for some i, $1 \leq i \leq \omega(x)$ by lemma 7.10 and corollary 7.11. Therefore

$$C(x') \leq C(x) - \beta_2(x) < 1$$

by (3) and we get $\gamma(x') \leq 1$. □

7.5 Monic expansions: the algorithm.

In this chapter, we prove theorem 5.1 when $\kappa(x) = 2$. This is restated as theorem 7.18 below. The strategy of the proof has much in common with the one used for theorem 6.1 or for Embedded Resolution of Singularities for surfaces [17]: roughly speaking, the invariant $\gamma(x)$ is in general nonincreasing by blowing up a point x, and drops at a nonrational exceptional point or exceptional point "at infinity" x'. Infinite chains of rational points not "at infinity" do not occur by corollary 3.9. This general idea is illustrated by the proof of proposition 7.20 below which provides the logical scheme of the proof.

Considering however the *precise* behaviour of the invariant $\gamma(x)$ under blowing up, the situation turns out to be more complicated than expected. Two phenomena contribute: on the one hand, the directrix vector space $\mathrm{Vdir}(x)$ is not well-behaved under blowing up; on the other hand, $\gamma(x)$ does not necessarily drop at a nonrational exceptional point or exceptional point "at infinity" and may also increase in some special situations (lemma 7.16(1')(2)). These phenomena make the proof very intricate when $\gamma(x) = 2$, especially when $p = 2$. One is then driven to a step by step proof where the main difficulty is to avoid loops (propositions 7.23 to 7.29). We also emphasize that most of these intricacies actually occur when S is equicharacteristic with algebraically closed residue field.

Let μ be a valuation of $L = k(\mathcal{X})$ centered at x and consider the quadratic sequence

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \tag{7.70}$$

along μ. We will show that x_r is resolved for some $r \geq 0$, hence x is good as explained in remark 5.2.

Theorem 7.18. *Projection Theorem 5.1 holds when $\kappa(x) = 2$. One may take all local blowing ups in (5.2) permissible (of the first kind or second kind) if $p = 2$ or if $\omega(x) \geq 3$.*

Proof. By proposition 7.8, it can be assumed that $\omega(x) \equiv 0 \bmod p$ and that x_r satisfies condition (*) for every $r \geq 0$. Under these assumptions, an invariant $\gamma(x_r) \in \mathbb{N}$ is defined for $r \geq 0$ (definition 7.4).

By proposition 7.20 below, there exists $r_0 \geq 0$ such that either x_{r_0} is resolved or $\gamma(x_{r_0}) \leq 2$.

If $\gamma(x_{r_0}) = 0$, then x_{r_0} is resolved by proposition 7.14.

Suppose that $\gamma(x_{r_0}) = 1$. If x_{r_0} satisfies condition (*1) (resp. (*2)), then x_{r_0} is resolved by proposition 7.21(1) (resp. proposition 7.22) below. If x_{r_0} satisfies condition (*3) and $\beta(x) < 1 - 1/\omega(x)$ (resp. and $\beta(x) = 1 - 1/\omega(x)$, $(p, \omega(x)) \neq (2, 2)$; resp. and $\beta(x) = 1/2$, $(p, \omega(x)) = (2, 2)$), then x_{r_0} is resolved by proposition 7.21(3) (resp. proposition 7.23(ii); resp. proposition 7.25(ii)).

Assume finally that $\gamma(x_{r_0}) = 2$. If x_{r_0} satisfies condition (*1) (resp. (*2); resp. (*3)), then x_{r_0} is resolved by proposition 7.23(i) or by proposition 7.26(i) (resp. by proposition 7.28; resp. by proposition 7.26(ii) or by proposition 7.29). □

Lemma 7.19. *With notations as above, assume that x_r satisfies condition (*2) for every $r \geq 0$. Then there exists $r_0 \geq 0$ such that $C(x_r) = 0$ for every $r \geq r_0$.*

Proof. We consider the points

$$\mathbf{y} := (A_1(x), A_2(x) + a(x)),\ \mathbf{y}' := (A_1(x) + a'(x), A_2(x)) \in \Delta_2(u_1, u_2; u_3; Z),$$

where $(u_1, u_2; u_3; Z)$ are well 2-adapted coordinates. By standard arguments on combinatorial blowing ups, we have $c(x_1) < c(x)$ for the lexicographical ordering whenever $C(x) > 0$, where

$$c(x) := (C(x) = \min\{a(x), a'(x)\}, \max\{a(x), a'(x)\}).$$

Since these numbers belong to $\frac{1}{\omega(x)!}\mathbb{N}^2$, we get $C(x_r) = 0$ for all $r >> 0$. □

Proposition 7.20. *With notations as above, there exists $r_0 \geq 0$ such that x_{r_0} is resolved or $\gamma(x_{r_0}) \leq 2$.*

Proof. Let $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates at x. We will name point "at infinity" for simplicity the origin x' of the second chart of the blowing up, i.e.

$$x' := (Z/u_2, u_1/u_2, u_2, u_3/u_2). \tag{7.71}$$

The notion is unambiguous if $E = \operatorname{div}(u_1)$, that is if x satisfies condition (*1) or (*3). If x satisfies condition (*2), the point "at infinity" furthermore depends on the numbering of u_1, u_2, where $E = \operatorname{div}(u_1 u_2)$.

We may assume that $\gamma(x) \geq 3$ for the whole proof. Note that the special situations described in lemma 7.15(3') and in lemma 7.16(1') occur only when $\gamma(x) \leq 2$. We may thus disregard them in this proof. To prove the proposition, it is sufficient to prove that there exists $r \geq 1$ such that x_r is resolved or $\gamma(x_r) < \gamma(x)$. We first bound $\gamma(x_1)$ in terms of $\gamma(x)$ at a nonrational point or at a point "at infinity".

Assume that $k(x_1) \neq k(x)$. We apply lemma 7.15(4) and lemma 7.16(1) with $d \geq 2$. Note that for $\alpha > 1$, we have

$$1 + \left\lfloor \frac{\alpha}{d} \right\rfloor \leq \lceil \alpha \rceil \tag{7.72}$$

and equality holds if and only if $\alpha = d = 2$. If x is in case (*1) or (*2), we deduce that

$$x_1 \text{ is resolved or } \gamma(x_1) < \gamma(x). \tag{7.73}$$

For $\alpha \in \mathbb{N}$, $\alpha \geq 3$, we have similarly

$$\left\lceil \frac{\alpha}{d} + \frac{1}{p} \right\rceil < \alpha.$$

If x is in case (*3), we deduce from lemma 7.16(1) that (7.73) also holds.

Assume that $x_1 = x'$ is the point at infinity (7.71). By lemma 7.17, x_1 is resolved or satisfies condition (*2).

If x is in case (*1), lemma 7.17(1) gives

$$\gamma(x_1) \leq 1 + \left\lfloor \frac{\beta(x)}{2} \right\rfloor < \gamma(x) \tag{7.74}$$

by (7.72), since $\beta(x) > 2$.

If x is in case (*2), lemma 7.17(2) gives $C(x_1) \leq C(x)$, so $\gamma(x_1) \leq \gamma(x)$.

If x is in case (*3), then lemma 7.17(3) similarly gives

$$\gamma(x_1) \leq 1 + \left\lfloor \frac{1+\beta(x)}{2} \right\rfloor < 1 + \lfloor \beta(x) \rfloor = \gamma(x)$$

since $\beta(x) \geq 2$. The conclusion is again (7.73).

Assume that $x_1 \neq x'$ and $k(x_1) = k(x)$. If x satisfies condition (*1) or (*3), the independence statement in theorem 7.12 shows that we may actually assume that $x_1 = (Z/u_1, u_1, u_2/u_1, u_3/u_1)$.

If x is in case (*1), then x_1 is resolved or satisfies again condition (*1) with $\beta(x_1) \leq \beta(x)$ by lemma 7.15(1).

If x is in case (*3), then x_1 is resolved or satisfies one of conditions (*1) or (*3). In the latter case, we have $\beta(x_1) \leq \beta(x)$ by lemma 7.16(3); in the former case, we have $\gamma(x_1) \leq \gamma(x)$ except if

$$\text{“}x \text{ satisfies the assumptions of lemma 7.16(2)”}. \tag{7.75}$$

This situation occurs only when $\beta(x) = \gamma(x) - 1/\omega(x)$ and gives

$$\beta(x_1) = \gamma(x) + 1/\omega(x), \ \gamma(x_1) = \gamma(x) + 1.$$

We first prove the proposition when x satisfies either condition (*1) or (condition (*3) with $\beta(x) < \gamma(x) - 1/\omega(x)$). By the above considerations, we are done except possibly if x_1 satisfies again condition (*1) or (*3) with ($k(x_1) = k(x)$ and $\gamma(x_1) = \gamma(x)$). Iterating, we conclude from corollary 3.9 that x_r is resolved or $\gamma(x_r) < \gamma(x)$ for some $r \geq 1$.

Assume now that x satisfies condition (*2). By the above considerations and lemma 7.15(4), we are done except possibly if x_1 satisfies again condition (*2). Iterating, we conclude from lemma 7.19 above that x_r is resolved or $\gamma(x_r) < \gamma(x)$ for some $r \geq 1$.

Assume finally that x satisfies condition (*3) with $\beta(x) = \gamma(x) - 1/\omega(x)$. By the above considerations, we are done except possibly if $k(x_1) = k(x)$ and (1) or (2) below holds:

(1) x_1 satisfies again condition (*3) with $\beta(x_1) = \beta(x)$;

(2) x_1 satisfies condition (*1) with $\beta(x_1) = \gamma(x) + 1/\omega(x)$, *viz.* (7.75).

Suppose that (2) holds; we now review the above proof with this extra assumption in mind. Since $\beta(x_1) > 3$, $\beta(x_1) \neq 4$, (7.72) or (7.74) applied to the point x_1 give the stronger

$$\gamma(x_2) < \gamma(x_1) - 1 = \gamma(x).$$

We conclude that either x_2 is resolved, either $\gamma(x_2) < \gamma(x)$, or x_2 satisfies again condition (*1) with $\beta(x_2) \leq \beta(x_1)$. If the latter inequality is strict, we have $\beta(x_2) \leq \gamma(x)$ and we are thus already done. Otherwise x_2 satisfies again (2).

Summing up, there exists $r_0 \geq 0$ such that either x_{r_0} is resolved, either $\gamma(x_{r_0}) < \gamma(x)$, or ($x_r$ satisfies one and the same property (1) or (2) above for every $r \geq r_0$). Iterating, we conclude again by corollary 3.9. □

Proposition 7.21. *Assume that $\kappa(x) = 2$ and one of the following properties holds:*

*(1) x satisfies condition (*1) with $\gamma(x) = 1$;*

*(2) x satisfies condition (*2) with $\beta(x) < 1$;*

*(3) x satisfies condition (*3) with $\beta(x) < 1 - 1/\omega(x)$.*

Then x is good.

Proof. Note that $A_1(x) > 0$ if x satisfies (2) or (3), since

$$1 \leq B(x) \leq A_1(x) + \beta(x)$$

in any case. If (x satisfies condition (1) with $A_1(x) = 0$), then x is good by proposition 7.7. Applying repeatedly lemma 7.13 if $A_1(x) \geq 1$, it can be assumed w.l.o.g. that

$$0 < A_1(x) < 1. \tag{7.76}$$

To prove the proposition, we first claim: x_1 is resolved or (x_1 satisfies again the assumptions of the proposition and $c(x_1) \leq c(x)$ for the lexicographical ordering), where

$$c(x) := (A_1(x), \beta(x)).$$

If x_1 belongs to the first chart, i.e. x_1 is distinct from the point x' at infinity (7.71), we apply lemma 7.15 and lemma 7.16. Note that the special situations described in lemma 7.15(3') and in lemma 7.16(1')(2) do not occur

under the assumptions of the proposition, so we may also disregard them in this proof. We obtain that x_1 is resolved or x_1 satisfies again condition (*) with

$$A_1(x_1) = B(x) - 1 \leq A_1(x) + \beta(x) - 1 \leq A_1(x). \tag{7.77}$$

Assume that x_1 belongs to the first chart and x satisfies (1). We have $C(x) \leq \beta(x) \leq 1$. If $k(x_1) = k(x)$, it can be assumed that x_1 is the origin of the chart by the independence statement in theorem 7.12. By lemma 7.15(1) we have $\beta(x_1) \leq \beta(x)$ and the claim follows. Note that we obtain $c(x_1) = c(x)$ only if $\beta(x) = 1$ by (7.77), in which case x_1 satisfies again (1). If $k(x_1) \neq k(x)$, the claim follows from lemma 7.15(4) with strict inequality $c(x_1) < c(x)$.

Assume that x_1 belongs to the first chart and x satisfies (2). Since $\beta(x) < 1$, inequality is strict in (7.77). The claim also follows from lemma 7.15(1)(4) with strict inequality $c(x_1) < c(x)$.

Assume that x_1 belongs to the first chart and x satisfies (3). Note that if x_1 satisfies condition (*1), then x_1 satisfies again the assumptions of the proposition since lemma 7.16(2) does not occur for $\beta(x) < 1 - 1/\omega(x)$; this is also true if x_1 satisfies condition (*3) by lemma 7.16(3) (note that $p = \omega(x) = 2$ does not occur: (7.76) gives $A_1(x) = 1/2$ while (3) gives $\beta(x) = 0$, a contradiction with $B(x) \geq 1$). The claim now follows with strict inequality $c(x_1) < c(x)$ by (7.77).

Assume that $x_1 = x'$. Turning to lemma 7.17, x' is resolved or x' satisfies condition (*2) with

$$A_1(x') = A_1(x),\ \beta(x') = A_1(x) + \beta(x) - 1 < \beta(x)$$

by (7.76). This proves the claim with $c(x_1) < c(x)$ in this case.

Summing up, we have proved the claim with strict inequality $c(x_1) < c(x)$ except possibly if both x and x_1 are in case (*1), $k(x_1) = k(x)$ and $\beta(x_1) = \beta(x) = 1$. One concludes the proof again by corollary 3.9. □

Proposition 7.22. *Assume that $\kappa(x) = 2$, x satisfies condition (*2) and $\gamma(x) = 1$. Then x is good.*

Proof. By lemma 7.15(4), x_1 is resolved or satisfies the assumptions of proposition 7.21(1) or (3) if x_1 is not a point at infinity. Therefore x_1 is resolved in this case. If x_1 is the origin of a chart, then x_1 is resolved or satisfies again the assumptions of this proposition by lemma 7.17(2).

Applying lemma 7.19, it can thus be assumed that $C(x) = 0$. Applying repeatedly lemma 7.13 if $A_1(x) \geq 1$ or if $A_2(x) \geq 1$, we then reduce to the case

$$0 \leq A_1(x), A_2(x) < 1, \ C(x) = 0.$$

Then $\beta(x) = A_2(x) < 1$ and the conclusion follows from proposition 7.21(2). □

Proposition 7.23. *Assume that $\kappa(x) = 2$ and one of the following properties holds:*

*(i) x satisfies condition (*1) with $\beta(x) < 2$;*

*(ii) x satisfies condition (*3), $\beta(x) = 1 - 1/\omega(x)$ and $(p, \omega(x)) \neq (2, 2)$.*

Then x is good.

Proof. Note that the special situations described in lemma 7.16(1')(2) do occur here.

Assume that x_1 belongs to the first chart. Under assumption (i), x_1 is resolved or x_1 satisfies condition (*1) or (*3); note that the latter occurs only if $k(x_1)$ is an inseparable extension of $k(x)$ (in particular $d \geq p$) and $d_1 \in \mathbb{N}$. By lemma 7.15(4), x_1 satisfies again assumption (i) of the proposition with $k(x_1) = k(x)$ or is resolved by proposition 7.21(1)(3).

Under assumption (ii), x_1 is resolved or x_1 satisfies condition (*1) or (*3). If x is as stated in lemma 7.16(1'), then x_1 is resolved or satisfies assumption (i) with $\beta(x_1) = p/(p-1) < 2$, since $(p, \omega(x)) \neq (2, 2)$.

Otherwise we may apply lemma 7.16(1)-(3): if x_1 satisfies condition (*1), we get $\beta(x_1) \leq 1 + 1/p$, $\beta(x_1) \leq 1$ if $k(x') \neq k(x)$, from lemma 7.16(1); if x_1 satisfies condition (*3), we get $\beta(x_1) \leq \beta(x)$, strict inequality if $k(x') \neq k(x)$, from lemma 7.16(2)(3). By proposition 7.21(1)(3), x_1 is resolved or satisfies again the assumptions of the proposition with $k(x_1) = k(x)$.

Assume that $x_1 = x'$ is the point at infinity. Then x_1 is resolved or x_1 satisfies condition (*2) with $C(x_1) < 1$ by lemma 7.17(1)(3); therefore x_1 is resolved in any case by proposition 7.22.

One concludes the proof again by corollary 3.9. □

Lemma 7.24. *Assume that $\kappa(x) = 2$ and one of the following properties holds:*

*(i) x satisfies condition (*1) with $\beta(x) = 2$;*

*(ii) x satisfies condition (*3) with $\beta(x) < 2$.*

Let $(u_1, u_2; u_3; Z)$ be well 2-adapted coordinates at x and

$$x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2)$$

*be the point at infinity. Then x' is resolved or (x' satisfies condition (*2) with $C(x') = 1$ and the following respectively hold:)*

(i') $p = 2$ and $d_1 \notin \mathbb{N}$;

(ii') $p \geq 3$.

Proof. By lemma 7.17, x' is resolved or x' satisfies condition (*2).

Under assumption (i), lemma 7.17(1) furthermore gives $C(x') \leq 1$; if $C(x') < 1$, we are done by proposition 7.22. If $C(x') = 1$, lemma 7.17(1) implies that $C(x) = 1$; moreover

$$A_1(x') = A_2(x') = A_1(x),\ C(x') = \beta(x') - A_2(x') = 1. \tag{7.78}$$

We now prove that x' is resolved unless ($p = 2$ and $d_1 \notin \mathbb{N}$). To prove this, it is sufficient to prove that any possible x_2 in (7.70) is resolved when $x_1 = x'$. Note that $(u_1', u_2; u_3'; Z')$ are well 2-adapted coordinates at x'. Let

$$\mathrm{in}_{\alpha'} h = Z'^p - G'^{p-1}_{\alpha'} Z' + F_{p,Z',\alpha'},$$

notations as in lemma 7.10(4) w.r.t. the face $\sigma_{2,\mathrm{in}}$ of $\Delta_2(h'; u_1', u_2; u_3'; Z')$. We expand

$$U_1'^{-pd_1} U_2^{-pd_2'} F_{p,Z',\alpha'} = \mu U_3'^{\omega(x)} + \sum_{i=1}^{\omega(x)} \mu_i U_3'^{\omega(x)-i} P_i(U_1', U_2), \tag{7.79}$$

where $d_2' := d_1 + \omega(x)/p - 1$ and

$$P_i(U_1', U_2) = U_1'^{a_i} U_2^{b_i} Q_i(U_1', U_2),$$

with $Q_i(U_1', U_2)$ zero or not divisible by either U_1' or U_2. Since $C(x') = 1$, we have by definition

$$a_i, b_i \geq iA_1(x_1),\ i \geq \deg Q_i(U_1', U_2)$$

whenever $Q_i(U_1', U_2) \neq 0$, $1 \leq i \leq \omega(x)$. Since $C(x) = C(x') = \beta(x') = 1$, we have

$$\deg_{U_1'} Q_{i_1} = i_1 \text{ and } \deg_{U_2} Q_{i_2} = i_2 \tag{7.80}$$

for some i_1, i_2, $1 \leq i_1, i_2 \leq \omega(x)$. Let

$$x_2' := (Z'/u_2, u_1'/u_2, u_2, u_3'/u_2),\ x_2'' := (Z'/u_1', u_1', u_2/u_1', u_3'/u_1')$$

be the points "at infinity". If $x_2 \in \{x_2', x_2''\}$, then lemma 7.15(1) implies that x_2 is resolved or x_2 satisfies condition (*2) with $C(x_2) = 0$ by (7.80). So x_2 is resolved in any case by proposition 7.22.

If $x_2 \notin \{x_2', x_2''\}$ and $k(x_2) \neq k(x')$, we apply lemma 7.15(4): then x_2 is resolved by proposition 7.21(1)(3).

If $x_2 \notin \{x_2', x_2''\}$ and $k(x_2) = k(x')$, we apply lemma 7.15(3')(3)(4). Note that the special situation in lemma 7.15(3') yields x_2 resolved if $(p, \omega(x)) \neq (2, 2)$ by proposition 7.23(i). Therefore x_2 is resolved or one of the following properties holds:

(A) x' satisfies the requirements in lemma 7.15(3') for $p = \omega(x) = 2$ and x_2 satisfies (7.56) (in particular $d_1 \notin \mathbb{N}$);

(B) x_2 satisfies condition (*3) with $\beta(x_2) \leq 1 + 1/p$.

Since $(d_1, 0, \omega(x)/p)$ is a vertex of $\Delta(h; u_1, u_2, u_3; Z)$ which is not solvable, we have $\mu \notin k(x)^p$ in (7.79) if $d_1 \in \mathbb{N}$. As $k(x_2) = k(x')$, x_2 satisfies condition (*3) only if

$$(d_1, d_2') \notin \mathbb{N}^2 \text{ and } d_1 + d_2' \in \mathbb{N}.$$

On the other hand $d_2' - d_1 = \omega(x)/p - 1 \in \mathbb{N}$, so the latter holds if and only if ($p = 2$ and $d_1 \notin \mathbb{N}$) as required.

Under assumption (ii), we are done by proposition 7.22 if $C(x') < 1$. Assuming that $C(x') \geq 1$, we have

$$1 \leq \max\{\beta(x) - C(x), C(x) - \beta_2(x)\} < 2$$

by lemma 7.17(3). It is easily deduced that

$$\beta(x') - A_2(x') = \beta(x) - C(x) < 2 \tag{7.81}$$

and that

$$\beta(x) \geq 1,\ 0 \leq C(x) \leq 1 - 1/\omega(x) \text{ and } \beta_2(x) \leq -1/\omega(x). \tag{7.82}$$

The proof is now a variation of that under assumption (i) and we explain now how it is to be adapted. To begin with, (7.79) holds with $d_2' := d_1 + (1+\omega(x))/p - 1$. Since $C(x) < 1$, $\beta_2(x) < 0$ and $C(x') \geq 1$, (7.80) is now replaced by

$$\deg_{U_1'} Q_{i_1} = i_1 \text{ for some } i_1,\ 1 \leq i_1 \leq \omega(x). \tag{7.83}$$

Note in particular that we have $C(x') = 1$.

If $x_2 \in \{x_2', x_2''\}$, we apply lemma 7.17: x_2' (resp. x_2'') is resolved or $C(x_2') < 1$ (resp. $C(x_2'') = 0$) by (7.81) (resp. by (7.83)). Therefore x_2 is resolved in any case by proposition 7.22.

If $x_2 \notin \{x_2', x_2''\}$ and $k(x_2) \neq k(x_1)$, then x_2 is resolved by the same argument as under assumption (i).

If $x_2 \notin \{x_2', x_2''\}$ and $k(x_2) = k(x_1)$, we first note that x' is *not* as specified in lemma 7.15(3'): since $C(x) < 1$, we have $A_1(x') = A_1(x) > 0$. Applying then lemma 7.15(3)(4), the argument used under assumption (i) gives x_2 resolved or $d_1 + d_2' \in \mathbb{N}$. Since $d_2' - d_1 - 1/p \in \mathbb{N}$, this can possibly hold only if $p \geq 3$. □

Lemma 7.25. *Assume that $\kappa(x) = 2$ and x satisfies one of the following properties:*

*(i) x satisfies condition (*1), $\beta(x) = 2$ and, given well 2-adapted coordinates $(u_1, u_2; u_3; Z)$, the polynomial $\mathrm{in}_\alpha h = Z^p - G_\alpha^{p-1} Z + F_{p,Z,\alpha}$, where*

$$U_1^{-pd_1} F_{p,Z,\alpha} = \mu U_3^{\omega(x)} + \sum_{i=1}^{\omega(x)} \mu_i U_3^{\omega(x)-i} U_1^{iy_1} U_2^{iy_2}, \tag{7.84}$$

notations as in lemma 7.10(4) w.r.t. the face

$$\sigma_2 = \mathbf{y} := (A_1(x), \beta(x)) \in \Delta_2(h; u_1, u_2; u_3; Z)$$

has $\mu_i \neq 0$ for some i with $1 \leq i \leq p-1$;

*(ii) x satisfies condition (*3) and $\beta(x) < 2 - 1/p$.*

Then x is good.

Proof. We again consider three cases.

Assume that $x_1 = x'$ is the point at infinity. We review the proof of lemma 7.24 with our extra assumptions and claim that x' is resolved.

Under assumption (i), we get $1 \leq i_2 \leq p-1$ in (7.80) by (7.84). Turning to (A) and (B) in the proof of lemma 7.24, note that (A) does not hold since $\mu_1 \neq 0$ in (7.84). Finally if (B) holds, then $\beta(x_2) \leq 1 - 1/(p-1)$ because $1 \leq i_2 \leq p-1$. Therefore x_2 is resolved by proposition 7.21(3).

Under assumption (ii), note that (7.82) is strengthened to

$$0 \leq C(x) < 1 - 1/p \text{ and } \beta_2(x) < -1/p$$

since $\beta(x) < 2 - 1/p$. We thus get $1 \leq i_1 \leq p-1$ in (7.83). We also get $\beta(x_2) \leq 1 - 1/(p-1)$ if (B) holds, so x_2 is resolved by proposition 7.21(3).

Assume that $k(x_1) \neq k(x)$. If x_1 satisfies condition (*1), lemma 7.15(4) and lemma 7.16(1) give $\beta(x) < 2$ in any case. Therefore x_1 is resolved by proposition 7.23(i).

If x_1 satisfies condition (*3), the same conclusion holds under assumption (i) except possibly if $C(x) = d = 2$. By (7.84), we then get $\beta(x_1) \leq 1 - 1/(p-1)$ and x_1 is resolved by proposition 7.21(3). Under assumption (ii), x_1 satisfies again the assumption (ii) in this lemma with $\beta(x_1) < \beta(x)$ by lemma 7.16(3).

Assume that $x_1 \neq x'$ and $k(x_1) = k(x)$. The independence statement in theorem 7.12 reduces to

$$x_1 = (Z' := Z/u_1, u_1, u_2' := u_2/u_1, u_3' := u_3/u_1).$$

Note that the extra assumption (7.84) is unaffected by this coordinate change.

Under assumption (i), lemma 7.15(1) shows that x_1 is resolved or x_1 satisfies again condition (*1) with $\beta(x_1) \leq \beta(x) = 2$. By proposition 7.23(i), x_1 is resolved unless equality holds. In this case, we have

$$C(x) = \beta(x) = \beta(x_1) = 2$$

and x_1 satisfies again assumption (i) of this lemma.

Under assumption (ii), lemma 7.16 shows that x_1 is resolved or satisfies condition (*1) or (*3). If one of lemma 7.16(1')(2) applies, we have $\gamma(x) = 1$ and x_1 satisfies condition (*1) with $\beta(x_1) \leq 2$. We are done if inequality is strict by proposition 7.23(i); otherwise $\omega(x) = p = 2$ and x_1 satisfies (i) of this lemma.

Any other situation yields $\gamma(x_1) \leq \gamma(x)$. If x_1 satisfies condition (*3), then x_1 satisfies again (ii) of this lemma with $\beta(x_1) \leq \beta(x)$ by lemma 7.16(3). If x_1 satisfies condition (*1), we have $\beta(x_1) \leq 2$. We are done if inequality is strict by proposition 7.23(i).

Assume then that (x_1 satisfies condition (*1) and $\beta(x_1) = 2$). We argue as in the proof of lemma 7.16. Let

$$U_1^{-pd_1} F_{p,Z,\alpha} = (\mu U_1 + U_2) U_3^{\omega(x)} + \sum_{j \in J_0} U_3^{\omega(x)-j} U_1^{b_j} \Psi_j(U_1, U_2), \tag{7.85}$$

where $\mu \in k(x)$, $1 + d_j := \deg_{U_2} \Psi_j(U_1, U_2)$, with $b_j \geq jA_1(x)$, $d_j \leq j\beta(x)$, notations as in lemma 7.10(5). By assumption (ii), we have

$$j \in J_0 \Longrightarrow \frac{d_j}{j} < 2 - 1/p.$$

Note that for $j \in J_0$, we then have $1 + d_j \leq 2j$, and inequality is strict if $j \geq p$. If $\min J_0 \geq p$, arguing as in the proof of lemma 7.16 ($B(x) > 1$, cases 1 to 4), we then get $\beta(x_1) < 2$: a contradiction. This proves that

$$1 \leq j_0 := \min J_0 \leq p - 1. \tag{7.86}$$

Let $\mathbf{y}' := (A_1(x_1), \beta(x_1)) \in \Delta_2(h'; u_1, u_2'; u_3'; Z')$, where $(u_1, u_2'; u_3'; Z')$ are well 2-adapted coordinates. With notations as in lemma 7.10(4), the initial form polynomial $\mathrm{in}_{\alpha'} h'$ w.r.t. the face $\sigma_2' = \mathbf{y}'$ satisfies an equation (7.84), say

$$U_1{}^{-pd_1'} F_{p,Z',\alpha'} = \mu' U_3'^{\omega(x)} + \sum_{j=1}^{\omega(x)} \mu_j' U_3'^{\omega(x)-j} U_1{}^{jA_1(x_1)} U_2'^{2j}, \tag{7.87}$$

with $d_1' := d_1 + (1 + \omega(x))/p - 1$, $\mu_{j_0}' \neq 0$ by (7.86). Therefore x_1 satisfies assumption (i) in this lemma.

Summing up, the following has been proved: if x satisfies (i), then x_1 is resolved or ($k(x_1) = k(x)$ and x_1 satisfies again (i)). If x satisfies (ii), then x_1 is resolved or x_1 satisfies (i) or (ii); if (ii) holds, then $\beta(x_1) \leq \beta(x)$ and inequality is strict if $k(x_1) \neq k(x)$.

Consider the quadratic sequence (7.70). By the previous considerations, there exists $r_0 \geq 0$ such that either x_{r_0} is resolved, or (x_r satisfies one and the same assumption in the lemma with $k(x_r) = k(x_{r_0})$ for every $r \geq r_0$). One concludes the proof again by corollary 3.9. $\square$

We will now conclude the proof of theorem 7.18. Note the interesting extra twist for $p = 2$.

Proposition 7.26. *Assume that $\kappa(x) = 2$ and one of the following properties holds:*

*(i) x satisfies condition (*1) with $\beta(x) = 2$;*

*(ii) x satisfies condition (*3) and $\beta(x) < 2 - 1/\omega(x)$.*

Then x is good.

Proof. This is a variation on the two previous lemmas. Note that we may disregard the special case stated in lemma 7.16(1') in this proof.

Assume that $x_1 = x'$ is the point at infinity. By lemma 7.24, x' is resolved under assumption (i) (resp. (ii)) if $p \geq 3$ (resp. if $p = 2$). Reviewing the proof of lemma 7.24, we are done except possibly when (A) or (B) stated therein hold. If (A) holds, then x_2 is resolved by lemma 7.25(i). If (B) holds, x_2 satisfies condition (*3) with $\beta(x_2) \leq 1 + 1/p$. If $p \geq 3$ or if ($p = 2$ and $\beta(x_2) < 3/2$), we have $\beta(x_2) < 2 - 1/p$ and the conclusion follows from lemma 7.25(ii). Therefore x' is resolved or

$$p = 2 \text{ and } \beta(x_2) = 3/2.$$

In the special case $p = \omega(x) = 2$, an explicit computation gives $\beta(x_2) \leq 1$ if x_2 satisfies condition (*3) (*cf.* (ii) of proof of lemma 7.27 below), so x' is resolved. This proves that x_2 is resolved or satisfies again the assumptions of the proposition in any case.

Assume that $k(x_1) \neq k(x)$. Under assumption (i), x_1 is resolved or

$$\beta(x) \leq \frac{C(x)}{d} + \frac{1}{p} \leq 1 + \frac{1}{p}$$

by lemma 7.15(3). Then x_1 is resolved by proposition 7.23(i) or by lemma 7.25(ii) except possibly if x_1 satisfies (ii) with ($p = 2$, $\beta(x) = 3/2$); in this case, note that (x satisfies condition (*1), x_1 satisfies condition (*3)) implies that $d_1 \in \mathbb{N}$.

Under assumption (ii), x_1 is resolved or

$$\beta(x) < \frac{2}{d} + \frac{1}{p} \leq 1 + \frac{1}{p}$$

by lemma 7.15(2). Then x_1 is resolved in any case by proposition 7.23(i) or by lemma 7.25(ii).

Assume that $x_1 \neq x'$ and $k(x_1) = k(x)$. We may assume once again that x_1 is the origin of the first chart of the blowing up.

Under assumption (i), x_1 is resolved or x_1 satisfies again assumption (i): same proof as in lemma 7.25(i).

Under assumption (ii), x_1 is resolved or satisfies again one of (i)(ii): same proof as in lemma 7.25(ii). If x_1 satisfies again (ii), we have $\beta(x_1) \leq \beta(x)$ by lemma 7.16(3).

Summing up, it has been proved that x_1 is resolved or x_1 satisfies again the assumptions of the proposition. Under assumption (i), x_1 is resolved or one of the following properties holds:

(1) $k(x_1) = k(x)$ and x_1 satisfies again (i);

(2) $p = 2$ and x_1 satisfies (ii) with $\beta(x_1) = 3/2$;

(3) $p = 2$ and x_2 satisfies (ii) with $\beta(x_2) = 3/2$.

Under assumption (ii), x_1 is resolved or one of the following properties holds:

(1') $k(x_1) = k(x)$ and x_1 satisfies (i);

(2') $k(x_1) = k(x)$ and x_1 satisfies again (ii) with $\beta(x_1) \leq \beta(x)$.

Consider the quadratic sequence (7.70) and suppose that (2) (resp. (3)) above occurs. Suppose that event (1') occurs again at x_r for $r \geq 1$ (resp. for $r \geq 2$). By (2') and lemma 7.25(ii), we may assume that $\beta(x_r) = 3/2$, so x_r is resolved by lemma 7.27 below. Therefore there exists $r_0 \geq 0$ such that either x_{r_0} is resolved, or (x_r satisfies one and the same assumption (i) or (ii) with $k(x_r) = k(x_{r_0})$ for every $r \geq r_0$). The proof now concludes once again by corollary 3.9. □

Lemma 7.27. *Assume that $p = 2$, $\kappa(x) = 2$ and x satisfies condition (*3) with $\beta(x) = 3/2$. If x_1 satisfies condition (*1), then x_1 is resolved.*

Proof. We argue as in the proof of lemma 7.25 (7.85) and (7.87): we have $\beta(x_1) = 2$ and, since $\beta(x) = 3/2$, there exist well 2-adapted coordinates $(u_1, u_2'; u_3'; Z')$ at x_1 such that

$$U_1{}^{-2d_1'} F_{2,Z',\alpha'} = \mu' U_3'^{\omega(x)} + \sum_{j=1}^{\omega(x)} \mu_j' U_3'^{\omega(x)-j} U_1{}^{jA_1(x_1)} U_2'^{2j}, \tag{7.88}$$

with $d_1' := d_1 + (1+\omega(x))/2 - 1$, $\mu_1' \neq 0$ or $\mu_2' \neq 0$. We conclude by lemma 7.25(i) if $\mu_1' \neq 0$.

Assume then that $\mu_1' = 0$ and let $a := \mathrm{ord}_2\omega(x)$. If ($a = 1$, $A_1(x) \in \mathbb{N}$ and $\mu_2'\mu'^{-1} = \lambda^2$ for some $\lambda \in k(x)$), we may perform the Tschirnhausen transform $U_3' \mapsto U_3' + \lambda U_1^{A_1(x)} U_2'^2$ and get $\mu_2' = 0$ in (7.88). Since $\beta(x_1) = 2$, we nevertheless obtain $\mu_{j_0}' \neq 0$ for some $j_0 \geq 3$ in (7.88). In other terms, we may assume that one of the following assumptions holds:

(i) $a \geq 2$ and $\mu_2' \neq 0$;

(ii) $a = 1$, ($A_1(x) \notin \mathbb{N}$ or $\mu_2'\mu'^{-1} \notin k(x)^2$) and $\mu_2' \neq 0$;

(iii) $a = 1$, $A_1(x) \in \mathbb{N}$, $\mu_2' = 0$ and $\mu_{j_0}' \neq 0$ for some $j_0 \geq 3$.

We consider three cases and review again the proof of lemma 7.25:

Assume that $x_2 = x_1'$ is the point at infinity. Situation (A) has been solved in lemma 7.25(i). Situation (B) does not hold by [27] proof of **I.8.3**: equality $\beta(x_3) = 3/2$ is achieved only in the situation of *ibid.* **I.8.3.6** case 2. This implies ($\mu_j' = 0$ for $1 \leq j \leq 2^a - 1$, and $\mu_{2^a}' \neq 0$): a contradiction with (i) and (iii) above. This also implies $B(x) = A_1(x) + \beta(x) \in \mathbb{N}$ *viz.* [27] **I.8.3.4** (so $A_1(x) \in \mathbb{N}$ since $\beta(x) = 2$), and

$$U_1'^{-2d_1'} \frac{\partial F_{2,Z',\alpha'}}{\partial \lambda_l} \in < U_1'^{-2d_1'} U_1' \frac{\partial F_{2,Z',\alpha'}}{\partial U_1'} >, \ l \in \Lambda_0$$

viz. [27] **I.8.3.5** where $d_1' \notin \mathbb{N}$ here: a contradiction with (ii). One gets $\beta(x_3) < 3/2$ (actually: $\beta(x_3) \leq 1$ if x_3 satisfies condition (*3)), so x' is resolved by lemma 7.25(ii).

Assume that $k(x_2) \neq k(x_1)$. Then x_2 is resolved.

Assume that $x_2 \neq x_1'$ and $k(x_2) = k(x_1)$. Then x_2 is resolved or x_2 satisfies again (7.88) with $\mu_j' \neq 0$ for some $j \geq 1$, $j \leq 2$ if $a \geq 2$.

Iterating, the conclusion follows again from corollary 3.9.

Proposition 7.28. *Assume that $\kappa(x) = 2$, x satisfies condition (*2) with $\gamma(x) = 2$. Then x is good.*

Proof. By lemma 7.15, x_1 is resolved or satisfies again condition (*) with $\gamma(x_1) \leq 2$.

If x_1 satisfies condition (*1), then x_1 is resolved by proposition 7.23(i) or by proposition 7.26(i).

If x_1 satisfies condition (*3), we have $\beta(x_1) < 2 - 1/\omega(x)$ by lemma 7.15(4). Therefore x_1 is resolved by proposition 7.26(ii).

If x_1 satisfies condition (*2) and $\gamma(x_1) = 1$, x_1 is resolved by proposition 7.22. Therefore x_1 is resolved or satisfies again the assumptions of the lemma. The conclusion follows from lemma 7.19. □

Proposition 7.29. *Assume that $\kappa(x) = 2$, x satisfies condition (*3) with $\beta(x) = 2 - 1/\omega(x)$. Then x is good.*

Proof. This is now a variation on proposition 7.20. By lemma 7.16, x_1 is resolved or satisfies again condition (*) with $\gamma(x_1) \leq 2$ except in the special situation specified in lemma 7.16(2). Applying the previous lemmas, we are done except possibly if $k(x_1) = k(x)$ and (1) or (2) below holds:

(1) x_1 satisfies again condition (*3) with $\beta(x_1) = \beta(x) = 2 - 1/\omega(x)$;

(2) x_1 satisfies condition (*1) with $\beta(x_1) = 2 + 1/\omega(x)$.

Suppose that (2) holds; by lemma 7.15(1)(4) and lemma 7.17(2), x_2 is resolved ($\gamma(x_2) \leq 2$, $\beta(x_2) < 2 - 1/\omega(x)$ if x_2 satisfies condition (*3)) or satisfies again (2) with $k(x_2) = k(x_1)$. We conclude once more by corollary 3.9. □

8 Projection theorem: transverse and tangent cases, reduction of $\kappa(x) = 3, 4$ to monic expansions.

In this chapter and the next one, we prove theorem 5.1 when $\kappa(x) = 3, 4$ (definition 5.1). This is restated as theorem 9.6 below. The structure of

the proof is similar to that of theorem 7.18: first getting a stable form for the equation of $\mathrm{in}_{m_S}h$ (i.e. monic expansions, definition 8.1 below), then introducing a projected polygon with secondary invariant $\gamma(x)$.

Two important differences with $\kappa(x) = 2$ arise. On the one hand, no simple reduction works for each of $\kappa(x) = 3, 4$ separately and we have to deal with both cases at the same time. On the other hand, the monic case is resolved by blowing up Hironaka-permissible centers $\mathcal{Y} \subset \mathcal{X}$ which are not necessarily permissible in the sense of definitions 3.1 and 3.2.

Given a valuation μ of $L = k(\mathcal{X})$ centered at x, we consider finite sequences of local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{8.1}$$

with Hironaka-permissible centers $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$, *viz.* (5.2).

Up to the end of this chapter, "resolved" stands for "resolved for $(p, \omega(x), 3)$*" (remark 5.2).*

Definition 8.1. (Monic expansion for $\kappa(x) \geq 3$). Assume that $\kappa(x) \geq 3$. We say that x satisfies condition (**) if there exists well adapted coordinates $(u_1, u_2, u_3; Z)$ at x such that the following conditions are fulfilled:

(i) $1 + \omega(x) \neq 0 \bmod(p)$;

(ii) $E = \mathrm{div}(u_1)$ (resp. $E = \mathrm{div}(u_1u_2)$), and $\mathbf{v} := (d_1, d_2, (1 + \omega(x))/p)$ is the only vertex (resp. is a vertex) of $\Delta_S(h; u_1, u_2, u_3; Z)$ in the region $x_1 = d_1$.

Assume $\kappa(x) = 4$, we say that x satisfies condition (T**) (for "towards (**)") if there exists well adapted coordinates $(u_1, u_2, u_3; Z)$ at x such that *one* of the following conditions is fulfilled:

(i) $\epsilon(x) = \omega(x)$, $\mathrm{div}(u_1) \subseteq E$ and $\mathrm{Vdir}(x) =< U_1 >$;

(ii) $\epsilon(x) = \omega(x)$, $\mathrm{div}(u_1u_2) \subseteq E$ and $\mathbf{v} := (d_1 + \omega(x)/p, d_2, d_3)$ is the only vertex of $\Delta_S(h; u_1, u_2, u_3; Z)$ in the region $x_2 = d_2$;

(iii) $E = \mathrm{div}(u_1u_2)$ and $\mathbf{v} := (d_1 + \omega(x)/p, d_2, 1/p)$ is the only vertex of $\Delta_S(h; u_1, u_2, u_3; Z)$ in the region $x_2 = d_2$.

When x satisfies any of (**) or (T**), we simply say that "h has a *monic expansion* for $(u_1, u_2, u_3; Z)$". In cases (**) and (T**)(iii), the nonexceptional variable u_3 will usually be denoted v.

Remark 8.1. If x satisfies (i)(ii) or ((iii) with $\epsilon(x) = \omega(x)$) above for (T**), we have $\kappa(x) \leq 2$ or $\kappa(x) = 4$. On the other hand, one may have (iii) with $\kappa(x) = 3$ if $\epsilon(x) = 1 + \omega(x)$. We however claim that $\tau'(x) = 3$ in this situation. Namely, by definition 2.16,

$$\kappa(x) = 3 \Leftrightarrow H^{-1}\frac{\partial TF_{p,Z}}{\partial U_3} \notin k(x)[U_1, U_2].$$

W.l.o.g. it can be assumed that $U_3 \in \mathrm{Vdir}(x)$. By (iii), we then have

$$H^{-1}\frac{\partial TF_{p,Z}}{\partial U_3} = \lambda U_1^{\omega(x)} + U_2\Phi(U_1, U_2, U_3),$$

with $\lambda \neq 0$ and $\Phi \notin k(x)[U_1, U_2]$. It is then obvious that $\tau'(x) = 3$.

As a consequence, it is sufficient for our purpose to check (i)(ii) or (iii) in order to check (T**), since x is already resolved if $\kappa(x) \leq 3$.

8.1 Preliminaries: transverse case.

Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x, where $\kappa(x) = 3$. In particular, we have $\epsilon(x) = \omega(x) + 1$. The initial form polynomial

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z]$$

has $H^{-1}G^p \subset k(x)[U_1, \ldots, U_e]_{\omega(x)+1}$ and an expansion

$$U_1^{-pd_1}U_2^{-pd_2}F_{p,Z} = cU_3^{\omega(x)+1} + \sum_{i=0}^{\omega(x)} U_3^{\omega(x)-i}\Phi_{i+1}(U_1, U_2), \tag{8.2}$$

with $U_3 \in \mathrm{Vdir}(x)$, $c \in k(x)$ and $\Phi_i \in k(x)[U_1, U_2]_{i+1}$, $0 \leq i \leq \omega(x)$. Since $\kappa(x) = 3$, note that we have

$$\begin{cases} (\omega(x) + 1 \not\equiv 0 \bmod p \text{ and } c \neq 0), \text{ or} \\ \\ \Phi_{i+1}(U_1, U_2) \neq 0 \text{ for some } i \leq \omega(x) - 2,\ \omega(x) - i \not\equiv 0 \bmod p \end{cases}. \tag{8.3}$$

Proposition 8.1. *Assume that* $\kappa(x) = 3$, $E = \mathrm{div}(u_1 u_2)$ *and*

$$\mathrm{Vdir}(x) =< U_3, \lambda_1 U_1 + U_2 >, \ \lambda_1 \neq 0.$$

Then x *is resolved.*

Proof. Take $\mathcal{Y}_0 := \{x\}$ in (8.1) and assume that x_1 is very near x. Since $U_1 \notin \mathrm{Vdir}(x)$, we have $G = 0$. Let $u'_j := u_j/u_1$, $j = 2, 3$. By theorem 3.6, we have

$$x_1 = (X' := Z/u_1, u_1, v := u'_2 + \gamma_1, u'_3), \ E' = \mathrm{div}(u_1), \ k(x_1) = k(x),$$

where $\gamma_1 \in S$ is a preimage of λ_1. By assumption,

$$\Psi := U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z}}{\partial U_3} = \sum_{i=0}^{\omega(x)} c_i (\lambda_1 U_1 + U_2)^i U_3^{\omega(x)-i}$$

with $c_i \in k(x)$ and $c_i \neq 0$ for some $i \neq \omega(x)$. Let $(u_1, v, u'_3; Z')$ be well adapted coordinates at x_1. Applying proposition 3.5(v) (with $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$), we have

$$(\Psi(1, \overline{v} - \lambda_1, \overline{u}'_3)) \subseteq J(F_{p,Z',W'}, E', W') \subseteq k(x)[\overline{u}'_2, \overline{u}'_3]_{(\overline{v}, \overline{u}'_3)}. \tag{8.4}$$

Since $\kappa(x_1) \geq 3$ is assumed, we have

$$\mathrm{ord}_{(\overline{v}, \overline{u}'_3)} U_1^{-pd'_1} F_{p,Z',W'} = \epsilon(x),$$

where

$$d'_1 = d_1 + d_2 - 1 + \epsilon(x)/p \in \mathbb{N}. \tag{8.5}$$

If $\epsilon(x_1) = \epsilon(x)$, we get $\mathrm{Vdir}(x_1) + < U_1 > =< U_1, V, U'_3 >$ by (8.4), so $\kappa(x_1) = 2$ by definition 5.1: a contradiction. Therefore $\epsilon(x_1) = \omega(x)$. Let

$$\Phi' := \mathrm{cl}_{\epsilon(x)} U_1^{-pd'_1} F_{p,Z',W'} \in k(x)[\overline{V}, \overline{U}'_3]_{\epsilon(x)}.$$

We deduce from (8.4) that

$$\frac{\partial \Phi'}{\partial \overline{U}'_3} = \Psi(1, \overline{V} - \lambda_1, \overline{U}'_3), \ \mathrm{Vdir}\left(\frac{\partial \Phi'}{\partial \overline{U}'_3}\right) =< \overline{V}, \overline{U}'_3 > . \tag{8.6}$$

The proof is now a variation of that of proposition 7.5, $\tau'(x) = 1$, which we state in the following lemma for further use. The assumptions are satisfied by (8.5)-(8.6) and this concludes the proof. □

Lemma 8.2. *Assume that $\epsilon(x) = \omega(x)$ and $E = \mathrm{div}(u_1)$. Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x. Assume furthermore that the initial form polynomial*

$$\mathrm{in}_E h = Z^p + U_1^{pd_1}\overline{F},\ \overline{F} \in S/(u_1)$$

of lemma 7.4 has $d_1 \in \mathbb{N}$ and

$$\mathrm{Vdir}\left(\frac{\partial\Phi}{\partial\overline{U}_2}, \frac{\partial\Phi}{\partial\overline{U}_3}\right) =< \overline{U}_2, \overline{U}_3 >,$$

where $\Phi := \mathrm{cl}_{\omega(x)+1}\overline{F} \in k(x)[\overline{U}_2, \overline{U}_3]_{\omega(x)+1}$. Then x is resolved.

Proof. It can be assumed that $\kappa(x) = 4$, i.e. $\mathrm{Vdir}(x) =< U_1 >$. We then review the proof of proposition 7.5 for $\tau'(x) = 1$, cases 1 and 2. We take $\mathcal{Y}_0 := \{x\}$ in (8.1).

Case 2 of *loc.cit.* gives $\iota(x_1) \leq (p, \omega(x), 2)$ after blowing up x, hence x_1 is resolved. Similarly, case 1 yields $\iota(x_1) \leq (p, \omega(x), 2)$ or after possibly changing well adapted coordinates:

$$\Phi \in < \overline{U}_3^{\omega(x)+1}, \overline{U}_2\overline{U}_3^{\omega(x)} >,$$

with $\omega(x) \not\equiv 0 \bmod p$ and

$$x_1 = (Z' := Z/u_2, u_1' = u_1/u_2, u_2, u_3' := u_3/u_2),\ E' = \mathrm{div}(u_1'u_2). \tag{8.7}$$

The case $\omega(x) = 1$ is dealt with as in proposition 7.5.

Assume that $\omega(x) \geq 2$. Let $E_1 := \mathrm{div}(u_1') \subset \mathrm{Spec}S'$ be the strict transform of E. We get an expansion

$$\mathrm{in}_{E_1} h' = Z'^p + U_1'^{pd_1}\overline{F_1},\ \overline{F} \in S'/(u_1'),$$

where $d_1' = d_1$, $d_2' = d_1 - 1 + \omega(x)/p$ and

$$(\overline{u}_2)^{-(pd_2'+1)}\overline{F_1} \equiv (\overline{u}_3'^{\omega(x)}) \bmod \overline{u}_2. \tag{8.8}$$

It can be furthermore assumed that $\kappa(x_1) = 4$. By lemma 7.3(ii), we have $\mathrm{Vdir}(x_1) =< U_1' >$ or $\mathrm{Vdir}(x_1) =< U_1', U_2 >$ since $\widehat{pd_1} = 0$ is assumed in this lemma. We take $\mathcal{Y}_1 := \{x_1\}$ in (8.1) and first consider the point

$$x'' := (Z'' := Z'/u_3', u_1'' = u_1'/u_3, u_2'' := u_2/u_3', u_3'),\ E'' = \mathrm{div}(u_1''u_2''u_3').$$

By (8.8), we obtain $\omega(x'') < \omega(x)$ (resp. $\tau'(x'') = 3$) if $\omega(x) \geq 3$ (resp. if $\omega(x) = 2$), so x'' is resolved in any case. By theorem 3.6 it can therefore be assumed that

$$\mathrm{Vdir}(x_1) =< U_1' > . \tag{8.9}$$

Applying again (8.8), we obtain

$$(\overline{u}_2)^{-(pd_2'+1)} \frac{\partial \overline{F_1}}{\partial \overline{u}_3'} \equiv (\overline{u}_3'^{\,\omega(x)-1}) \bmod \overline{u}_2.$$

Once again, we obtain $\iota(x_2) \leq (p, \omega(x), 2)$ or after possibly changing well adapted coordinates:

$$x_2 = (Z'/u_2, u_1'' := u_1'/u_2, u_2, u_3'/u_2),\ E'' = \mathrm{div}(u_1'' u_2).$$

It is now clear that (8.8)-(8.9) are stable by blowing up. Iterating, we obtain that x_r is resolved for some $r \geq 1$ in (8.1) or there exists a formal curve $\hat{\mathcal{Y}} = V(\hat{Z}, u_1, \hat{u}_3)$ whose strict transform passes through all points x_r, $r \geq 1$. By proposition 3.8(1), it can be assumed that $\mathcal{Y} = V(Z, u_1, u_3)$ is permissible of the first kind. Then x is resolved by blowing up $\mathcal{Y}$ and the conclusion follows. □

Lemma 8.3. *Assume that $\kappa(x) = 3$. Then x is good, or there exist well adapted coordinates $(u_1, u_2, u_3; Z)$ at x and an expansion (8.2) such that one of the following properties holds.*

(1) we have

$$\begin{cases} \Phi_{i+1} \in k(x)[U_1],\ 0 \leq i \leq \omega(x) - 1, \text{ and} \\ \\ \Phi_{\omega(x)+1} = (\lambda_1 U_1 + \lambda_2 U_2) U_1^{\omega(x)},\ \lambda_1, \lambda_2 \in k(x) \end{cases} . \tag{8.10}$$

Furthermore ($\Phi_i = 0$ for every $i \geq 0$) or ($x_1 = x'$ in (8.1)), where

$$\mathcal{Y}_0 := \{x\} \text{ and } x' := (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_2);$$

(2) we have $E = \mathrm{div}(u_1 u_2)$, $\tau'(x) = 1$ and x satisfies condition (**) (definition 8.1).

Proof. We always take $\mathcal{Y}_0 := \{x\}$ in (8.1) and discuss according to x_1. It can be assumed that $\iota(x_1) \geq \iota(x)$ (in particular $\omega(x_1) = \omega(x)$).

First suppose that $x_1 = x'$. By proposition 2.6, $(u'_1, u_2, u'_3; Z')$ are well adapted coordinates at x'. Since $\epsilon(x') \geq \omega(x)$ by assumption, we deduce that $\deg_{U_2} \Phi_{i+1} \leq 1$, $0 \leq i \leq \omega(x)$. Similarly, $\Phi_{i+1} \in k(x)[U_1]$ for $\omega(x) - i \not\equiv 0 \bmod p$ (resp. for $\omega(x) - i \equiv 0 \bmod p$, $i \neq \omega(x)$) because $\omega(x') = \omega(x)$ (resp. because $\kappa(x') > 2$). Therefore (8.10) holds if $\iota(x') \geq \iota(x)$.

Assume now that $x_1 \neq x'$. By theorem 3.6, x_1 is resolved if

$$< U_1, U_3 > \subseteq \mathrm{Vdir}(x).$$

If ($E = \mathrm{div}(u_1 u_2)$ and $\tau'(x) = 2$), it can thus be assumed by symmetry on u_1, u_2 that $\mathrm{Vdir}(x) =< U_3, \lambda_1 U_1 + U_2 >$, $\lambda_1 \neq 0$. Then x is resolved by proposition 8.1. Since x_1 is very near x, it can be assumed from now on that

$$\mathrm{Vdir}(x) =< U_3 > . \tag{8.11}$$

We get in (8.2): $G = 0$ and $\Psi_{i+1} = 0$ for $\omega(x) - i \not\equiv 0 \bmod p$. By (8.3), we furthermore have

$$c \neq 0 \text{ and } \omega(x) + 1 \not\equiv 0 \bmod p. \tag{8.12}$$

If $E = \mathrm{div}(u_1 u_2)$, we therefore have (2) and the proof is complete.

Assume now that $E = \mathrm{div}(u_1)$. Let $I := \{i : \Phi_{i+1} \neq 0\}$. To conclude the proof, we will prove that

$$I \neq \emptyset \Longrightarrow x_1 \text{ is resolved.}$$

Let $i \in I$. By (8.11) and (8.12), we have

$$\omega(x) - i \equiv 0 \bmod p, \ i + 1 \not\equiv 0 \bmod p. \tag{8.13}$$

There is an expansion

$$\Phi_{i+1}(U_1, U_2) = U_1^{a_i} \Psi_{i+1}(U_1, U_2), \ a_i \geq 0. \tag{8.14}$$

where U_1 does not divide Ψ_{i+1}. By (8.11), we have $\frac{\partial \Phi_{i+1}}{\partial U_2} = 0$, therefore $\Psi_{i+1} \in k(x)[U_1^p, U_2^p]$, whence $a_i \geq 1$ by (8.13). Expand

$$\Psi_{i+1}(U_1, U_2) =: \mu_i U_2^{pb_i} + \cdots, \ \mu_i \neq 0, \ b_i \in \mathbb{N}.$$

After possibly changing Z with $Z-\phi$, $\phi \in S$, it can be assumed that $pd_1+a_i \not\equiv 0 \bmod p$ or $\mu_i \notin k(x)^p$. In particular (8.10) holds for $i=0$.

If $I=\{0\}$, $\kappa(x_1)>2$ implies that $\epsilon(x_1)=\omega(x_1)$: x_1 satisfies the assumptions of lemma 7.1 (or of lemma 7.2) and the conclusion follows.

Suppose that $i \geq 1$ in what follows. We can take a unitary polynomial $P(t) \in S[t]$, whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible and

$$x_1 = (X' := Z/u_1, u_1, u_2' := P(u_2/u_1), u_3' := u_3/u_1).$$

Let $(u_1, u_2', u_3'; Z')$ be well adapted coordinates at x_1. Given

$$D \in \left\{ U_1 \frac{\partial}{\partial U_1},\ \{\frac{\partial}{\partial \lambda_l}\}_{l \in \Lambda_0} \right\},$$

we let $\phi_{i,D}(t) := U_1^{-(pd_1+i+1)}(D \cdot U_1^{pd_1}\Phi_{i+1}) \in k(x)[t]_{\leq pb_i}$. By proposition 3.5(v), we have

$$\omega(x_1) \leq \min_{i,D}\{\omega(x) - i + \mathrm{ord}_{\overline{u}_2'}\phi_{i,D}(t)\} \leq \omega(x),$$

where equality holds only if $a_i = 1$ and $k(x_1) = k(x)$ by lemma 6.3(2). In particular we have $I \subset p\mathbb{N}$. Since $k(x_1) = k(x)$, it can be assumed w.l.o.g. that $P(t) = t$ and

$$\Phi_{i+1}(U_1, U_2) = \mu_i U_1 U_2^i, \text{ for every } i \geq 0$$

after possibly changing well adapted coordinates (including $i=0$, *cf.* above). Then $(u_1, u_2', u_3'; X')$ are well adapted coordinates at x_1 by proposition 2.6. We obtain: $\epsilon(x_1) = \omega(x)$ and

$$H'^{-1}F_{p,X'} = \sum_{k=0}^{\omega(x)/p} \mu_{kp} U_3'^{\omega(x)-kp} U_2'^{kp} + U_1\Phi',$$

for some $\Phi' \in k(x)[U_1, U_2', U_3']$. But then $\kappa(x_1) \leq 2$: a contradiction. This completes the proof when $E = \mathrm{div}(u_1)$. □

8.2 Preliminaries: tangent case.

Let $(u_1, u_2, u_3; Z)$ be well adapted coordinates at x, where $\kappa(x) = 4$. This splits into two different situations:

• if $\omega(x) = \epsilon(x)$, the initial form polynomial is of the form

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z} \in G(m_S)[Z], \tag{8.15}$$

where $H^{-1}F_{p,Z} \subset k(x)[U_1, \ldots, U_e]_{\omega(x)}$, $1 \leq e \leq 3$.

• if $\omega(x) = \epsilon(x) - 1$, the initial form polynomial is of the form

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z] \tag{8.16}$$

with $H^{-1}G^p \subset k(x)[U_1, \ldots, U_e]_{\omega(x)+1}$, $1 \leq e \leq 2$. By definition 2.16, we have

$$(0) \neq V(TF_{p,Z}, E, m_S) \subseteq k(x)[U_1, \ldots, U_e]_{\omega(x)}. \tag{8.17}$$

Definition 8.2. Assume that $\kappa(x) = 4$ and $\epsilon(x) = \omega(x)$. We say that $\mathrm{Vdir}(x)$ is skew if for every subset $J \subseteq \{1, \ldots, e\}$, we have

$$\mathrm{Vdir}(x) \neq < \{u_j\}_{j \in J} > .$$

Assume that $\mathrm{Vdir}(x)$ is skew and first note that $e = 2$ or $e = 3$. Elementary casuistics, similar to that performed in the proof of proposition 7.8, yield the following types up to reordering exceptional variables:

(T0) $E = \mathrm{div}(u_1u_2u_3)$ and

$$\mathrm{Vdir}(x) = < \lambda_1 U_1 + \lambda_2 U_2 + U_3 >, \ \lambda_1\lambda_2 \neq 0. \tag{8.18}$$

(T1) $E = \mathrm{div}(u_1u_2u_3)$ and

$$\mathrm{Vdir}(x) = < \lambda_1 U_1 + U_2, \lambda_2 U_2 + U_3 >, \ \lambda_1\lambda_2 \neq 0. \tag{8.19}$$

(T2) $E = \mathrm{div}(u_1u_2u_3)$ and

$$\mathrm{Vdir}(x) = < \lambda_1 U_1 + U_2, U_3 >, \ \lambda_1 \neq 0. \tag{8.20}$$

(T3) $\mathrm{div}(u_1u_2) \subseteq E \subseteq \mathrm{div}(u_1u_2u_3)$ and

$$\mathrm{Vdir}(x) = < \lambda_1 U_1 + U_2 >, \ \lambda_1 \neq 0. \tag{8.21}$$

Proposition 8.4. *Assume that* $\mathrm{Vdir}(x)$ *is skew. Assume furthermore that*

$$J(F_{p,Z}, E, m_S) \nsubseteq (U_3) \cap G(m_S)_{\epsilon(x)}$$

if x is of type (T2) above. Take (8.1) to be the quadratic sequence along μ.

Then there exists $r \geq 0$ such that either x_r is resolved or x_r satisfies condition (**). If $\omega(x) < p$, then x is resolved.

Proof. We discuss according to x_1 in (8.1), where $x_0 = x$ is of type (Tk) for some $k \in \{0,1,2,3\}$. It can be assumed w.l.o.g. that $\iota(x_1) \geq (p, \omega(x), 3)$. Let $u'_j := u_j/u_1$, $j = 2, 3$.

• *Assume that* $k = 0$. By (8.18), we have

$$J(F_{p,Z}, E, m_S) =< (\lambda_1 U_1 + \lambda_2 U_2 + U_3)^{\omega(x)} > . \tag{8.22}$$

We expand

$$U_1^{-pd_1} U_2^{-pd_2} U_3^{-pd_3} F_{p,Z} = \lambda U_3^{\omega(x)} + (\lambda'_1 U_1 + \lambda'_2 U_2) U_3^{\omega(x)-1} + \cdots$$

where $\lambda \neq 0$ by (8.22).

Suppose that $\omega(x) \not\equiv 0 \bmod p$. Since $\lambda_1\lambda_2 \neq 0$, we also have $\lambda'_1\lambda'_2 \neq 0$ by identifying the coefficient of $U_3^{\omega(x)-1}$ in (8.22). By lemma 7.3(ii) with $i = 1$, we deduce that

$$d_3 + \frac{\omega(x) - 1}{p} \in \mathbb{N}.$$

But then

$$U_1^{-pd_1} U_2^{-pd_2} U_3^{-pd_3} \left(U_3 \frac{\partial F_{p,Z}}{\partial U_3} \right) = \lambda U_3^{\omega(x)} + \Phi,$$

with $\deg_{U_3} \Phi \leq \omega(x) - 2$: a contradiction with (8.22) since $\lambda \neq 0$.

This proves that $\omega(x) \equiv 0 \bmod p$ (in particular $\omega(x) \geq p$). By lemma 7.3(i), it can thus be assumed that

$$U_1^{-pd_1} U_2^{-pd_2} U_3^{-pd_3} F_{p,Z} = \lambda(\lambda_1 U_1 + \lambda_2 U_2 + U_3)^{\omega(x)}$$

after possibly changing Z with $Z - \phi$, $\phi \in S$. After possibly reordering exceptional variables, we have

$$x_1 = (X' := Z/u_1, u_1, v := P(u'_2), w := u'_3 + \gamma_2 u'_2 + \gamma_1),$$

where $\gamma_1, \gamma_2 \in S$ are preimages of λ_1, λ_2 and $P(t) \in S[t]$ is a unitary polynomial whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible. Applying proposition 3.5(v) (with $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$), we have

$$J(F_{p,X',W'}, E', W') = (\overline{w}^{\omega(x)}) \subseteq k(x_1)[\overline{u}'_2, \overline{u}'_3]_{(\overline{v},\overline{w})}. \tag{8.23}$$

Since $\iota(x_1) \geq (p, \omega(x), 3)$ is assumed, (8.23) reads

$$U_1^{-pd'_1} \left(\frac{\partial F_{p,X',W'}}{\partial \overline{v}}, \frac{\partial F_{p,X',W'}}{\partial \overline{w}} \right) = (\overline{w}^{\omega(x)})$$

when $E' = \mathrm{div}(u_1)$. If (8.23) is achieved by $\frac{\partial}{\partial \overline{v}}$, we then have $\epsilon(x_1) = \omega(x)$ and x_1 satisfies the assumptions of lemma 7.1; hence x_1 is resolved. Otherwise (8.23) gives

$$U_1^{-pd'_1} U_2^{-pd'_2} F_{p,Z',W'} = (\overline{w}^{1+\omega(x)}),$$

for $E' = \mathrm{div}(u_1)$ or $E' = \mathrm{div}(u_1 v)$. This proves that x_1 satisfies condition (**).

• *Assume that* $k = 1$. By theorem 3.6 and (8.19), we have

$$x_1 = (X' := Z/u_1, u_1, v := u'_2 + \gamma_1, w := u'_3 + \gamma_2),\ E' = \mathrm{div}(u_1)$$

where $\gamma_1, \gamma_2 \in S$ are preimages of λ_1, λ_2.

Assume that $\epsilon(x_1) = \omega(x)$. By proposition 3.5(v), x_1 satisfies the assumptions of lemma 8.2 and the conclusion follows.

Assume now that $\epsilon(x_1) = 1 + \omega(x)$. Let $(u_1, v, w; Z')$ be well adapted coordinates at x_1. By proposition 3.5(v) and (8.19), we have

$$\mathrm{Vdir}(x_1) + < U_1 > = < U_1, V, W > .$$

This is a contradiction with definition 5.1, since $\kappa(x_1) \geq 3$ by assumption.

• *Assume that* $k = 2$. By theorem 3.6 and (8.20), we have

$$x_1 = (X' := Z/u_1, u_1, v := u'_2 + \gamma_1, u'_3),\ E' = \mathrm{div}(u_1 u'_3),\ k(x_1) = k(x),$$

where $\gamma_1 \in S$ is a preimage of λ_1. By assumption, there exists

$$\Phi := \sum_{i=0}^{\epsilon(x)} \Phi_i(U_1, U_2) U_3^{\omega(x)-i} \in J(F_{p,Z}, E, m_S)$$

with $\Phi_i \in k(x)[U_1, U_2]_i$ and $\Phi_{\omega(x)} = c(\lambda_1 U_1 + U_2)^{\omega(x)}$, $c \neq 0$. Applying proposition 3.5(v) (with $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$), we have

$$(\Phi(1, \overline{v} - \lambda_1, \overline{u}'_3)) \subseteq J(F_{p,Z,W'}, E', W') \subseteq k(x)[\overline{u}'_2, \overline{u}'_3]_{(\overline{v}, \overline{u}'_3)}. \tag{8.24}$$

Therefore x_1 satisfies condition (**) since $E' = \mathrm{div}(u_1 u'_3)$, $c \neq 0$.

Assume now that $\omega(x) < p$. By lemma 7.3(ii), we have

$$d_1, d_2 \notin \mathbb{N},\ d_3 \in \mathbb{N},\ \widehat{pd_1} + \widehat{pd_2} + \omega(x) = p. \tag{8.25}$$

If $d_j \geq 1$, $j = 1, 2, 3$, the center $\mathcal{Y}_j := V(Z, u_j)$ is Hironaka-permissible w.r.t. E. Blowing up finitely many times, we reduce to the case $d_3 = 0$, $0 < d_1, d_2 < 1$. By (8.25), we thus have

$$p\delta(x) = p(d_1 + d_2) + \omega(x) = p, \ \omega(x) \leq p - 2.$$

We thus deduce that $m(x_1) \leq 1 + \omega(x) < p$, hence x_1 is resolved.

• *Assume that* $k = 3$. If $\omega(x) < p$, we may assume to begin with that $\delta(x) = 1$ arguing as in (8.25) *sqq.* Let

$$x' := (Z' := X/u_3, v_1 := u_1/u_3, v_2 := u_2/u_3, u_3), \ E' := \mathrm{div}(v_1 v_2 u_3).$$

First assume that $x_1 \neq x'$. We have

$$x_1 = (Z/u_1, u_1, v := u_2' + \gamma_1, w := P(u_3')),$$

where $\gamma_1 \in S$ is a preimage of λ_1 and $P(t) \in S[t]$ is a unitary polynomial whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible. Let $(u_1, v, w; Z_1')$ be well adapted coordinates. Applying proposition 3.5(v) (with $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$), we have

$$J(F_{p,Z_1',W'}, E', W') = (\overline{v}^{\omega(x)}) \subseteq k(x_1)[\overline{u}_2', \overline{u}_3']_{(\overline{v},\overline{w})}. \tag{8.26}$$

The conclusion follows as for type (T0): x_1 satisfies condition (**) or x_1 is resolved by lemma 7.1. The latter holds if $\omega(x) < p$.

Assume now that $x_1 = x'$. By proposition 2.6, $(v_1, v_2, u_3; Z')$ are well adapted coordinates at x'. We deduce that $\epsilon(x') = \omega(x)$. Furthermore, (8.21) implies that

$$J(F_{p,Z'}, E', m_{S'}) \equiv < (\lambda_1 V_1 + V_2)^{\omega(x)} > \ \mathrm{mod}(U_3) \cap G(m_{S'})_{\epsilon(x')}. \tag{8.27}$$

Suppose that $\mathrm{Vdir}(x')$ is *not* skew. By (8.27), we have $\tau'(x') = 3$, hence x' is resolved.

Suppose that $\mathrm{Vdir}(x')$ is skew. By (8.27), x' is of type (Tk) for some $k \in \{0, 1, 2, 3\}$. Furthermore if $k = 2$, then x' satisfies again the extra assumption in the proposition also by (8.27). We are already done if $k \leq 2$, so we may assume again that x' is of type (T3) and iterate. In particular, we have $e = 3$. In case $\omega(x) < p$, we again have $d_j' = d_j$, $1 \leq j \leq 3$.

By proposition 3.8, it can be assumed that $\mathcal{Y} := V(Z, u_1, u_2)$ is permissible of the first kind. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along $\mathcal{Y}$ and $x_1' \in \pi^{-1}(x)$ satisfy $\iota(x_1') \geq (p, \omega(x), 3)$. By theorem 3.6, we have

$$x_1' = (X/u_1, u_1, v := u_2/u_1 + \gamma_1, u_3),\ E_1' = \mathrm{div}(u_1 u_3),$$

where $\gamma_1 \in S$ is a preimage of λ_1. Then x_1' satisfies condition (**). If $\omega(x) < p$, then $m(x_1') < p$ and x is resolved. □

Proposition 8.5. *Assume that $\kappa(x) = 4$, $\epsilon(x) = \omega(x)$ and $E = \mathrm{div}(u_1 u_2)$. Assume furthermore that the following properties are satisfied:*

(i) $\mathrm{Vdir}(x) =< U_1, U_2 >$*;*

(ii) the polyhedron $\Delta_S(h; u_1, u_2, u_3; Z)$ has a vertex of the form

$$\mathbf{v} := (v_1, d_2, v_3),\ v_1 + v_3 = \frac{1 + \omega(x)}{p},\ v_3 > \frac{1}{p},$$

where $(u_1, u_2, u_3; Z)$ are well adapted coordinates at x.

Take (8.1) to be the quadratic sequence along μ. There exists $r \geq 0$ such that either x_r is resolved or x_r satisfies condition (**). If $\omega(x) < p$, then x is resolved.

Proof. Suppose that x_1 is very near x. By (i) and theorem 3.6, we have

$$x_1 := (Z' := Z/u_3, u_1' := u_1/u_3', u_2' := u_2/u_3, u_3), E' := \mathrm{div}(u_1' u_2' u_3),$$

and the polyhedron $\Delta_{S'}(h'; u_1', u_2', u_3; Z')$ is minimal by proposition 2.6. Since $v_3 > 0$ in (ii), $\mathbf{v}$ is induced by $f_{p,Z}$ by theorem 2.14, and $f_{p,Z}$ has an expansion

$$f_{p,Z} = \sum_{\mathbf{x} \in \mathbf{S}} \gamma(\mathbf{x}) \prod_{j=1}^{3} u_j^{p x_j},\ \gamma(\mathbf{x}) \in S$$

such that $\gamma(\mathbf{v})$ is a unit. By (ii), x_1 is very near x only if $v_3 = 2/p$, i.e.

$$U_1'^{-pd_1'} U_2'^{-pd_2'} U_3^{-pd_3'} F_{p,Z'} = U_3(\lambda' U_1'^{\omega(x)-1} + U_3 \Phi') + \Phi(U_1', U_2') \tag{8.28}$$

for some $\Phi' \in k(x)[U_1', U_2', U_3]$, where $\lambda' \neq 0$ is induced by $\mathbf{v}$, and

$$(d_1', d_2', d_3') = (d_1, d_2, d_1 + d_2 - 1 + \frac{\omega(x)}{p}). \tag{8.29}$$

To conclude the proof, we compute $\mathrm{Vdir}(x_1)$. First note that

$$\mathrm{Vdir}(x_1) + < U_3 > = < U_1', U_2', U_3 > \tag{8.30}$$

by (i). If $\tau'(x_1) = 3$, then x_1 is resolved by theorem 3.6.

Suppose that $\tau'(x_1) \leq 2$. This gives

$$\mathrm{Vdir}(x_1) = < U_1' + \lambda_1' U_3, U_2' + \lambda_2' U_3 >, \ \lambda_1', \lambda_2' \in k(x). \tag{8.31}$$

Since $\lambda' \neq 0$, we have $(\lambda_1', \lambda_2') \neq (0,0)$. We are done by proposition 8.4 if $\lambda_1'\lambda_2' \neq 0$ (type (T1)) or if $\lambda_2' = 0$ (type (T2) where the extra assumption holds by (8.28)-(8.31)).

Suppose finally that

$$\mathrm{Vdir}(x_1) = < U_1', U_2' + \lambda_2' U_3 >, \ \lambda_2' \neq 0.$$

We now apply lemma 7.3(ii) to the $U_1'^{\omega(x)-1}$-term in (8.28), i.e. for the variables (U_3, U_2', U_1') respectively and $i = 1$. We deduce from (7.8) that

$$d_1' + \frac{\omega(x) - 1}{p} \in \mathbb{N}, \ d_2', d_3' \notin \mathbb{N} \text{ and } \widehat{pd_2'} + \widehat{pd_3'} + 1 = p.$$

Turning back to (8.29), we get

$$\widehat{pd_3'} = \widehat{pd_2'} + 1, \ 2(\widehat{pd_2'} + 1) = p.$$

This is a contradiction, since $p \geq 3$, and the proof is complete. □

8.3 Reduction to monic expansions (**) and (T**).

We can now conclude the reduction to monic expansions.

Proposition 8.6. *Assume that $\kappa(x) = 3$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent sequence of local permissible blowing ups of the first kind (8.1) along μ such that one of the following holds for some $r \geq 0$:*

*(i) x_r is resolved or satisfies condition (T**);*

(ii) x_r satisfies condition (**).

If $\omega(x) < p$ and $\tau'(x) = 2$, then (i) holds.

Proof. It can be assumed that the conclusion of lemma 8.3(1) above holds.

If $\Phi_{i+1} = 0$ for every $i \geq 0$, then x_1 satisfies condition (**) and we are done. Otherwise, we may furthermore assume that

$$x_1 = x' = (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3' := u_3/u_1),\ E' = \mathrm{div}(u_1' u_2)$$

with $\iota(x') \geq \iota(x)$. Note that when $E = \mathrm{div}(u_1 u_2)$, (8.10) marks an exceptional component $\mathrm{div}(u_1)$ of E.

If ($c \neq 0$ and $\omega(x) + 1 \not\equiv 0 \bmod p$), then x' satisfies condition (**) and we are done for $\omega(x) \geq p$. Otherwise (i.e. if either $\omega(x) < p$, either $c = 0$, or $\omega(x) + 1 \equiv 0 \bmod p$), we have $E' = \mathrm{div}(u_1' u_2)$ and $(u_1', u_2, u_3'; Z')$ are well adapted coordinates at x'. Furthermore $\mathrm{Vdir}(x) =< U_1, U_3 >$ (by (8.10) if $\omega(x) \geq p$, by assumption $\tau'(x) = 2$ if $\omega(x) < p$). Let

$$\Phi(U_1, U_3) := U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} \in k(x)[U_1, U_3]_{\omega(x)+1}$$

and consider two cases:

Case 1: $\epsilon(x') = \omega(x)$. We have $\kappa(x') = 4$ and

$$\Phi'(U_1', U_2) := U_1'^{-pd_1'} U_2^{-pd_2'} F_{p,Z'} = \lambda_2 U_1'^{\omega(x)} + U_2 \Phi_1'(U_1', U_2), \tag{8.32}$$

with $\Phi_1' \in k(x)[U_1', U_2]_{\omega(x)-1}$.

If $\tau'(x') = 1$ (i.e. $\Phi_1' = 0$ or $\mathrm{Vdir}(x) =< U_2 + \lambda U_1 >$, $\lambda \in k(x)$), then x' satisfies condition (T**) or x' satisfies the assumptions of proposition 8.4 type (T3) respectively, and the proof is complete. We may thus furthermore assume that

$$\mathrm{Vdir}(x') =< U_1', U_2 > . \tag{8.33}$$

Since $\kappa(x) = 3$, we have at this point:

$$\frac{\partial \Phi}{\partial U_3}(U_1, U_3) \notin k(x)[U_1]. \tag{8.34}$$

Therefore $\Delta_{S'}(h'; u_1', u_2, u_3'; Z')$ has a vertex of the form

$$\mathbf{v}' := (v_1', d_2', v_3'),\ v_1' + v_3' = \frac{1 + \omega(x)}{p},\ v_3' = \frac{\deg_{U_3} \Phi}{p},$$

where $(u_1, u_2, u_3; Z)$ are well adapted coordinates at x. The proposition follows from proposition 8.5 whose assumptions are satisfied by (8.33)-(8.34).

Case 2: $\epsilon(x') = \epsilon(x)$. We again have $\kappa(x') = 3$ and may iterate. Note that for $\omega(x) < p$, we have $\lambda_2 = 0$ in (8.10) and

$$H'^{-1}\frac{\partial F_{p,Z'}}{\partial U_3'} \equiv \Phi(U_1', U_3') \bmod(U_2) \cap G(m_{S'})_{\epsilon(x')}.$$

Then $\mathrm{Vdir}(x') + < U_2 > = < U_1', U_2, U_3' >$, so $\tau'(x') \geq 2$. We are done if $\tau'(x') = 3$ and may iterate if $\tau'(x') = 2$ as asserted.

Since the exceptional component $\mathrm{div}(u_1)$ of E has been marked (*cf.* beginning of the proof), the theorem holds except possibly if x_r is in case 2 for every $r \geq 0$. In this situation, we apply proposition 3.8(1): w.l.o.g. it can be assumed that $\mathcal{Y} := V(Z, u_1, u_3)$ is permissible of the first kind. Since $\mathrm{Vdir}(x) = < U_1, U_3 >$, it follows from theorem 3.6 that x is resolved by blowing up $\mathcal{Y}$. □

Lemma 8.7. *Assume that $\kappa(x) = 4$ and $\epsilon(x) = \omega(x)$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent sequence of local permissible blowing ups of the first kind (8.1) along μ such that one of the following holds for some $r \geq 0$:*

*(i) x_r is resolved or satisfies condition (T**);*

(ii) x_r satisfies condition (**).

If $\omega(x) < p$, then (i) holds.

Proof. By proposition 8.4, it can be assumed that one of the following conditions holds:

(1) $\mathrm{Vdir}(x)$ is skew and satisfies condition (T2);

(2) $\mathrm{div}(u_1u_2) \subseteq E$ and $\mathrm{Vdir}(x) = < U_1, U_2 >$.

Take (8.1) to be the quadratic sequence along μ. Under assumption (1), we have $E = \mathrm{div}(u_1u_2u_3)$ and $\mathrm{Vdir}(x) = < U_1, \lambda_2 U_2 + U_3 >$, $\lambda_2 \neq 0$ up to renumbering variables. By proposition 8.4, it can be assumed that

$$J(F_{p,Z}, E, m_S) \subseteq (U_1)G(m_S)_{\epsilon(x)}.$$

By theorem 3.6, we have

$$x_1 = (Z/u_2, u_1' := u_1/u_2, u_2, v := u_3/u_1 + \gamma),\ E' = \mathrm{div}(u_1' u_2),$$

where $\gamma \in S$ is a preimage of λ. Let $W' := \mathrm{div}(u_2) \subset \mathrm{Spec} S'$ and $(u_1', u_2, v; Z')$ be well adapted coordinates at x_1. By proposition 3.5(v), we have

$$J(F_{p,Z',W'}, E', W') = U_2^{-pd_2'} J(F_{p,Z}, E, m_S) \subseteq k(x_1)[\overline{u}_1', \overline{u}_3']_{(\overline{u}_1', \overline{v})}.$$

If $\mathrm{ord}_{(\overline{u}_1', \overline{v})} H_{W'}^{-1} F_{p,Z',W'} = \omega(x)$, we have $\kappa(x_1) \leq 2$ (so x is resolved) or

$$H'^{-1} F_{p,Z'} \equiv < U_1'^{\omega(x)} > \ \mathrm{mod}(U_2) \cap G(m_{S'})_{\omega(x)}.$$

In this last situation, the conclusion follows in each of the following possible cases:

• x_1 satisfies condition (T**) if $\mathrm{Vdir}(x_1) =< U_1' >$;

• x_1 satisfies the assumptions of proposition 8.4 if $\mathrm{Vdir}(x_1) =< U_1' + \lambda' U_2 >$, $\lambda' \neq 0$;

• x_1 satisfies the assumptions of proposition 8.5 if $\mathrm{Vdir}(x_1) =< U_1', U_2 >$.

If $\mathrm{ord}_{(\overline{u}_1', \overline{v})} H_{W'}^{-1} F_{p,Z',W'} = \omega(x) + 1$, we are also done by proposition 8.6 if $\kappa(x_1) = 3$, since $\tau'(x_1) \geq 2$. Assume finally that $\kappa(x_1) = 4$, i.e.

$$\epsilon(x_1) = \omega(x) = \mathrm{ord}_{(\overline{u}_1', \overline{v})} H_{W'}^{-1} \frac{\partial F_{p,Z',W'}}{\partial \overline{v}} < \mathrm{ord}_{(\overline{u}_1', \overline{v})} H_{W'}^{-1} F_{p,Z',W'} = \epsilon(x).$$

Similarly, x_1 satisfies condition (T**) unless $\mathrm{Vdir}(x_1) =< U_1', U_2 >$. The conclusion then follows again from proposition 8.5.

Under assumption (2), it can be assumed that $x_1 = x'$, $\iota(x') = \iota(x)$, where

$$x' := (Z' := Z/u_3, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3),\ E' := \mathrm{div}(u_1' u_2' u_3).$$

By proposition 2.6, $(u_1', u_2', u_3; Z')$ are well adapted coordinates at x'. We get $\epsilon(x') = \omega(x)$ and

$$\mathrm{Vdir}(x') + < U_3 > =< U_1', U_2', U_3 > .$$

If $\tau'(x') = 3$, then x' is resolved by theorem 3.6. Otherwise, x' satisfies again the assumptions of the proposition, with (1) up to renumbering variables or (2) above.

Iterating, the proof concludes as in the proof of proposition 8.6: x is resolved or the curve $\mathcal{Y} := V(Z, u_1, u_2)$ is permissible of the first kind; then x is resolved by blowing up $\mathcal{Y}$, since $\mathrm{Vdir}(x) =< U_1, U_2 >$. □

Proposition 8.8. *Assume that $\kappa(x) = 4$. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent sequence of local permissible blowing ups of the first kind (8.1) along μ such that one of the following holds for some $r \geq 0$:*

*(i) x_r is resolved or satisfies condition (T**);*

(ii) x_r satisfies condition (**).

If $\omega(x) < p$, then (i) holds.

Proof. By lemma 8.7, we are done if $\epsilon(x) = \omega(x)$. Otherwise one of the following conditions holds up to reordering exceptional variables:

(1) $E = \mathrm{div}(u_1 u_2)$, $\mathrm{Vdir}(x) =< U_1, U_2 >$;

(2) $E = \mathrm{div}(u_1 u_2)$, $\mathrm{Vdir}(x) =< \lambda_1 U_1 + U_2 >$, $\lambda_1 \neq 0$;

(3) $\mathrm{div}(u_1) \subseteq E \subseteq \mathrm{div}(u_1 u_2)$, $\mathrm{Vdir}(x) =< U_1 >$.

Take (8.1) to be the quadratic sequence along μ. We may always assume that

$$\iota(x_1) \geq (p, \omega(x), 3) \text{ and } \epsilon(x_1) = 1 + \omega(x) \tag{8.35}$$

in this proof. Let

$$x' := (Z/u_3, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3),\ \mathrm{div}(u_1' u_3) \subseteq E',$$

where $\frac{\partial TF_{p,Z}}{\partial U_3} \neq 0$. If $x_1 = x'$, we have $\epsilon(x') = \omega(x)$: a contradiction with (8.35). This concludes the proof under assumption (1) by theorem 3.6.

Assume that $x_1 \neq x'$. Under assumption (2), we can take a unitary polynomial $P(t) \in S[t]$, whose reduction $\overline{P}(t) \in k(x)[t]$ is irreducible, and

$$x_1 = (X' := Z/u_1, u_1, v := u_2/u_1 + \gamma_1, w := P(u_3/u_1)),\ E' = \mathrm{div}(u_1),$$

where $\gamma_1 \in S$ is a preimage of λ_1.

Let $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$ and $(u_1, v, w; Z')$ be well adapted coordinates at x_1, $Z' := X' - \phi$, $\phi \in S'$. By proposition 3.5(v), we deduce that

$$\mathrm{in}_{W'} h' = Z'^p + F_{p,Z',W'} \in G(W')[Z'],$$

where $G(W') = k(x_1)[\overline{u}'_1, \overline{u}'_3]_{(\overline{v},\overline{w})}[U_1]$ and

$$(\overline{v}^{\omega(x)}) \subseteq J(F_{p,Z',W'}, E', W'). \tag{8.36}$$

If $\omega(x) < p$, assumption (2) reads:

$$H^{-1}F_{p,Z} = U_3(U_2 + \lambda_1 U_1)^{\omega(x)} + \Phi(U_1, U_2),\ G = 0.$$

If $\Phi = 0$, this is a contradiction since then $\kappa(x_1) = 2$ by (8.36). After possibly performing a linear change of coordinates in u_3, then picking again well adapted coordinates, we reduce to:

$$\Phi(U_1, U_2) = U_1 U_2 \Psi(U_1, U_2).$$

Since $\Delta(h; u_1, u_2, u_3; Z)$ is minimal, we have $U_1^{pd_1} U_2^{pd_2} \Phi \notin G(m_S)^p$ and obtain

$$\mathrm{ord}_{(\overline{v},\overline{w})} J(F_{p,Z',W'}, E', W') \leq \deg \Psi = \omega(x) - 1,$$

also a contradiction, since $\omega(x_1) = \omega(x)$ is assumed.

If $\omega(x) \geq p$, we may then furthermore assume that $\epsilon(x_1) = \epsilon(x)$ by (8.35), so $\kappa(x_1) = 3$ by (8.36). We conclude by proposition 8.6.

Under assumption (3), we define a refinement $\mathcal{C}$ of the function $x \mapsto (m(x), \omega(x))$, *cf.* chapter 6. Let $\pi :\ \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along a permissible center of the first kind $\mathcal{Y} \subseteq \mathrm{div}(u_1)$, $x_1 \in \pi^{-1}(x)$. We set:

$$\mathcal{C}(x_1) < \mathcal{C}(x) \Leftrightarrow x_1 \text{ satisfies the conclusion of the proposition.}$$

By theorem 3.6, we have $\mathcal{C}(x_1) < \mathcal{C}(x)$ unless x_1 belongs to the strict transform $\mathrm{div}(u'_1) \subseteq E' = \mathrm{div}(u'_1 u_2)$ of $\mathrm{div}(u_1)$. Otherwise, we let $\mathcal{C}(x_1) = \mathcal{C}(x)$.

With notations as in chapter 6, we claim that $\mathrm{div}(u_1)$ has maximal contact for the condition $\mathcal{C}$ (definition 6.1). To see this, suppose that $\mathcal{C}(x_1) = \mathcal{C}(x)$ and apply proposition 8.6, lemma 8.7 and (1) and (2) above. It can be assumed that

$$\epsilon(x_1) = \epsilon(x),\ \kappa(x_1) \geq 3 \text{ and } \mathcal{Y} = \{x\}.$$

If $\omega(x) \geq p$, we are done unless x_1 satisfies again (3) and the claim is proved; if $\omega(x) < p$, we must still check that the situation

$$\kappa(x_1) = 3,\ \tau'(x_1) = 1$$

does not occur. By assumption (3), (8.17) with $G = 0$ gives an expansion

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = L(U_1, U_2, U_3) U_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} Q_{i+1}(U_2, U_3) U_1^{\omega(x)-i}$$

with $L(0, 0, U_3) \neq 0$, $Q_{\omega(x)+1}(U_2, U_3) \in k(x)[U_2^p, U_3^p]$. Therefore

$$(0) \neq V(F_{p,Z'}, E', m_{S'}) \subseteq k(x')[U_1', U_2]_{\omega(x)}$$

after blowing up, where $(u_1', u_2, v'; Z')$ are well adapted coordinates at x': a contradiction with $\kappa(x_1) = 3$, $\tau'(x_1) = 1$. This concludes the proof of the claim when $\omega(x) < p$. The proposition now follows from theorem 6.1. $\square$

9 Resolution of $\kappa(x) = 3, 4$ with monic expansions.

In this chapter, we prove projection theorem 5.1 in the case where $\kappa(x) \geq 3$.

9.1 Basic notations, an exit case.

In the particular case where $\omega(x) < p$, there may appear a very special kind of points x.

Definition 9.1. The point x is *combinatoric* if $\omega(x) < p$ and if the following algorithm starts and stops with success.
(i) if there exists $\mathrm{div}(u_i) \subset E$ such that $\mathrm{div}(u_i) \cap X$ is Hironaka-permissible, choose one and blow up X along this one,
(ii) if the center $x' \in X'$ of our valuation is not ω-near x: success,
(iii) if $x' \in X'$ ω-near x, and $\bar{e}(x') \leq 2$: success,
(iv) if $x' \in X'$ ω-near x, and $\bar{e}(x') = 3$ and there exists $\mathrm{div}(u_i) \subset E'$ such that $\mathrm{div}(u_i) \cap X'$ is Hironaka-permissible, go to (i),
(v) else failure.

Remark 9.1. The reader sees easily that:
(i) if $\mathrm{div}(u_1) \cap X$ is Hironaka-permissible, and if we blow up X along $\mathrm{div}(u_1) \cap X$, there is at most one near point (in Hironaka's sense): the point x' of parameters $X/u_1, u_1, u_2, u_3$,
(ii) if it is so, $\omega(x') = \omega(x)$.

9.2 From (T**) to (**), resolution for $\epsilon(x) = \omega(x) < p$.

The purpose of this section is to reduce theorem 5.1 for $\kappa(x) = 3, 4$ to points satisfying condition (**) in definition 8.1. We prove the following proposition.

Proposition 9.1. *Let x be in the case (T**) of definition 8.1, and μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent sequence of permissible blowing ups of the first kind*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r),$$

where x_i is the center of μ in $\mathcal{X}_i$, $0 \leq i \leq r$, such that x_r is resolved or (x_r satisfies condition (**) and $\omega(x) \geq p$).

Proof. By proposition 9.3 below, there is weak maximal contact (definition 6.1) for the condition

$$\mathcal{C} := \{(T^{**}) \text{ and } \iota(x) \geq (p, \omega(x), 3)\}.$$

Furthermore nonresolved points created by blowing up along closed points satisfy condition (**) with $\omega(x) \geq p$ (proposition 9.3(i)).

Theorem 6.1 does not apply directly since maximal contact does not necessarily hold (proposition 9.4 below). We must check that its proof remains valid when using only those blowing ups of the first kind which are well behaved w.r.t. $\mathcal{C}$. Blowing ups along permissible curves $\mathcal{Y}$ of the first kind are used in:

Proof of theorem 6.1(b): $\mathcal{Y} = V(Z, u_1, u_3)$, $E = \operatorname{div}(u_1 u_2)$. Then $\mathcal{Y}$ satisfies assumption (3) of proposition 9.4 except possibly in case (T**)(i). Let $W := \eta(\mathcal{Y})$ and expand:

$$U_1^{-pd_1} \overline{u}_2^{-pd_2} F_{p,Z,W} = \overline{\gamma}_0 U_1^{\omega(x)} + \overline{u}_2 \sum_{i=1}^{\omega(x)} \overline{\gamma}_i U_1^{\omega(x)-i} U_3^i \in G(W)_{\omega(x)},$$

with $\overline{\gamma}_i \in S/(u_1, u_3)$, $\overline{\gamma}_0$ a unit. We are done by proposition 9.4(1) if $\overline{\gamma}_i = 0$ for $1 \leq i \leq \omega(x)$. Otherwise, blow up along x. There is nothing to prove except at the point $x' := (Z/u_2, u_1' := u_1/u_2, u_2, u_3/u_2)$ on the strict transform $\mathcal{Y}'$ of $\mathcal{Y}$, $E' = \operatorname{div}(u_1' u_2)$. Then x' is now in case (T**)(ii) and the conclusion follows from proposition 9.4, assumption (2).

Proof of proposition 6.9(a): $\mathcal{Y} = V(Z, u_1, u_2)$, $E = \mathrm{div}(u_1 u_2)$. We use notations as in proposition 6.9. Assumption (1) in proposition 9.4 is equivalent to $A_2(x) > 1$. If $A_2(x) = 1$, there is an expansion

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z,W} = \overline{\gamma}_0 U_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} \overline{\gamma}_i U_1^{\omega(x)-i} U_2^i, \ W := \eta(\mathcal{Y}),$$

with $\overline{\gamma}_i \in S/(u_1, u_2)$, $\overline{\gamma}_0$ a unit. Then

$$\min_{1 \le i \le \omega(x)} \left\{ \frac{\mathrm{ord}_{\overline{u}_3} \overline{\gamma}_i}{i} \right\} = \beta(x) \le 1,$$

since $\gamma(x) = 1$ is assumed here. We prove that proposition 9.1 holds in this situation.

If $\beta(x) > 0$, we have $\mathrm{VDir}(x) =< U_1 >$ and get $\iota(x') \le (p, \omega(x), 2)$ after blowing up, so x is resolved by blowing up along $\mathcal{Y}$.

If $\beta(x) = 0$, we blow up along x. By proposition 9.3 below (proof in case (T**)(ii)), we get x' resolved or (x' satisfies condition (**) with $\omega(x) \ge p$) except if $x' = (Z/u_3, u_1' := u_1/u_3, u_2' := u_2/u_3, u_3)$ is the point on the strict transform $\mathcal{Y}'$ of $\mathcal{Y}$, $E' = \mathrm{div}(u_1' u_2' u_3)$. We now have $\mathrm{VDir}(x') =< U_1', U_2' >$ or $\mathrm{VDir}(x') =< \lambda_1 U_1' + U_2' >$, $\lambda_1 \ne 0$. Blowing up along $\mathcal{Y}'$ gives x'' resolved or (x'' satisfies (**) with $\omega(x) \ge p$), arguing as in the proof of proposition 9.4 below, assumption (2).

Proof of proposition 6.9(b)(c): $\mathcal{Y} = V(Z, u_1, u_j)$, $E = \mathrm{div}(u_1 u_2 u_3)$, $j = 2$ or $j = 3$. Assumption (1) (resp. assumption (2)) of proposition 9.4 is equivalent to $A_j(x) > 1$ (resp. to: $j = 3$ and $A_2(x) > 0$). By symmetry, there remains to deal with the case $\mathcal{Y} = V(Z, u_1, u_3)$ with $A_2(x) = 0$, $A_3(x) = 1$. There is an expansion

$$u_1^{-pd_1} u_2^{-pd_2} u_3^{-pd_3} f_{p,Z} = \gamma u_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} f_i u_1^{\omega(x)-i} u_3^i, \ f_i \in S$$

with $\gamma \in S$ a unit. Let $\overline{f}_i \in S/(u_1)$ be the residue of f_i. Then

$$\min_{1 \le i \le \omega(x)} \left\{ \frac{\mathrm{ord}_{(\overline{u}_2, \overline{u}_3)} \overline{f}_i}{i} \right\} = C(x) < 1,$$

since $\gamma(x) = 1$ is assumed here. Arguing as in (a) above, we consider two cases: $C(x) > 0$ and $C(x) = 0$. Blowing up along $\mathcal{Y}$, we get respectively

x resolved; x' resolved or (x' satisfies (**) with $\omega(x) \geq p$). Proposition 9.1 holds in any case. $\square$

This proposition leads to:

Corollary 9.2. *Assume that $\omega(x) < p$ and either $\kappa(x) = 4$, or ($\kappa(x) = 3$ and $\tau'(x) = 2$). Then x is resolved.*

Proof. Indeed, by propositions 8.6 and 8.8, there exists an independent sequence of local blowing ups (8.1) along μ such that x_r is resolved or x_r satisfies condition (T**). In the last case, apply proposition 9.1. $\square$

Proposition 9.3. *Let x be in the case (T**) of definition 8.1. Then* $\mathrm{div}(u_1)$ *has weak maximal contact (definition 6.1) for the condition (T**) and $\kappa(x) \geq 3$. More precisely, let $\pi : \mathcal{X}' \longrightarrow (\mathcal{X}, x)$ be the blowing up along x and $x' \in \pi^{-1}(x)$, with $\iota(x') \geq (p, \omega(x), 3)$:*

(i) if x' is not on the strict transform of div(u_1), then x' is resolved or satisfies (**) with $\omega(x) \geq p$;

*(ii) if x' is on the strict transform of div(u_1), then x' satisfies (T**).*

Proof. In the case (T**)(i), the reader sees that $< U_1 >= \mathrm{Vdir}(x)$ and, if we blow up along x, any point x' with $\iota(x') \geq (p, \omega(x), 3)$ verifies (T**)(ii) or (iii).

In the case (T**)(ii) and not (i), we have

$$U_1^{-pd_1} U_2^{-pd_2} U_3^{-pd_3} F_{p,Z} = \lambda_0 U_1^{\omega(x)} + U_2 P(U_1, U_2, U_3),$$

by (8.15), with $0 \neq \lambda_0 \in k(x)$, $0 \neq P \in k(x)[U_1, \ldots, U_e]_{\omega(x)-1}$.

Either

$$\mathrm{VDir}(x) =< U_1, U_2 >,$$

then $\iota(x') > (p, \omega(x), 1)$ only if $x' = (Z/u_3, u_1/u_3, u_2/u_3, u_3)$ by theorem 3.6. Clearly $\iota(x') < (p, \omega(x), 3)$ or x' satisfies (T**)(ii). Or we have

$$\mathrm{VDir}(x) =< \lambda_1 U_1 + U_2 >, \ \lambda_1 \neq 0.$$

This is case (T3) of proposition 8.4. Arguing as in its proof, *cf.* (8.26), x' satisfies condition (**) with $\omega(x) \geq p$ or x' is resolved by lemma 7.1 except possibly if $x' = (Z/u_3, u_1/u_3, u_2/u_3, u_3)$. Then $\iota(x') < (p, \omega(x), 3)$ or x' satisfies again (T**)(ii).

In the case (T**)(iii), we apply lemma 7.1 when $\epsilon(x) = \omega(x)$: x is resolved for $\iota = (p, \omega(x), 2)$. Assume that $\epsilon(x) = 1 + \omega(x)$. By remark 8.1, we may assume $\kappa(x) = 4$.

If $x' = (Z/u_3, u_1/u_3, u_2/u_3, u_3)$, we have $\omega(x') \leq \omega(x)$ and in case of equality, $\epsilon(x') = \omega(x)$ and x' satisfies (T**)(ii). In particular, we are done if $\mathrm{VDir}(x) =< U_1, U_2 >$ by theorem 3.6. There remains to deal with the case $\tau'(x) = 1$.

Case 1: $\mathrm{VDir}(x) =< U_1 >$. Expand

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = U_3 U_1^{\omega(x)} + U_2 Q,\ Q \in k(x)[U_1, U_2, U_3]_{\omega(x)}. \tag{9.1}$$

If $Q = 0$, the reader sees that x' satisfies (T**)(ii) or (T**)(iii) if $\iota(x') \geq (p, \omega(x), 3)$. The difficult case is $Q \neq 0$. By (8.17), we have

$$V(TF_{p,Z}, E, m_S) = H^{-1} \frac{\partial TF_{p,Z}}{\partial U_3} \subseteq < U_1^{\omega(x)} > .$$

This gives $\frac{\partial Q}{\partial U_3} = 0$, i.e. $Q \in k(x)[U_1, U_2, U_3^p]$ in both cases $G = 0$ and $G \neq 0$. Expand again

$$Q = \sum_{i=0}^{i_0} U_1^{\omega(x)-i} Q_i(U_2, U_3^p),\ Q_{i_0}[U_2, U_3^p] \neq 0. \tag{9.2}$$

If $i_0 = 0$, we reduce to $Q = 0$ after possibly picking new well adapted coordinates $(u_1, u_2, v; Z')$ at x.

If $i_0 \geq 1$, we apply proposition 3.5(v) to those elements of $J(F_{p,Z}, E, m_S)$ of the form:

$$U_1^{-pd_1} U_2^{-pd_2} D \cdot F_{p,Z} = \lambda_D U_3 U_1^{\omega(x)} + U_2 \sum_{i=0}^{i_0} U_1^{\omega(x)+1-i} Q_{i,D}(U_2, U_3^p),$$

where $D \in \{U_1 \frac{\partial}{\partial U_1}, U_2 \frac{\partial}{\partial U_2}, \{\frac{\partial}{\partial \lambda_l}\}_{l \in \Lambda_0}\}$.

By lemma 6.3(2) (applied to $F := Q_{i_0}(U_2, U_3^p)$), we get $\omega(x') \leq \omega(x)$ with strict equality if $k(x') \neq k(x)$. If $k(x') = k(x)$, it can be assumed w.l.o.g. that $x' = (Z/u_2, u_1/u_2, u_2, u_3/u_2)$. Then $\iota(x') \leq (p, \omega(x), 2)$ by (9.1)-(9.2) and the conclusion follows.

Case 2: $\mathrm{VDir}(x) =< \lambda_1 U_1 + U_2 >$, $\lambda_1 \neq 0$. We now have $G = 0$ and expand

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = U_3 (\lambda_1 U_1 + U_2)^{\omega(x)} + U_2 Q,\ Q \in k(x)[U_1, U_2, U_3^p]_{\omega(x)}.$$

If $Q \neq 0$, we expand again

$$Q = \sum_{i=0}^{i_0} U_3^{pi} Q_{\omega(x)-i}(U_1, U_2), \;\; Q_{\omega(x)-i_0}[U_1, U_2] \neq 0.$$

Since $(u_1, u_2, u_3; Z)$ are well adapted coordinates, we have

$$U_1^{pd_1} U_2^{pd_2+1} Q_{\omega(x)-i_0}[U_1, U_2] \notin G(m_S)^p.$$

If $i_0 = 0$, we argue as in the proof of proposition 8.8, *cf.* (8.36) *sqq.*: after possibly picking new well adapted coordinates $(u_1, u_2, v; Z')$ at x, it can be assumed that U_1 divides $Q = Q_{\omega(x)}[U_1, U_2]$. We get $\omega(x') < \omega(x)$ if $Q \neq 0$; if $Q = 0$, we obtain $\iota(x') \leq (p, \omega(x), 2)$ or x' satisfies the assumptions of lemma 7.1 (lemma 7.2 if $\omega(x) = 1$), so x' resolved. In particular, the proof is complete if $\omega(x) < p$.

If $i_0 \geq 1$, arguing as in case 1, we obtain $\omega(x') < \omega(x)$ except possibly if $k(x') = k(x)$ and

$$a(1) := pd_1, \;\; a(2) := pd_2 + 1, \;\; a(3) := 0, \;\; F_0 := Q_{\omega(x)-i_0}[U_1, U_2]$$

satisfies the assumptions of lemma 7.3 with $\lambda = \lambda_1^{-1}$. Then it can be assumed w.l.o.g. that $x' = (Z/u_1, u_1, \gamma_1 + u_2/u_1, u_3/u_1)$, where $\gamma_1 \in S$ is a unit with residue λ_1. We obtain $\iota(x') \leq (p, \omega(x), 2)$ or x' satisfies the assumptions of lemma 8.2. Then x' is resolved and this concludes the proof.

□

Proposition 9.4. *Let x be in the case (T**) of definition 8.1 and $\mathcal{Y} \subset (\mathcal{X}, x)$ be a permissible curve of the first kind, $\eta(\mathcal{Y}) \subset \mathrm{div}(u_1)$, with generic point y. Let*

$$\pi : \;\; \mathcal{X}' \longrightarrow (\mathcal{X}, x)$$

be the blowing up along $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$, $\iota(x') \geq (p, \omega(x), 3)$. Assume furthermore that one of the following extra assumptions holds:

(1) $\mathrm{VDir}(y) =< U_1 >$*;*

(2) $\mathcal{Y} = V(Z, u_1, u_3)$ *and x satisfies (T**)(ii) or (iii),*

where $(u_1, u_2, u_3; Z)$ are well adapted coordinates. Then one of the following holds:

(i) x' is resolved, or (x' satisfies (**) with $\omega(x) \geq p$);

*(ii) x' maps to the strict transform of div(u_1) and satisfies (T**).*

Proof. As $\mathcal{Y}$ has normal crossings with E, we can choose in any case well adapted coordinates $(u_1, u_2, u_3; Z)$ at x such that $\mathcal{Y} = V(Z, u_1, u_i)$, $i = 2$ or $i = 3$.

Let us see the case where $\mathcal{Y} = V(Z, u_1, u_2)$, $\mathrm{div}(u_1 u_2) \subseteq E$, up to renumbering u_2, u_3. As $\mathcal{Y}$ is a permissible curve of the first kind, we have

$$U_1^{-pd_1} U_2^{-pd_2} U_3^{-pd_3} F_{p,Z} \in k(x)[U_1, U_2]_{\epsilon(x)}$$

by proposition 3.1, in particular $\epsilon(x) = \omega(x)$.

If $< U_1 > \subseteq \mathrm{VDir}(x)$, we are done by theorem 3.6 unless equality holds and $x' = (Z' := Z/u_2, u_1' := u_1/u_2, u_2, u_3)$. We may therefore assume that x satisfies (T**)(i). Note that $(u_1', u_2, u_3; Z')$ are well adapted coordinates at x' by proposition 2.6. The proof is trivial under assumption (1) and we get x' resolved or (T**)(ii). Under assumption (2) (with u_2, u_3 relabeled), we have $E = \mathrm{div}(u_1 u_2 u_3)$ and there is an expansion

$$u_1^{-pd_1} u_2^{-pd_2} u_3^{-pd_3} f_{p,Z} \equiv \gamma u_1^{\omega(x)} \bmod u_3 (u_1, u_2)^{\omega(x)},$$

with $\gamma \in S$ a unit. We get x' resolved or (T**)(ii).

Finally if $\mathrm{VDir}(x) =< \lambda_1 U_1 + U_2 >$, $\lambda_1 \neq 0$, x is in case (T**)(ii) with $E = \mathrm{div}(u_1 u_2 u_3)$, assumption (2) (with u_2, u_3 relabeled). We are done by theorem 3.6 unless

$$x' = (X' := Z/u_1, u_1, v := \gamma_1 + u_2/u_1, u_3), \ E' = \mathrm{div}(u_1 u_3),$$

where $\gamma_1 \in S$ is a unit with residue λ_1. Applying proposition 3.5(v) (with $W' := \mathrm{div}(u_1) \subset \mathrm{Spec} S'$), we get

$$J(F_{p,X',W'}, E', W') = (\overline{v}^{\omega(x)}) \subseteq S/(u_1, u_2)[\overline{u}_2']_{(\overline{v}, \overline{u}_3)}. \tag{9.3}$$

If $\iota(x_1) \geq (p, \omega(x), 3)$, (9.3) thus reads

$$U_1^{-pd_1'} \overline{u}_3^{-pd_3} \frac{\partial F_{p,X',W'}}{\partial \overline{v}} = (\overline{v}^{\omega(x)}),$$

where $d_1' := d_1 + d_2 + \omega(x)/p - 1$, i.e. x' satisfies condition (**). This situation occurs only if $(d_1', d_3) \in \mathbb{N}^2$; therefore x' is combinatoric if $\omega(x) < p$.

Let us now see the case where $\mathcal{Y} = V(Z, u_1, u_3)$, $E = \mathrm{div}(u_1 u_2)$. If $\epsilon(x) = \omega(x)$, we thus have $\mathrm{VDir}(x) =< U_1 >$ by proposition 3.1, in particular x satisfies (T**)(i) or (ii). We are done by theorem 3.6 unless $x' = (Z/u_3, u_1/u_3, u_2, u_3)$. The reader ends the proof easily as above, under either assumption (1) or (2): we get x' resolved or (T**)(ii).

If $\epsilon(x) = 1+\omega(x)$, x satisfies (T**)(iii) by assumption (2). By proposition 3.1, we have $H^{-1}F_{p,Z} =< U_3 U_1^{\omega(x)} >$. Since $\mathrm{VDir}(x) =< U_1 >$, we are done by theorem 3.6 unless $x' = (Z/u_3, u_1/u_3, u_2, u_3)$. The reader ends the proof easily as before. □

9.3 Resolution for (**), the end.

The purpose of this section is to prove the following proposition and theorem which end the proof of projection theorem 5.1.

Proposition 9.5. *Assume that x is in case (**) (definition 8.1), then x is resolved for $\iota = (p, \omega(x), 3)$.*

Proof. This follows from corollary 9.8 and propositions 9.17 and 9.18 below. □

Theorem 9.6. *Assume that $\kappa(x) \geq 3$, then x is resolved.*

Proof. By propositions 8.6 and 8.8, it can be assumed that

$$\kappa(x) \geq 3, \ x \text{ satisfies } (**) \text{ or } (\mathrm{T}**).$$

By proposition 9.1, the remaining case is when x satisfies (**). This case is just the assumption of proposition 9.5. □

9.3.1 An extra assumption on the singular locus.

The following extra assumption **(E)'** is used as a shortcut in order to ensure that certain exceptional curves on $\mathcal{X}$ are Hironaka-permissible and can be blown up in order to reduce $\omega(x)$. Such blowing up centers are not used in [19] and the authors do not know if such blowing ups are relevant in dimension $n \geq 4$.

Definition 9.2. We say that (S, h, E) satisfies condition **(E)'** if it satisfies condition **(E)** and if

$$\omega(x) \geq p \Longrightarrow \eta^{-1}(E) = \mathrm{Sing}_p \mathcal{X}.$$

where $\eta^{-1}(m_S) =: \{x\}$.

As stated after definition 2.11, we have in any case $\mathrm{Sing}_p\mathcal{X} \subseteq \eta^{-1}(E)$ whenever (S, h, E) satisfies condition **(E)**.

Proposition 9.7. *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a permissible blowing up (of the first or second kind) at $x \in \eta^{-1}(m_S)$ and $x' \in \pi^{-1}(x)$. If (S, h, E) satisfies condition **(E)'**, then (S', h', E') satisfies again **(E)'** at x'.*

Proof. This reduces to proposition 2.13 if $\omega(x) \leq p - 1$. Assume that $\omega(x) \geq p$, so we have $d_j \geq 1$, $1 \leq j \leq e$, by assumption **(E)'**. Let $\mathcal{Y} \subset \mathcal{X}$ be permissible with generic point y, $W := \eta(\mathcal{Y}) = V(\{u_j\}_{j\in J}) \subset E$ and $I(W)S' =: (u)$, where

$$\eta' : \ (\mathcal{X}', x') \longrightarrow \mathrm{Spec}S'$$

is the projection. By definition 3.1 or proposition 3.3, we have $\epsilon(y) \geq \omega(x) \geq p$. Applying proposition 3.5(iv), we have $H(x') = u^{\epsilon(y)-p}H(x)S'$, therefore

$$\mathrm{ord}_{(u)}H(x') = \epsilon(y) - p + \mathrm{ord}_W H(x) \geq \min\{pd_j : j \in J_E\} \geq p$$

and the conclusion follows. □

Corollary 9.8. *It can be assumed that condition **(E)'** holds in the proof of proposition 9.5 and theorem 9.6.*

Proof. All blowing ups used in the proofs of propositions 8.6, 8.8 and 9.1 are permissible of the first kind. □

Lemma 9.9. *Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent composition of local permissible blowing ups of the first kind:*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r),$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that x_r is resolved or $H(x_r) \neq (1)$.

Proof. It can be assumed that $\omega(x) \geq p$. Since resolved means "resolved for $(p, \omega(x), 3)$" in this section, it can be assumed that

$$\omega(x_i) = \omega(x), \ \kappa(x_i) \geq 3$$

for every $i \geq 0$ along the process to be defined. Note that $\mathrm{ord}_{m_{S_1}} H(x_1) > 0$ is achieved by blowing up x if $\delta(x) > 1$.

Assume now that $\delta(x) = 1$, i.e. $\tau(x) \geq 2$ (definition 2.15). Since $\kappa(x) \geq 3$ and $\epsilon(x) = \omega(x) = p$, we actually have $\kappa(x) = 4$, i.e.

$$\mathrm{in}h = Z^p + F_{p,Z},\ 0 \neq F_{p,Z} \in k(x)[U_1, \ldots, U_e]_p, \tag{9.4}$$

where $(u_1, u_2, u_3; Z)$ are well adapted coordinates.

• if $\tau'(x) = 3$, let $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along x. Then x is resolved by theorem 3.6.

• if $\tau'(x) = 2$, let also $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along x. W.l.o.g. we have

$$\mathrm{VDir}(x) =< U_1 + \alpha_1 U_3, U_2 + \alpha_2 U_3 >,\ \alpha_1, \alpha_2 \in k(x), \tag{9.5}$$

where $\mathrm{div}(u_1 u_2) \subseteq E$, and $E = \mathrm{div}(u_1 u_2 u_3)$ if $(\alpha_1, \alpha_2) \neq (0, 0)$. By theorem 3.6, we have $k(x_1) = k(x)$. By proposition 3.5(v), we deduce that

$$< \{\frac{\partial F_{p,Z}}{\partial \lambda_l}\}_{l \in \Lambda_0} > \subseteq k(x)[\{U_j : \alpha_j = 0\}], \tag{9.6}$$

where $\eta_1 : (\mathcal{X}_1, x_1) \to \mathrm{Spec} S_1$ is the projection, since $\kappa(x_1) \geq 3$.

If $\alpha_1 \alpha_2 \neq 0$, we therefore have $F_{p,Z} \in k(x)^p[U_1, U_2, U_3]$. In particular,

$$0 < d := \deg_{U_1} F_{p,Z} < p,$$

since $\Delta_S(h; u_1, u_2, u_3; Z)$ is minimal. Lemma 7.3(ii) applied to the term in U_1^d of $F_{p,Z}$ gives a contradiction with (9.5), since $d \not\equiv 0 \bmod p$. We now assume that $\alpha_1 = 0$.

If $\alpha_2 \neq 0$, we derive a contradiction in a similar way: by (9.6), the coefficient of degree 0 in U_1 in $F_{p,Z}$ must be zero; lemma 7.3(ii) applied to the term of minimal degree d in U_1 of $F_{p,Z}$ gives again a contradiction, since $0 < d < p$. This proves that $\mathrm{VDir}(x) =< U_1, U_2 >$.

By proposition 2.6, we have $\delta(x_1) = 1$ and may iterate. By proposition 3.8, this process is finite or the curve $\mathcal{Y} := V(Z, u_1, u_2)$ is permissible of the first kind. Since $\mathrm{VDir}(x) =< U_1, U_2 >$, blowing up along $\mathcal{Y}$ then completes the proof.

• if $\tau'(x) = 1$, it can be assumed that (9.4) has the form

$$\mathrm{in}h = Z^p + \lambda(U_1 + \alpha_2 U_2 + \alpha_3 U_3)^p,\ \lambda \notin k(x)^p, \tag{9.7}$$

with $\mathrm{div}(u_1) \subseteq E$, and $\mathrm{div}(u_j) \subseteq E$ if $\alpha_j \neq 0$, $j = 2, 3$.

If $\alpha_2\alpha_3 \neq 0$, let $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along x. We get a contradiction with $\kappa(x_1) \geq 3$ unless $\lambda \in k(x_1)^p$; but then $\delta(x_1) = 1$ implies that x_1 satisfies the assumptions of lemma 7.1 from which the conclusion follows. We now assume that $\alpha_3 = 0$.

If $\alpha_2 \neq 0$, let also $\mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along x. The previous argument works in the same way unless $x_1 = (Z/u_3, u_1/u_3, u_2/u_3, u_3)$. Then x_1 satisfies again (9.7) for some $\alpha_3 \in k(x)$ and we may iterate. By proposition 3.8, this process is finite or the curve $\mathcal{Y} := V(Z, u_1, u_2)$ is permissible of the first kind and we blow up along $\mathcal{Y}$. But then $k(x_1) = k(x)$, and this gives a contradiction with $\kappa(x_1) \geq 3$. Therefore the lemma is proved unless

$$\mathrm{in}h = Z^p + \lambda U_1^p,\ \lambda \notin k(x)^p,\ \mathrm{div}(u_1) \subseteq E. \tag{9.8}$$

We now define a refinement $\mathcal{C}$ of the function $x \mapsto (m(x), \omega(x))$, *cf.* chapter 6. Let $\pi :\ \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along a permissible center of the first kind $\mathcal{Y} \subseteq \mathrm{div}(u_1)$, $x' \in \pi^{-1}(x)$. We set:

$$\mathcal{C}(x') < \mathcal{C}(x) \Leftrightarrow x' \text{ satisfies the conclusion of the lemma.}$$

By theorem 3.6, we have $\mathcal{C}(x') < \mathcal{C}(x)$ unless $x' \in \mathrm{div}(u_1')$, where $\mathrm{div}(u_1') \subseteq E'$ is the strict transform of $\mathrm{div}(u_1)$. Otherwise, we let $\mathcal{C}(x') = \mathcal{C}(x)$.

With notations as in chapter 6, we claim that $\mathrm{div}(u_1)$ has maximal contact for the condition $\mathcal{C}$ (definition 6.1). To see this, suppose that $\mathcal{C}(x') = \mathcal{C}(x)$. Note that $\delta(x') > 1$ or x' satisfies the assumptions of lemma 7.1 if $\lambda \in k(x')^p$: a contradiction. If $\delta(x') = 1$ and $\lambda \notin\in k(x')^p$, we get an expansion

$$\mathrm{in}h' = Z'^p + F_{p,Z'},\ 0 \neq F_{p,Z'} \in k(x)[U_1', \ldots, U_{e'}']_p,$$

where $(u_1', u_2', u_3'; Z')$ are well adapted coordinates at x', and the leading coefficient of $F_{p,Z'}$ in U_1' is $\lambda U_1'^p$. Since $\mathcal{C}(x') = \mathcal{C}(x)$ is assumed, we actually have

$$\mathrm{in}h' = Z'^p + \lambda U_1'^p$$

by (9.8) and the claim is proved. The conclusion now follows from theorem 6.1. $\square$

Proposition 9.10. *Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent composition of local permissible blowing ups of the first kind:*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{9.9}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that x_r is resolved or x_r satisfies condition ***(E)'****.*

Proof. It can also be assumed that $\omega(x) \geq p$ and that

$$\omega(x_i) = \omega(x),\ \kappa(x_i) \geq 3$$

for every $i \geq 0$ along the process to be defined. By lemma 9.9, we may assume that $H(x) \neq (1)$ to begin with. Order

$$d_1 \geq \cdots \geq d_e \geq 0 =: d_{e+1},\ d_1 > 0,$$

where $E = \mathrm{div}(u_1 \cdots u_e)$. We define e_0, $1 \leq e_0 \leq e$, by:

$$\min\{1, d_{e_0}\} = \min\{1, d_1\} \text{ and } d_{e_0+1} < \min\{1, d_1\}.$$

The invariant is:

$$d(x) := (d'(x) := \max\{0, 1 - d_1\}, d''(x) := e - e_0)_{\mathrm{lex}}.$$

Note that $d(x) = (0, 0)$ if and only if x satisfies condition **(E)'**.

Let $\pi :\ \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along a permissible center of the first kind $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$. We refine the function $x \mapsto (m(x), \omega(x))$, *cf.* chapter 6, by setting:

$$\mathcal{C}(x') < \mathcal{C}(x) \Leftrightarrow d(x') < \min\{d(x), (d'(x), 1)\}.$$

Otherwise, we let $\mathcal{C}(x') = \mathcal{C}(x)$. To prove the proposition, it is sufficient to prove that there exists a sequence (9.9) such that $\mathcal{C}(x_r) < \mathcal{C}(x)$. We claim the following: assume that

$$\eta(\mathcal{Y}) \subset \mathrm{div}(u_j) \text{ for some } j,\ 1 \leq j \leq e_0. \tag{9.10}$$

Then $d(x') \leq d(x)$; if $d(x') = d(x)$ (resp. if $\mathcal{C}(x') = \mathcal{C}(x)$), then x' belongs to the strict transform of $\mathrm{div}(u_j)$ for every j (resp. for some j) such that $e_0 < j \leq e$.

To prove this claim, let $W := \eta(\mathcal{Y})$ and $I(W)S' =: (u)$, where

$$\eta' :\ (\mathcal{X}', x') \longrightarrow \mathrm{Spec} S'$$

is the projection. By proposition 3.5(iv), we have $H(x') = u^{\epsilon(y)-p}H(x)S'$, therefore

$$d_1' \geq \frac{\mathrm{ord}_u H(x')}{p} = \frac{\epsilon(x)}{p} - 1 + \frac{\mathrm{ord}_W H(x)}{p} \geq \min\{1, d_1\} \tag{9.11}$$

by (9.10). We get

$$d'(x') = \max\{0, 1 - d_1'\} \leq \max\{0, 1 - d_1\} = d'(x).$$

If equality holds, (9.11) implies that $\min\{1, d_1'\} = \mathrm{ord}_u H(x')/p$, i.e.

$$\frac{\mathrm{ord}_u H(x')}{p} = d_{j'}' \text{ for some } j',\ 1 \leq j' \leq e_0' := e_0(x').$$

The claim follows immediately.

We now define $\Omega(x) \subset (\mathcal{X}, x)$ to be the Zariski closure of the set:

$$\Omega^\circ(x) := \{y \in \mathcal{X} : m(y) = p,\ \omega(y) > 0 \text{ and } \forall j,\ 1 \leq j \leq e_0,\ y \notin \mathrm{div}(u_j)\}.$$

By proposition 3.13, $\Omega(x)$ is a (possibly empty) curve. Note that

(1) $\Omega(x')$ is the strict transform of $\Omega(x)$ in $(\mathcal{X}', x')$ if $\mathcal{Y}$ satisfies (9.10), and

(2) $\Omega(x) = \emptyset$ if $e_0 \geq 2$ or if $d''(x) = 0$.

We consider two cases:

Case 1: $\Omega(x) = \emptyset$. This implies that any permissible center of the first kind $\mathcal{Y}$ satisfies (9.10). By the above claim, there exists j, $e_0 < j \leq e$ such that $\mathrm{div}(u_j)$ has maximal contact for the condition $\mathcal{C}$. By theorem 6.1, we obtain a sequence (9.9) such that $\mathcal{C}(x_r) < \mathcal{C}(x)$.

Case 2: $\Omega(x) \neq \emptyset$. Consider the quadratic sequence along μ. By the above claim and (1), we either obtain $\mathcal{C}(x_r) < \mathcal{C}(x)$ (in particular if we reach case 1), or achieve that $\Omega(x_{r_1})$ is irreducible for some $r_1 \geq 0$; by proposition 3.8, it can be furthermore assumed that $\Omega(x_{r_1})$ is permissible of the first kind when the latter holds. Let then $y_1 \in (\mathcal{X}_{r_1}, x_{r_1})$ be the generic point of $\Omega(x_{r_1})$. By (2), we also have:

$$e_0(x_r) = e_0 = 1 \text{ and } d''(x_{r_1}) \geq 1. \tag{9.12}$$

Let $\pi_1 : \mathcal{X}' \to (\mathcal{X}_{r_1}, x_{r_1})$ be the blowing up along $\Omega(x_{r_1})$ and $x' \in \pi_1^{-1}(x)$. Since $d'(x') \leq d'(x_{r_1}) = d'(x)$, we have $\mathcal{C}(x') < \mathcal{C}(x)$ or are done by (1) and case 1 unless

$$d'(x') = d'(x),\ e'_0 := e_0(x') = 1 \text{ and } d''(x') \geq 1.$$

Then π_1 restricts to a finite morphism

$$\Omega(x') \longrightarrow \Omega(x_{r_1}). \tag{9.13}$$

We now iterate this construction: this constructs a sequence

$$(\mathcal{X}_{r_1}, x_{r_1}) \leftarrow (\mathcal{X}_{r_2}, x_{r_2}) \leftarrow \cdots \leftarrow (\mathcal{X}_{r_k}, x_{r_k}) \leftarrow \cdots$$

where $x_{r_i} \in \mathcal{X}_{r_i}$ is the center of μ. If $\mathcal{C}(x_{r_k}) = \mathcal{C}(x)$, there is an induced two-dimensional quadratic sequence

$$(\mathcal{X}_{r_1}, y_1) \leftarrow (\mathcal{X}_{r_2}, y_2) \leftarrow \cdots \leftarrow (\mathcal{X}_{r_k}, y_k) \leftarrow \cdots$$

where $y_k \in (\mathcal{X}_{r_k}, x_{r_k})$ is the generic point of the permissible curve $\Omega(x_{r_k})$ by (9.13). By two-dimensional resolution, we have $(m(y_k), \omega(y_k)) < (p, \omega(x))$ for $k >> 0$: a contradiction with permissibility. Therefore the above sequence achieves $\mathcal{C}(x_{r_k}) < \mathcal{C}(x)$ for some $k \geq 0$ and the proof is complete. $\square$

9.3.2 Proof of proposition 9.5.

From now on, we assume that **(E)'** is satisfied.

Definition 9.3. (Preparation). Assume that x is in case (**) (definition 8.1). We define

$$\mathrm{pr} : \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} |\ c < \frac{1 + \omega(x)}{p}\} \longrightarrow \mathbb{R}^2_{\geq 0},$$

as the translation by the vector $(-d_1, -d_2, 0)$ followed by projection from the point $(0, 0, \frac{1+\omega(x)}{p})$ over the (x_1, x_2)-plane, followed by the homothety of ratio $\frac{p}{1+\omega(x)}$. We will write $\Delta_2(h; u_1, u_2; v; Z)$, even Δ_2 if no confusion is possible instead of $\mathrm{pr}\Delta_S(h; u_1, u_2, v; Z)$ for short.

Let $\mathbf{x}$ be a vertex of Δ_2. We say that $\mathbf{x}$ is a left vertex if its ordinate is bigger or equal the ordinate of the vertex of bigger ordinate of the side of slope -1.

Let $\mathbf{x}$ be a vertex of Δ_2. Let $\mathrm{pr}^{-1}(\mathbf{x})$ the edge of $\Delta(h; u_1, u_2, v; Z)$ giving $\mathbf{x}$ by projection, this edge is defined by an equation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 1$, $\alpha_1\alpha_2\alpha_3 > 0$, as usual we define the monomial valuation $v_{\alpha_\mathbf{x}}$ by

$$v_{\alpha_\mathbf{x}}(Z) = 1,\ v_{\alpha_\mathbf{x}}(u_1) = \alpha_1,\ v_{\alpha_\mathbf{x}}(u_2) = \alpha_2,\ v_{\alpha_\mathbf{x}}(v) = \alpha_3.$$

We say that $\mathbf{x}$ is prepared if

$$Z^p - G_\mathbf{x}^{p-1} Z + F_{p,Z,\mathbf{x}} := \mathrm{in}_{\alpha_\mathbf{x}}(h) \in k(x)[Z, U_1, U_2, V]$$

verifies one of the following:
1- either $G_\mathbf{x} \neq 0$,
2- either $H^{-1}\frac{\partial F_{p,Z,\mathbf{x}}}{\partial V}$ is not proportional to an $\omega(x)$-power,
3- or $H^{-1}\frac{\partial F_{p,Z,\mathbf{x}}}{\partial V} = \lambda V^{\omega(x)}$, $\lambda \in k(x)^*$.

We say that (Z, u_1, u_2, v) is totally prepared if

(i) $\Delta_S(h; u_1, u_2, v; Z)$ minimal,

(ii) when $pd_2 = 0$ (f.i. when $E = \mathrm{div}(u_1)$), all the left vertices of $\Delta_2(h; u_1, u_2; v; Z)$ are prepared,

(iii) when $pd_1 > 0$ and $pd_2 > 0$ ($\Leftrightarrow$ $E = \mathrm{div}(u_1u_2)$ when $\omega(x) \geq p$), all the vertices of $\Delta_2(h; u_1, u_2; v; Z)$ are prepared.

Proposition 9.11. *Assume that x is in case (**) definition 8.1. There exists $v \in S$, $\phi \in S$ such that $(Z - \phi, u_1, u_2, v)$ is totally prepared. Furthermore x is resolved for $m(x) = p$ if $\Delta_2(h; u_1, u_2; v; Z - \phi) = \emptyset$.*

Proof. We apply a strategy similar to Hironaka's strategy of minimizing in [42]. Let us start by the a vertex $\mathbf{x} = (x_1, x_2)$ not prepared. With the notations as above, we have $\mathrm{in}_{\alpha_\mathbf{x}}(h) = Z^p + F_{p,Z,\mathbf{x}}$, with

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z,\mathbf{x}} = \lambda V^{1+\omega(x)} + \sum_{1 \leq j \leq 1+\omega(x)} \lambda_j V^{1+\omega(x)-j} U_1^{jx_1} U_2^{jx_2},\ \lambda \in k(x)^*,$$

$$U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z,\mathbf{x}}}{\partial V} = (1+\omega(x))\lambda(V + \lambda' U_1^{x_1} U_2^{x_2})^{\omega(x)},\ \lambda' \in k(x)^*,$$

in particular, $x_i \in \mathbb{N}$, $i = 1, 2$. We take any invertible $\gamma_\mathbf{x} \in S$ whose residue is λ' and we define

$$w := v + \gamma_\mathbf{x} u_1^{x_1} u_2^{x_2}.$$

Then (Z, u_1, u_2, w) is a regular system of parameters of S.

$$\Delta_2(h; u_1, u_2; w; Z) \subset \Delta_2(h; u_1, u_2; v; Z).$$

Furthermore, let $\mathbf{y} = (y_1, y_2)$ another vertex of $\Delta_2(h; u_1, u_2; w; Z)$, let

$$\alpha'_1 x_1 + \alpha'_2 x_2 + \alpha'_3 x_3 = 1$$

be an equation of the edge of $\Delta_S(h; u_1, u_2, v; Z)$ defined by $\mathbf{y}$, of course $v_{\alpha_\mathbf{y}}(u_1^{x_1} u_2^{x_2}) > 1$, so $\mathrm{in}_{\alpha_\mathbf{y}}(v) = \mathrm{in}_{\alpha_\mathbf{y}}(w)$. In particular, y is still a vertex of $\Delta_2(h; u_1, u_2; w; Z)$ and, if it was prepared for $(u_1, u_2, v; Z)$, it is still prepared for $(u_1, u_2, w; Z)$. Furthermore, if we make an eventual translation on $Z \leftarrow Z - \phi$, $\phi \in S$ to minimize $\Delta_S(h; u_1, u_2, w; Z)$, as $\mathrm{in}_{\alpha_\mathbf{y}}(v) = \mathrm{in}_{\alpha_\mathbf{y}}(w)$, in the of expansion $\mathrm{in}_{\alpha_\mathbf{y}}(h)$, we just change $\mathrm{in}_{\alpha_\mathbf{y}}(v)$ by $\mathrm{in}_{\alpha_\mathbf{y}}(w)$: we can choose ϕ with $v_{\alpha_\mathbf{y}}(\phi) > 1$. So

$$\Delta_2(h; u_1, u_2; w; Z - \phi) \subset \Delta_2(h; u_1, u_2; v; Z),$$

any prepared vertex $\mathbf{y} = (y_1, y_2)$ of $\Delta_2(h; u_1, u_2; v; Z)$ is a prepared vertex of $\Delta_2(h; u_1, u_2; w; Z - \phi)$.

We apply this process to each $\mathbf{x} = (x_1, x_2)$ to be prepared, starting by those of smallest modules. When this process is finite, we get the announced result.

When this process is infinite, we get $\phi, \psi \in \hat{S}$ such that $(u_1, u_2, v - \psi; Z - \phi)$ is totally prepared. Let us remark that x is resolved if

$$\Delta_2(h; u_1, u_2; w; Z - \phi) \neq \emptyset.$$

The contrary would mean that $\Delta(h; u_1, u_2; w; Z - \phi)$ has only one vertex $(d_1, d_2, 1 + \omega(x))$: $\mathbf{V}(Z - \phi, w)$ would be a component of dimension *two* of the locus of multiplicity $\min\{p, 1 + \omega(x)\}$, $\eta(\mathbf{V}(Z - \phi, w)) \not\subseteq E$. This contradicts **(E)** if $\omega(x) \geq p$ or if h is separable (assumption (ii) in theorem 1.4). If $\omega(x) < p$ and $h = Z^p + f_{p,Z}$, $\mathrm{char} S = p$, x is resolved for $m(x) = p$ by a combinatorial algorithm, *vid.* proof of theorem 2.23.

The remark above implies that, after a finite number of steps, we apply infinitely the process to vertices of smallest abscissa or (smallest ordinate and $E = \mathrm{div}(u_1 u_2)$) of $\Delta_2(h; u_1, u_2, v; Z)$ and this smallest abscissa or smallest ordinate remains constant.

Let us study the very special case where $\mathbf{x} := (A, \beta)$ is the vertex of smallest abscissa of Δ_2 and that the process dissolves it, creating a new vertex (A, β'), $\beta' > \beta$ infinitely times. This implies $A, \beta, \beta' \in \mathbb{N}$.

Let $\alpha = (\alpha_1, 0, \alpha_3)$, such that $\alpha_1 x_1 + \alpha_3 x_3 = 1$ is the equation of the non compact face of $\Delta_S(h; u_1, u_2, v; Z)$ whose image by pr is the non compact

face $x_1 = A$ of Δ_2. We get $\alpha_1 p d_1 + \alpha_3(1+\omega(x)) = p$, $\alpha_1 A - \alpha_3 = 0$, and

$$\mathrm{in}_\alpha h = Z^p - G_{\mathbf{x}}^{p-1} Z + F_{p,\mathbf{x}} \in \mathrm{gr}_\alpha(S[Z]) = \frac{S}{(v,u_1)}[U_1,V][Z].$$

Let $\mathcal{C} := \mathrm{Spec}\frac{S}{(v,u_1)}$. By quasi-homogeneity and the uniqueness of the solution [42] Corollary (4.1.1), there exists $\Phi \in \widehat{\mathcal{O}_{\mathcal{C}}[U_1,V]} = \widehat{\mathcal{O}_{\mathcal{C}}}[[U_1,V]]$ with

$$\Phi^p \in U_1^{pd_1}(V, U_1^A)^{1+\omega(x)},\ \Psi \in \bar{u_2}^\beta U_1^A \widehat{\mathcal{O}_{\mathcal{C}}}[[U_1,V]],$$

such that

$$\mathrm{in}_\alpha h = (Z-\Phi)^p + U_1^{pd_1}\overline{\gamma}(V-\Psi)^{1+\omega(x)}. \tag{9.14}$$

Lemma 9.12. *There exists*

$$\phi \in S,\ \phi^p \in u_1^{pd_1}(v, u_1^A)^{1+\omega(x)} \text{ and } w \in S,\ v - w \in (v^2, u_1^A u_2^\beta)S$$

such that

$$\mathrm{in}_\alpha h = (Z - \mathrm{in}_\alpha \phi)^p + U_1^{pd_1}\overline{\gamma}W^{1+\omega(x)}.$$

When $\omega(x) \geq p$, (9.14) means that $\mathbf{V}(Z-\Phi, V-\Psi)$ is the only component in the locus of multiplicity p of

$$\Xi := \mathrm{Spec}(\widehat{\mathcal{O}_{\mathcal{C}}}[[U_1,V]]/(\mathrm{in}_\alpha h))$$

not contained in $\mathrm{div}(U_1)$. Since $\mathcal{O}_{\mathcal{C}}[U_1,V]$ is excellent and Noetherian, by [25] lemma **1.37**, this component is algebraic and the conclusion follows.

When $\omega(x) < p$, $\mathbf{V}(Z-\Phi, V-\Psi)$ is the only component in the locus of multiplicity $1+\omega(x)$ of Ξ not contained in $\mathrm{div}(U_1)$: we conclude as above. This ends the proof of lemma 9.12.

Let us remark that, if there exists another vertex $\mathbf{x}_1$ which is already prepared, then

$$\mathrm{in}_{\alpha_{\mathbf{x}_1}}(Z) = \mathrm{in}_{\alpha_{\mathbf{x}_1}}(Z-\phi),\ \mathrm{in}_{\alpha_{\mathbf{x}_1}}(v) = \mathrm{in}_{\alpha_{\mathbf{x}_1}}(w),$$

so $\mathbf{x}_1$ is still prepared for $(u_1, u_2, w; Z-\phi)$.

By applying lemma 9.12, we see that there exists $\phi \in S$ and $w \in S$ such that the vertex of smallest abscissa of $\Delta_2(h; u_1, u_2; w; Z-\phi)$ is prepared.

The case where the process is infinite along points of smallest ordinates is, mutatis mutandis, the same: by applying the remark above, we see that, when $E = \mathrm{div}(u_1 u_2)$, there exists $\phi \in S$ and $w \in S$ such that both the vertices of smallest abscissa and smallest ordinate of $\Delta_2(h; u_1, u_2; w; Z-\phi)$ are prepared. This ends the proof of proposition 9.11. $\square$

Definition 9.4. (Invariants). Suppose $\kappa(x) = 3$, suppose that (Z, u_1, u_2, v) is totally prepared. In the case where $E = \mathrm{div}(u_1u_2)$, we choose u_1 so that $d_1 > 0$ and let

(i) $(A_1(Z, u_1, u_2, v), \beta(Z, u_1, u_2, v))$ is the vertex of smallest abscissa of Δ_2;

(ii) $B(Z, u_1, u_2, v) = \inf\{|\mathbf{x}| \ |\mathbf{x} \in \mathrm{pr}\Delta\}$;

(iii) $A_2(Z, u_1, u_2, v)$ is the inf of the ordinates of points in Δ_2,

$$C(Z, u_1, u_2, v) = B(Z, u_1, u_2, v) - A_1(Z, u_1, u_2, v) - A_2(Z, u_1, u_2, v);$$

(iv) $\gamma(Z, u_1, u_2, v) \in \mathbb{N}$ is given by:

$$\gamma(Z, u_1, u_2, v) := \begin{cases} \lceil \beta(Z, u_1, u_2, v) \rceil & \text{if} \quad E = \mathrm{div}(u_1) \\ \\ 1 + \lfloor C(Z, u_1, u_2, v) \rfloor & \text{if} \quad E = \mathrm{div}(u_1u_2) \end{cases}.$$

For sake of simplicity, most of the time, we will skip (Z, u_1, u_2, v) and write $A_1(x), A_2(x), B(x), C(x), \beta(x), \gamma(x)$.

Proposition 9.13. *Suppose x satisfies conditions (**) and **(E)'** with $\kappa(x) = 3$ and (Z, u_1, u_2, v) is totally prepared. The following holds:*

(i) $V \in \mathrm{Vdir}(x)$ or x is resolved;

(ii) if $B(x) = 1$ and $E = \mathrm{div}(u_1)$, x is resolved or

$$x' := (Z', u_1', u_2', v') = (Z/u_2, u_1/u_2, u_2, v/u_2)$$

*is the unique closed point $x_1 \in \pi^{-1}(x)$ in the blowing up $\pi : \mathcal{X}' \to \mathcal{X}$ along x such that $\iota(x_1) \geq \iota(x)$, and x' then satisfies conditions (**) and **(E)'**;*

(iii) if $B(x) = 1$ and $\omega(x) < p$, x is resolved.

Proof. When $B(x) > 1$, clearly $V \in \mathrm{Vdir}(x)$. When $B(x) = 1$, then

$$U_1^{-pd_1}U_2^{-pd_2}F_{p,Z} = \lambda V^{1+\omega(x)} + \sum_{1 \leq i \leq 1+\omega(x)} V^{1+\omega(x)-i}Q_i(U_1, U_2), \ \lambda \neq 0.$$

Suppose $V \notin \mathrm{Vdir}(x)$, then

$$U_1^{-pd_1}U_2^{-pd_2}\frac{\partial F_{p,Z}}{\partial V} \neq (1 + \omega(x))\lambda V^{\omega(x)},$$

so $\tau'(x) \geq 2$ by total preparedness. By proposition 8.1 and lemma 8.3, x is resolved except possibly if

$$\mathrm{Vdir}(x) =< V + aU_2, U_1 >, \ a \in k(x)$$

up to renumbering u_1, u_2 if $E = \mathrm{div}(u_1 u_2)$.

Suppose $a \neq 0$, then it would mean that $\mathbf{x} = (0,1)$ is a vertex of Δ_2. This implies that

$$U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z}}{\partial V} = (1+\omega(x))\lambda(V+aU_2)^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} \lambda_i (V+aU_2)^{1+\omega(x)-i} U_1^i,$$

so $H^{-1}\frac{\partial F_{p,Z,\mathbf{x}}}{\partial V} = (1+\omega(x))\lambda(V+aU_1)^{\omega(x)}$ with notations as in definition 9.3: a contradiction with total preparedness and (i) is proved.

Assume that $E = \mathrm{div}(u_1)$, so we have

$$\mathrm{VDir}(x) =< V > \ \text{ or } \mathrm{VDir}(x) =< V, U_1 >$$

by (i). Apply now lemma 8.3(1) and note that the form (8.10) is automatically achieved when (Z, u_1, u_2, v) is totally prepared: if $\mathrm{VDir}(x) =< V >$, we have

$$U_1^{-pd_1} F_{p,Z} \in k(x)[U_1, V]_{\epsilon(x)} \tag{9.15}$$

by (8.10); if $\mathrm{VDir}(x) =< V, U_1 >$, we have

$$U_1^{-pd_1} F_{p,Z} \in k(x)[U_1, V]_{\epsilon(x)} \oplus < U_1^{\omega(x)} U_2 >$$

by (8.10). Therefore (ii) follows from lemma 8.3(1) and proposition 9.7.

To prove (iii), it can be assumed that $\mathrm{VDir}(x) =< V >$ by (i) and corollary 9.2. In particular, we have

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z}, \ U_1^{-pd_1} U_2^{-pd_2} F_{p,Z} = \lambda V^{1+\omega(x)} + Q(U_1, U_2),$$

with $\lambda \neq 0$, $Q \neq 0$, and $Q \in k(x)[U_1]$ if $E = \mathrm{div}(u_1)$. We blow up along x and let $x' := (Z/u_2, u_1/u_2, u_2, v/u_2)$.

Assume that $E = \mathrm{div}(u_1)$. By (ii) and (9.15), the only point to consider is x'. By corollary 9.2, we are done unless $\iota(x') = \iota(x)$, so x' satisfies again assumption (iii) of the proposition with $E' = \mathrm{div}(u_1' u_2')$. Note that we have $A_1(x') > 0$ by (**).

Assume that $E = \mathrm{div}(u_1 u_2)$ and let $x_1 \in \pi^{-1}(x)$ with $\iota(x_1) \geq \iota(x)$. By corollary 9.2, we are done unless $\iota(x_1) = \iota(x)$. If $E' = \mathrm{div}(u_1')$, we have $B(x') = 1$ except possibly if

$$a(1) := pd_1,\ a(2) := pd_2,\ F_0 := Q(U_1, U_2)$$

satisfies the assumptions of lemma 7.3(ii). This holds only if

$$d_1' := d_1 + d_2 + \frac{1+\omega(x)}{p} \in \mathbb{N}.$$

Then x_1 is resolved for $m(x) = p$ by blowing up d_1' times along codimension two centers of the form (Z', u_1'). Otherwise, we have $< Q >=< U_1^{1+\omega(x)} >$, $x_1 = x'$ up to renumbering u_1, u_2, so $B(x') = 1$ and x' satisfies again assumption (iii) of the proposition. Note that no renumbering is necessary if $A_1(x) > 0$.

Summing up, x is resolved or we construct a sequence of infinitely near points lying on the successive strict transforms of a formal curve

$$\hat{\mathcal{Y}} = V(\hat{Z}, u_1, u_2, \hat{v}) \subset \hat{\mathcal{X}} = \mathcal{X} \times_S \mathrm{Spec}\hat{S}.$$

By proposition 3.8 we may assume that $\mathcal{Y}$ is permissible of the first kind, so x is resolved by blowing up along $\mathcal{Y}$. □

Proposition 9.14. *Assume that x satisfies conditions (**) and **(E)'** with $\kappa(x) = 4$ and let (Z, u_1, u_2, v) be totally prepared. Let us call $\mathcal{Y} := V(Z, u_1, v)$ with generic point y.*

(1) if $\omega(x) < p$, x is resolved;

(2) if $\omega(x) \geq p$ and $\epsilon(y) \geq 2$, then $(d_1, d_2, \frac{1+\omega(x)}{p})$ is the only vertex of $\Delta_S(h; u_1, u_2, v; Z)$ in the region $x_1 = d_1$. Furthermore $\mathcal{Y}$ is Hironaka-permissible and x is resolved.

(3) if $\omega(x) \geq p$ and $E = \mathrm{div}(u_1)$, let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along x and $x' \in \pi^{-1}(x)$ with $\iota(x') \geq (p, \omega(x), 3)$. Then x' is resolved or there is a Hironaka-permissible line

$$D' = V(Z', u_1', u_2'),\ E' = \mathrm{div}(u_1' u_2').$$

Let $\pi' : \mathcal{X}'' \to \mathcal{X}'$ be the blowing up along D' and $x'' \in \pi'^{-1}(x')$ with $\omega(x'') \geq \omega(x')$. Then:

(i) x'' satisfies again ***(E)'*** *and $\omega(x'') = \omega(x)$;*

(ii) x'' satisfies condition (**), $E'' = \mathrm{div}(u_1'' u_2'')$ and $\kappa(x'') = 3$;

(iii) $C(x'') < 1 - \frac{1}{1+\omega(x)}$, $A_1(x'') < 1$, $A_2(x'') < 1$.

Proof. Statement (1) has been proved in corollary 9.2. From now on, we assume that $\omega(x) \geq p$.

Let us prove (3). As $\kappa(x) = 4$, $E = \mathrm{div}(u_1)$, we have $\mathrm{Vdir}(x) =< U_1 >$. By (**):

$$f_{p,Z} = u_1^{-pd_1}(\gamma v^{1+\omega(x)} + \gamma' u_1^{\omega(x)} + u_1\phi),\ \gamma, \gamma' \text{ invertible},\ \phi \in m_S^{\omega(x)}.$$

We blow up along x: if x' is ω-near x, x' is on the strict transform of $\mathrm{div}(u_1)$. In the chart of origin $(Z', u_1', u_2', v) := (Z/v, u_1/v, u_2/v, v)$, we get, before any preparation:

$$f_{p,Z'} = {u_1'}^{pd_1} v^{pd_1+\omega(x)-p}(\gamma v + u_1'\phi'),\ \phi' \in S',\ E' = \mathrm{div}(u_1' v).$$

As $1 + \omega(x) \neq\ 0 \bmod(p)$, the monomial ${u_1'}^{pd_1} v^{pd_1+\omega(x)-p} v$ is not a pth-power, it cannot be spoilt by any translation on Z': $\omega(x') = 1 < p \leq \omega(x)$. The only difficult point is the point

$$x' = (Z', u_1', u_2', v') := (Z/u_2, u_1/u_2, u_2, v/u_2),\ E' = \mathrm{div}(u_1' u_2').$$

There is an expansion $h' = Z'^p + \sum_{i=1}^p f_{i,Z'} Z'^{p-i}$, with

$$f_{p,Z'} = {u_1'}^{pd_1} {u_2'}^{pd_1+\omega(x)-p}(\gamma {v'}^{1+\omega(x)} u_2' + \gamma' {u_1'}^{\omega(x)} + u_1' u_2' \psi'),\ \psi' \in S'. \quad (9.16)$$

As we are at the origin of a chart, (Z', u_1', u_2', v) are well adapted: $\epsilon(x') \leq \omega(x)$. As $\omega(x) \geq p$, we keep condition **(E)'** at x' (proposition 9.7). We are done unless

$$\iota(x') = \iota(x),\ \mathrm{ord}_{x'}(u_1' u_2' \psi') \geq \omega(x).$$

In particular, we have $\mathrm{in}_{m_{S'}} h' = Z'^p + F_{p,Z'}$.

• Case $\mathrm{ord}_{x'}(u_1' u_2' \psi') = \omega(x)$. Since $\kappa(x') = 4$, we have

$$\mathrm{Vdir}(x') \subseteq < U_1', U_2' > .$$

By (9.16), we have $< U_1' > \subsetneq \mathrm{Vdir}(x')$, so $\mathrm{Vdir}(x') =< U_1', U_2' >$.

Then we blow up along x', the only possible ω-near point is

$$x'' = (Z'', u_1'', u_2'', v'') := (Z'/v', u_1'/v', u_2'/v', v'),\ E'' = \mathrm{div}(u_1''u_2''v'').$$

There is an expansion

$$f_{p,Z''} = u_1''^{pd_1} u_2''^{pd_1+\omega(x)-p} v''^{2(pd_1+\omega(x)-p)}(\gamma v''^2 u_2'' + \gamma' u_1''^{\omega(x)} + u_1''u_2''\psi''),\ \psi'' \in S''$$

and we get $\omega(x'') \leq 3$: we are done for $\omega(x) \geq 4$.

When $\omega(x) = 3$, in $J(F_{p,Z''}, E'', m_{S''})$, there is an homogeneous polynomial

$$P := V''^2 U_2'' + U_1''U_2''(\lambda U_1'' + \mu U_2'' + \nu V'') + \delta U_1''^3,\ \lambda, \mu, \nu, \delta \in k(x) = k(x'').$$

Applying the Hasse-Schmidt derivation $2 \times \frac{\partial^2 P}{\partial V''^2} = U_2''$ gives $U_2'' \in \mathrm{Vdir}(x'')$. The reader ends the computation and sees that $\tau'(x'') = 3$: x'' is is resolved.

When $\omega(x) = 2$, ψ'' is invertible, we have $\mathrm{VDir}(x'') =< U_1'', U_2'' >$. We blow up along x'', at the only possible ω-near points, we have, with suitable variables:

$$f_{p,Z'''} = u_1'''^{pd_1} u_2'''^{pd_1+\omega(x)-p} v'''^{3(pd_1+\omega(x)-p)}(\gamma v''' u_2''' + \gamma' u_1'''^{\omega(x)} + u_1'''u_2'''\psi''').$$

A quick computation shows that $\tau'(x''') = 3$, so x''' is resolved.

• Case $\mathrm{ord}_{x'}(u_1'u_2'\psi') > \omega(x)$. We get $\mathrm{Vdir}(x') =< U_1' >$. We may decompose in (9.16):

$$\psi' = \psi_1' + v\psi_2',\ \psi_1' \in (u_1', u_2')^{\omega(x)-1},\ \psi_2' \in (u_1', u_2').$$

By condition **(E)'**, the line $D' := \mathrm{V}(Z', u_1', u_2')$ with generic point y' is Hironaka-permissible, $\epsilon(y') = 1$.

Let us blow up along D'. Let us begin with the point x_2' at infinity, i.e.

$$x_2' := (Z'', u_1'', u_2'', v'') = (Z'/u_1', u_1', u_2'/u_1', v'),\ E'' = \mathrm{div}(u_1''u_2'').$$

We get $H(x_2') = (u_1''^{2pd_1+\omega(x)+1-2p} u_2''^{pd_1+\omega(x)-p})$ and

$$H(x_2')^{-1} f_{p,Z''}'' = \gamma v''^{1+\omega(x)} u_2'' + \gamma' u_1''^{\omega(x)-1} + u_1''^{\omega(x)}\psi_1' + u_2''u_1''v''\psi_2'.$$

As we are at the origin of a chart, the coordinates (Z'', u_1'', u_2'', v'') are well adapted, so $\epsilon(x_2') \leq \omega(x) - 1$.

For $x_2 \in \pi'^{-1}(x')$ in the chart of origin

$$x'' := (Z'', u_1'', u_2'', v'') := (Z'/u_2', u_1'/u_2', u_2', v'),\ E'' = \mathrm{div}(u_1''u_2''),$$

we get $H(x_2) = (u_1''^{pd_1} u_2''^{2pd_1+\omega(x)+1-2p})$ (in particular **(E)'** holds) and

$$H(x_2)^{-1} f_{p,Z''} = \gamma v''^{1+\omega(x)} + u_2''^{\omega(x)-1}\gamma' u_1''^{\omega(x)} + u_2''^{\omega(x)} u_1''\psi_1' + u_2''u_1''v''\psi_2'.$$

As $1+\omega(x) \neq 0 \bmod(p)$, the monomial $H(x_2)v''^{1+\omega(x)}$ cannot be spoilt by any translation on Z'': we have $(m(x_2), \omega(x_2)) \leq (p, \omega(x))$. Because of the monomial $H(x_2)u_2''^{\omega(x)-1}u_1''^{\omega(x)}$, we must have $u_1''(x_2) = 0$: therefore $x_2 = x''$ is the origin of the chart. We have

$$\min\{\mathrm{ord}_{m_{S''}}(u_2''^{\omega(x)}u_1''\psi_1'), \mathrm{ord}_{m_{S''}}(u_2''u_1''v''\psi_2')\} \geq \omega(x)+1$$

if $(m(x''), \omega(x'')) = (p, \omega(x))$: x'' is in case (**) with $\kappa(x'') = 3$. This proves (i) and (ii).

Let us prove assertion (iii) which is valid only for the point x'' of parameters

$$(Z'', u_1'', u_2'', v'') := (Z'/v', u_1'/u_2', u_2', v') = (Z/u_2^2, u_1/u_2^2, u_2, v/u_2).$$

In the expansion of $f_{p,Z}$, the monomial $(u_1^{pd_1}) \times u_1^a u_2^b v^c = H(x)u_1^a u_2^b v^c$ becomes

$$u_2''^{2p} u_1''^{pd_1} u_2''^{2pd_1+\omega(x)+1-2p} \times u_1''^a u_2''^{2a+b+c-(\omega(x)+1)} v''^c.$$

As $f_{p,Z''} = u_2''^{-2p} f_{p,Z}$, to the monomial $H(x)u_1^a u_2^b v^c$ corresponds the monomial $H(x'')u_1''^a u_2''^{2a+b+c-(\omega(x)+1)}v''^c$ in the expansion of $f_{p,Z''}$. The point

$$\left(\frac{a}{1+\omega(x)-c}, \frac{b}{1+\omega(x)-c}\right) \in \mathrm{pr}(\Delta(h; u_1, u_2, v; Z))$$

gives the point $(\frac{a}{1+\omega(x)-c}, \frac{2a+b}{1+\omega(x)-c} - 1)$ of $\mathrm{pr}(\Delta(h; u_1'', u_2'', v''; Z''))$. For example, the monomial $H(x)\gamma' u_1^{\omega(x)}$ becomes

$$H(x'')u_1''^{\omega(x)}u_2''^{\omega(x)-1}.$$

Choose (a_0, b_0, c_0) such that $(\frac{a_0}{1+\omega(x)-c_0}, \frac{b_0}{1+\omega(x)-c_0} - 1)$ is a vertex of of $\mathrm{pr}(\Delta(h; u_1, u_2, v; Z))$ with $\frac{2a_0+b_0}{1+\omega(x)-c_0}$ minimal. Then, because of the monomial $H(x)\gamma' u_I^{\omega(x)}$,

$$\frac{2a_0+b_0}{1+\omega(x)-c_0} - 1 \leq \frac{2\omega(x)}{\omega(x)+1} - 1 = 1 - \frac{2}{\omega(x)+1}, \tag{9.17}$$

in particular

$$\frac{a_0}{1+\omega(x)-c_0} \leq \frac{a_0+b_0/2}{1+\omega(x)-c_0} \leq \frac{\omega(x)}{\omega(x)+1} < 1, \tag{9.18}$$

so the point

$$(\frac{a_0}{1+\omega(x)-c_0}, \frac{2a_0+b_0}{1+\omega(x)-c_0} - 1)$$

has both coordinates < 1, it is the vertex of $\Delta_2(h''; u_1'', u_2''; v''; Z'')$ of smallest ordinate.

Let us note that if $(a_0, b_0, c_0) \neq (\omega(x), 0, 0)$, then, as $\mathrm{Idir}(x) =< U_1 >$, we have $a_0 + b_0 \geq 1 + \omega(x) - c_0$, so $\frac{2a_0+b_0}{1+\omega(x)-c_0} - 1 \geq \frac{a_0}{1+\omega(x)-c_0} > 0$, the last inequality because u_1 divides g. When $(a_0, b_0, c_0) = (\omega(x), 0, 0)$, we get

$$\frac{2a_0+b_0}{1+\omega(x)-c_0} - 1 = \frac{2\omega(x)}{1+\omega(x)} - 1 = \frac{\omega(x)-1}{1+\omega(x)} \geq \frac{p-1}{1+\omega(x)} > 0,$$

$$f_{p,Z''} = H(x'')(\gamma {v''}^{1+\omega(x)} + u_1'' u_2'' \vartheta), \ \vartheta \in S''. \tag{9.19}$$

As we saw above, $\epsilon(x'') = \omega(x'') + 1$, $\kappa(x'') = 3$ and we have (**). Then $(\frac{a_0}{1+\omega(x)-c_0}, \frac{2a_0+b_0}{1+\omega(x)-c_0} - 1)$ is the vertex of $\Delta_2(h''; u_1'', u_2''; v''; Z'')$ of smallest ordinate, both coordinates are < 1 and positive. As x' and x'' are origins of chart, (Z'', u_1'', u_2'', v'') are well prepared and no translation on v'' can spoil this vertex. By (9.17)(9.18), we get:

$$C(x'') \leq \frac{a_0}{1+\omega(x)-c_0} - A_1(x'') < 1 - \frac{1}{1+\omega(x)},$$

$$0 < A_2(x'') = \frac{2a_0+b_0}{1+\omega(x)-c_0} - 1 < 1,$$

$$A_1(x'') \leq \frac{a_0}{1+\omega(x)-c_0} \leq \frac{2a_0+b_0}{1+\omega(x)-c_0} - 1 < 1.$$

Note that $A_1(x'') > 0$ because of (9.19). This proves (iii).

Let us prove (2). Since $\epsilon(y) > 0$, we have $A_1(x) > 0$ and $(d_1, d_2, \frac{1+\omega(x)}{p})$ is the only vertex of $\Delta_S(h; u_1, u_2, v; Z)$ in the region $x_1 = d_1$, $U_1 \in \mathrm{Vdir}(x)$. If $\mathrm{Vdir}(x) =< U_1, U_2 >$, then, if we blow up along x, as $\omega(x) \geq p \geq 2$, there is no ω-near point. The only case we have to look at is $\mathrm{Vdir}(x) =< U_1 >$.

As $\omega(x) \geq p$, by condition **(E)'** at x: $pd_1 \geq p$, $\mathcal{Y}$ is Hironaka-permissible. Let us denote by $d := \epsilon(y) \geq 2$. Then $\gamma v^{1+\omega(x)} + g \in (v, u_1)^d$ with $g =$

$\gamma' u_1^{\omega(x)} + u_1\phi$, $\phi \in m_S^{\omega(x)} \cap (v, u_1)^{d-1}$, γ' invertible. Up to change γ' modulo m, there is a decomposition: $\phi = v\phi_1 + u_2\phi_2$, $\phi_1 \in (u_1, v)^{\omega(x)-1}$, $\phi_2 \in (u_1, v)^{d-1}$.

$$f_{p,Z} = {u_1}^{pd_1} {u_2}^{pd_2} (\gamma v^{1+\omega(x)} + \gamma' u_1^{\omega(x)} + u_1 v\phi_1 + u_1 u_2 \phi_2).$$

Let us blow up along $\mathcal{Y}$. In the first chart of origin

$$(Z', u_1', u_2', v') := (Z/u_1, u_1, u_2, v/u_1),$$

we get

$$f_{p,Z'} = {u_1'}^{pd_1+d-p} {u_2'}^{pd_2} (\gamma {v'}^{1+\omega(x)} {u_1'}^{\omega(x)+1-d} + \gamma' {u_1'}^{\omega(x)-d} + {u_1'}^{\omega(x)-d+1} \phi_1' + u_2' \phi_2'),$$

$\phi_1', \phi_2' \in S'$. Because of the monomial

$${u_1'}^{pd_1+d-p} {u_2'}^{pd_2} \gamma' {u_1'}^{\omega(x)-d} = H(x')\gamma' {u_1'}^{\omega(x)-d},$$

we get $\omega(x_1) \leq \omega(x) - d < \omega(x) - 1$ for any x_1 in this chart.

Let us see the point at infinity $x' = (Z', u_1', u_2', v') := (Z/v, u_1/v, u_2, v)$, we get

$$f_{p,Z'} = {u_1'}^{pd_1} {u_2'}^{pd_2} {v'}^{pd_1+d-p} (\gamma v^{\omega(x)+1-d} + u_1' \phi'),$$

$\phi_1', \phi_2', \phi_3' \in S'$. As we are at the origin of a chart, (Z', u_1', u_2', v) are well adapted: $\epsilon(x') \leq \omega(x) + 1 - d \leq \omega(x) - 1$. □

Proposition 9.15. *Assume that x satisfies conditions (**) and **(E)'** with $\kappa(x) = 4$, $E = \mathrm{div}(u_1)$ and let (Z, u_1, u_2, v) be totally prepared. With the notations of proposition 9.14, assume furthermore that*

$$\epsilon(y) = 1 \text{ and } \beta(Z, u_1, u_2, v) < 1.$$

Then x is resolved.

Proof. By proposition 9.14(1), we may assume $\omega(x) \geq p$. As $A_1(x) > 0$ by condition (**), $\epsilon(y) = 1$ implies that $\Delta_S(h; u_1, u_2, v)$ has a vertex

$$\mathbf{x} = (\frac{d_1+1}{p}, \frac{b}{p}, 0),\ b \in \mathbb{N}.$$

This leads to

$$A_1(x) = \frac{1}{1+\omega(x)},\ \beta(x) = \frac{b}{1+\omega(x)}.$$

On the other hand, since $\kappa(x) = 4$, we have $b \geq \omega(x)$, i.e. $b = \omega(x)$.

Let us come back to the proof of proposition 9.14(2). The only point to consider is the point x' at infinity, $E' = \mathrm{div}(u_1' v)$. We get an expansion

$$f_{p,Z'} = {u_1'}^{pd_1} {v'}^{pd_1+1-p}(\gamma v^{\omega(x)} + u_1'\phi'),\ (\phi') \equiv (u_2^{\omega(x)}) \bmod (v'). \tag{9.20}$$

The conclusion follows from lemma 7.1 applied to the well prepared coordinates $(v', u_1', u_2; Z')$. □

The following proposition produces bounds identical to those occurring for embedded resolution of surfaces [17].

Proposition 9.16. *Assume that x satisfies conditions (**) and **(E)'** with $\kappa(x) \geq 3$. Consider Hironaka-permissible blowing ups $\pi : \mathcal{X}' \to (\mathcal{X}, x)$ of the following kinds:*

Case 1: $E = \mathrm{div}(u_1 u_2)$ and $\omega(x) \geq p$; we blow-up along $D := (Z, u_1, u_2)$.

Case 2: $\kappa(x) = 3$; we blow up along x.

Let $x' \in \pi^{-1}(x)$ with $(m(x'), \omega(x')) \geq (p, \omega(x))$. Then $\omega(x') \leq \omega(x)$ and (x' is resolved or the following holds):

*(i) conditions (**) and **(E)'** are satisfied at x' and we have*

$$\gamma(x') \leq \max\{\gamma(x), 1\}.$$

(ii) if $E = \mathrm{div}(u_1 u_2)$ and $\eta'(x') \in \mathrm{Spec} S[u_2']$ (resp. $\eta'(x') \in \mathrm{Spec} S[u_2', v']$), where

$$(u_1, u_2' := \frac{u_2}{u_1}, v)\ (\text{resp. } (u_1, u_2', v' := \frac{v}{u_1}))$$

in case 1 (resp. case 2), then $A_1(x') = B(x)$, (resp. $A_1(x') = B(x) - 1$) and,

$$\beta(x') \leq A_2(x) + C(x) \leq \beta(x);$$

if ($k(x') \neq k(x)$ and $\beta(x) \geq 1$), we have $\beta(x') < \beta(x)$;

if $u_2' \in m_{S'}$, then $C(x') \leq \min\{C(x), \beta(x) - C(x)\}$, so $C(x') \leq \frac{\beta(x)}{2}$;

if $u_2' \notin m_{S'}$, then $\beta(x') < 1 + \lfloor C(x) \rfloor$;

(iii) if x' is the origin of the second chart, i.e.

$$x' = (Z' := \frac{Z}{u_2}, u_1' = \frac{u_1}{u_2}, u_2, v)\ (\text{resp. } (Z', u_1', u_2, \frac{v}{u_2}))$$

in case 1 (resp. case 2), then $A_1(x) = A_1(x')$, $C(x') \leq \frac{\beta(x)}{2}$ *and*

$$\beta(x') = A_1(x) + \beta(x) \text{ (resp. } \beta(x') = A_1(x) + \beta(x) - 1);$$

(iv) if $E = \mathrm{div}(u_1)$, $E' = \mathrm{div}(u_1')$ *and* $\beta(x) > 0$, *then* $\beta(x') \leq \beta(x)$, *with strict inequality if* ($k(x') \neq k(x)$ *and* $\beta(x) \geq 1$).

Proof. We first prove the proposition in case 1. Let x' be in the chart with origin $(X' := \frac{Z}{u_1}, u_1, u_2', v)$. In the expansion of $f_{p,Z}$ the monomial $u_1^{pd_1} u_2^{pd_2} v^{1+\omega(x)-i} u_1^a u_2^b$ transforms into $u_1^{pd_1+pd_2-p} {u_2'}^{pd_2} v^{1+\omega(x)-i} u_1^{a+b} {u_2'}^b$ in the expansion of $f_{p,Z'}$, $0 \leq i \leq 1 + \omega(x) - i$. This leads to:

$$f_{p,Z'} = u_1^{pd_1'} {u_2'}^{pd_2} (\gamma v^{1+\omega(x)} + u_1 \phi), \ d_1' := d_1 + d_2 - 1.$$

As $1 + \omega(x) \not\equiv 0 \bmod p$, the monomial $u_1^{pd_1'} {u_2'}^{pd_2} \gamma v^{1+\omega(x)}$ will not be spoilt by any translation on Z': x' satisfies (**) and $(m(x'), \omega(x')) \leq (p, \omega(x))$. If $\omega(x) \geq p$, we have $d_1, d_2 \geq 1$, so x' satisfies condition **(E)'**. Statement $\gamma(x') \leq \gamma(x)$ follows from (ii). There remains to prove (ii).

The monomials defining $B(x)$ in the expansion of $f_{p,Z}$ are minimal for the monomial valuation v_α defined by the weight vector $\alpha := (a, a, aB(x))$:

$$v_\alpha(Z) = 1, \ v_\alpha(u_1) = v_\alpha(u_2) = a, \ v_\alpha(v) = aB(x),$$

with

$$a := \frac{p}{pd_1 + pd_2 + B(x)(1 + \omega(x))}.$$

Let us denote by

$$\mathrm{in}_{v_\alpha} h = Z^p - G_\alpha^{p-1} Z + F_{p,Z,\alpha} \in \mathrm{gr}_\alpha S[Z]$$

At x', there exists $P(t) \in S[t]$, unitary of degree $d := [k(x') : k(x)]$, whose reduction modulo m_S is irreducible and $w := P(u_2')$ is such that (X', u_1, w, v) is a system of coordinates at x'. Of course, we take $w = u_2'$ when x' is the origin of the chart. In this special case where x' is the origin, the reader verifies that (X', u_1', w, v) is totally prepared and that all the statements of (ii) are true.

From now on, $E' = \mathrm{div}(u_1)$. Monomials defining $B(x)$ become the monomials defining $A_1(x') = B(x)$. The monomials defining the vertices of smaller

abscissa of $\Delta_2(h'; u'_1, w, v'; X')$ are those minimal for the valuation $v_{\alpha'}$ given by

$$v_{\alpha'}(X') = 1,\ v_{\alpha'}(u_1) = a,\ v_{\alpha'}(w) = 0,\ v_{\alpha'}(v) = aB(x).$$

Let us denote by

$$\mathrm{in}_{v_{\alpha'}} h = X'^p - G_{\alpha'}^{p-1} X' + F_{p,X',\alpha'} \in \mathrm{gr}_{\alpha'} S = k(x)[\overline{u}'_2]_{(\overline{w})}[U_1, V, X'].$$

When $G_{\alpha'} \neq 0$, we have $A_1(x') = B(x)$, $\beta(x') = 0$, so (ii) holds. Assume now that $G_{\alpha'} \neq 0$.

Subcase 1.1: when

$$U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z,\alpha}}{\partial V} \notin < V^{\omega(x)} > .$$

We expand

$$U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z,\alpha}}{\partial V} = \lambda V^{\omega(x)} + \sum_{1 \leq i \leq \omega(x)} V^{\omega(x)-i} U_1^{a_1(i)} U_2^{a_2(i)} Q_i(U_1, U_2), \quad (9.21)$$

with $\lambda \neq 0$, $Q_i = 0$ or Q_i divisible neither by U_1, nor by U_2. For $Q_i \neq 0$:

$$a_j(i) \geq iA_j(x),\ \deg(Q_i) \leq iC(x).$$

By proposition 3.5(v), $\frac{\partial F_{p,X',\alpha'}}{\partial V}$ is the transform of $\frac{\partial F_{p,Z,\alpha}}{\partial V}$. Then, by [17] lemma 6.2.3 a and page 92, the lowest abscissa of the vertices of the polygon

$$\Delta(\frac{\partial F_{p,X',\alpha'}}{\partial V}; U_1, \overline{w}; V; X')$$

is $B(x)$. The non compact face of lowest abscissa is not solvable and, after a possible translation:

$$Z' = X' + \phi',\ \phi' \in U_1'^{\lceil B(x) \rceil} k(x)[\overline{u}'_2]_{(\overline{w})}[V],$$

the ordinate β' of the vertex of lowest abscissa of

$$\Delta(\frac{\partial F_{p,Z',\alpha'}}{\partial V}; U_1, \overline{w}; V; Z')$$

satisfies

$$\beta' < 1 + \lfloor \frac{C(x)}{d} \rfloor,\ \beta' \leq \beta_2(x),$$

where $\beta_2(x)$ is the ordinate of the left vertex of the initial face of the polygon $\Delta(\frac{\partial F_{p,Z,\alpha}}{\partial V}; U_1, U_2; V; Z)$. Then we have

$$\beta(x') \leq \beta' < 1 + \lfloor \frac{C(x)}{d} \rfloor, \ \beta(x') \leq \beta' \leq \beta_2(x) \leq \beta(x). \tag{9.22}$$

This implies all the assertions in subcase 1-1.

Subcase 1.2: when

$$U_1^{-pd_1} U_2^{-pd_2} \frac{\partial F_{p,Z,\alpha}}{\partial V} \in < V^{\omega(x)} > .$$

We now have an expansion

$$U_1^{-pd_1} U_2^{-pd_2} F_{p,Z,\alpha} = \lambda V^{1+\omega(x)} + \sum_{i=0}^{\lfloor \frac{1+\omega(x)}{p} \rfloor} V^{pi} U_1^{a_1(i)} U_2^{a_2(i)} Q_i(U_1, U_2), \tag{9.23}$$

with $\lambda \neq 0$, $Q_i = 0$ or Q_i divisible neither by U_1, nor by U_2. For $Q_i \neq 0$:

$$a_j(i) \geq (1 + \omega(x) - pi) A_j(x), \ \deg(Q_i) \leq (1 + \omega(x) - pi) C(x).$$

Take i_0, $1 \leq i_0 < (1 + \omega(x))/p$ maximal such that $U_1^{pd_1 + a_1(i_0)} U_2^{pd_2 + a_2(i_0)} Q_{i_0}$ is not a pth-power. This i_0 exists by total preparation. By (9.23), the transform of $\frac{\partial F_{p,Z,\alpha}}{\partial V}$ now reads

$${U_1}^{-pd'_1} \frac{\partial F_{p,X',\alpha'}}{\partial V} = \lambda' V^{\omega(x)}, \ \lambda' \text{ a unit.} \tag{9.24}$$

Preparation along the face of abscissa $B(x)$ will thus be a translation $Z' = X' + \phi'$ on X', no translation on v: this will just add a pth-power to the term ${U_1}^{pd'_1 + (1+\omega(x) - pi_0)B(x)} {\overline{u}'_2}^{pd_2} Q_{i_0}(1, \overline{u}'_2)$ in (9.23), which will become of the form

$$\overline{\gamma}' {U_1}^{pd'_1 + (1+\omega(x) - pi_0)B(x)} \overline{w}^c, \ \overline{\gamma}' \in k(x)[\overline{u}'_2]_{(\overline{w})}, \ \overline{\gamma}' \text{ invertible.}$$

By the usual computations ([17] page 92 or the blowing up formula applied to $U_1^{pd_1 + a_1(i_0)} U_2^{pd_2 + a_2(i_0)} Q_{i_0}(U_1, U_2)$), we have

$$c \leq 1 + \frac{\deg(Q_{i_0})}{d}; \text{ when } d_2 = 0, \ c \leq a_2(i_0) + \deg(Q_{i_0}) \leq \beta_2(x) \leq \beta(x). \tag{9.25}$$

This implies all the assertions in subcase 1-2, x' not the origin and (ii) is proved. Permuting u_1 and u_2 gives (iii).

We now turn to case 2. Let x' be in the chart of origin $(X' := \frac{Z}{u_1}, u_1, u_2', v')$. By proposition 9.13(ii), we may assume that $B(x) > 1$, i.e. $< V >= \mathrm{Vdir}(x)$, so $v' \in m_{S'}$. In the expansion of $f_{p,Z}$ the monomial

$$u_1^{pd_1} u_2^{pd_2} v^{1+\omega(x)-i} u_1^a u_2^b,\ 0 \leq i \leq 1 + \omega(x) - i$$

becomes $u_1^{pd_1+pd_2+1+\omega(x)-p} {u_2'}^{pd_2} v^{1+\omega(x)-i} u_1^{a+b} {u_2'}^b$ in the expansion of $f_{p,Z'}$. This leads to:

$$f_{p,Z'} = u_1^{pd_1'} {u_2'}^{pd_2} (\gamma v^{1+\omega(x)} + u_1\phi),\ d_1' := d_1 + d_2 + \frac{1+\omega(x)}{p} - 1.$$

Then x' is resolved or x' satisfies conditions (**) and **(E)'** as in case 1. Then the proof runs along the same lines as above: equations (9.22) and (9.25) remain true.

The case where x' is the origin of the second chart is given by a permutation of u_1 and u_2 in the computations above and the fact that the vertices of $\Delta_2(h'; u_1/u_2, u_2; v/u_2; Z/u_2)$ are the transforms of the **left vertices** of $\Delta_2(h; u_1, u_2; v; Z)$ by the affinity $(x_1, x_2) \mapsto (x_1, x_1 + x_2 - 1)$: they are totally prepared. □

Proposition 9.17. *Assume that x satisfies conditions (**) and **(E)'**. Let μ be a valuation of $L = k(\mathcal{X})$ centered at x. There exists a finite and independent composition of local Hironaka-permissible blowing ups w.r.t. E:*

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{9.26}$$

*where $x_i \in \mathcal{X}_i$ is the center of μ, such that x_r is resolved or (x_r satisfies again conditions (**) and **(E)'** together with one of the following):*

(i) $E_r = \mathrm{div}(u_{1,r})$, $\beta(x_r) < 1$;

(ii) $E_r = \mathrm{div}(u_{1,r} u_{2,r})$, $C(x_r) = 0$.

Proof. Let (Z, u_1, u_2, v) be totally prepared. Let $\mathcal{Y} = V(Z, u_1, v)$ with generic point y. We define by induction on $i \geq 0$ a sequence of local Hironaka-permissible blowing ups w.r.t. E, or composition of two such local blowing ups. Take $i = 0$ w.l.o.g. in the following definition.

(1) if ($E = \mathrm{div}(u_1)$, $\kappa(x) = 3$), blow up along x (proposition 9.16, case 2);

(2) if ($E = \mathrm{div}(u_1)$, $\kappa(x) = 4$, $\epsilon(y) \leq 1$), blow up along x, then along $D' = V(Z', u_1', u_2')$ (notations of proposition 9.14(3));

(3) if ($E = \mathrm{div}(u_1)$, $\kappa(x) = 4$, $\epsilon(y) \geq 2$), blow up along $\mathcal{Y}$ (proposition 9.14(2));

(4) if ($E = \mathrm{div}(u_1 u_2)$, $\omega(x) \geq p$), blow up along $D = (Z, u_1, u_2)$ (proposition 9.16, case 1);

(5) if ($E = \mathrm{div}(u_1 u_2)$, $\omega(x) < p$), blow up along x (proposition 9.16, case 2).

We must prove that (A) this algorithm is well defined, i.e. x_1 is resolved or satisfies again conditions (**) and **(E)'**, so it builds up a sequence (9.26), then (B) this sequence is finite.

Note that any x fits into some of (1)-(5). To prove (A)(B), we recollect results from the previous propositions. By proposition 9.14, applying (2) produces x_1 satisfying again the assumptions of the lemma and fitting into (4) with $\kappa(x_1) = 3$, $\gamma(x_1) = 1$; applying (3) shows that x is resolved.

We now turn to proposition 9.16. Statement (i) shows that x_1 is resolved or satisfies again the assumptions of the lemma. The proof of (A) is thus complete and we turn to (B). Assume w.l.o.g. that x neither satisfies (i) nor (ii). In particular $\gamma(x) \geq 1$. We first claim that there exists $r_0 \geq 0$ such that x_{r_0} is resolved or

$$\gamma(x_r) = 1 \text{ for all } r \geq r_0. \tag{9.27}$$

By proposition 9.16(i), we have $\gamma(x_1) \leq \gamma(x)$; by proposition 9.16(iii), inequality is strict if:

$$E = \mathrm{div}(u_1),\ E_1 = \mathrm{div}(u_{1,1} u_{2,1})$$

provided $\gamma(x) \geq 2$, $\beta(x) \neq 2$. In case $\beta(x) = 2$, we obtain $C(x_1) \leq 1$. Then any further occurrence of $E_r = \mathrm{div}(u_{1,r})$ along the algorithm will satisfy $\beta(x_r) < 2$ by proposition 9.16(ii)-(iv). Therefore it can be assumed that E and E_i have *the same number* of irreducible components for every $i \geq 0$ in order to prove (9.27) (note that we are done if (2) is applied).

If $E = \mathrm{div}(u_1)$, we reach (i) or $k(x_i) = k(x)$ for $i >> 0$ by proposition 9.16(iv). The claim follows from corollary 3.9.

If $E = \mathrm{div}(u_1 u_2)$, we get (9.27) by standard arguments on combinatorial blowing ups.

To conclude the proof, we may hence assume that ($E = \mathrm{div}(u_1)$, $\beta(x) = 1$) or ($E = \mathrm{div}(u_1 u_2)$, $C(x) < 1$).

When ($E = \mathrm{div}(u_1)$, $\beta(x) = 1$), this is stable by blowing up or yields $E_1 = \mathrm{div}(u_{1,1}u_{2,1})$ (proposition 9.14(3) and proposition 9.16(iii)). Stability ends after finitely many steps by proposition 9.16(iv) and corollary 3.9.

When ($E = \mathrm{div}(u_1u_2)$, $C(x) < 1$), this is stable by blowing up or yields (i) (proposition 9.16(ii)). Stability ends up in (ii) for $r >> 0$ by standard arguments on combinatorial blowing ups. □

Proposition 9.18. *Assume that x satisfies conditions (**) and **(E)'** together with one of the following:*

(i) $E = \mathrm{div}(u_1)$, $\beta(x) < 1$;

(ii) $E = \mathrm{div}(u_1u_2)$, $A_1(x) < 1$, $C(x) < \frac{1}{2}$, $\beta(x) < 1 - \frac{1}{1+\omega(x)}$;

(iii) $E = \mathrm{div}(u_1u_2)$ *and* $C(x) = 0$.

Then x is resolved for $(p, \omega(x), 3)$.

Proof. We assume that (Z, u_1, u_2, v) is totally prepared. Let

$$c(x) := (A_1(x), \beta(x))$$

with lexicographical ordering. First suppose that

$$A_1(x) < 1 \text{ and } (x \text{ is in case (iii)} \Longrightarrow A_2(x) < 1). \tag{9.28}$$

If $E = \mathrm{div}(u_1u_2)$ and $\kappa(x) = 3$, we blow up along x. Let x' be a point ω near x. When x' is the origin of a chart, by proposition 9.16(i)-(iii), x' satisfies again the assumptions of the proposition with $c(x') < c(x)$. When x' is in the first chart with $E' = \mathrm{div}(u_1)$, proposition 9.16(ii) gives

$$A_1(x') = B(x) - 1 \leq A_1(x) + \beta(x) - 1 < A_1(x) \text{ and } \beta(x') < 1.$$

In both cases, x' satisfies again the assumptions of the proposition together with (9.28) and $c(x') < c(x)$.

If $E = \mathrm{div}(u_1u_2)$ and $\kappa(x) = 4$, we let $\mathcal{Y}_j := V(Z, v, u_j)$ with generic point y_j, $j = 1, 2$. The condition $\epsilon(y_j) \geq 2$ is equivalent to $A_j(x) > \frac{1}{1+\omega(x)}$. We apply proposition 9.14(1)(2): then x is resolved except possibly if $A_j(x) \leq \frac{1}{1+\omega(x)}$, $j = 1, 2$. Then

$$1 - \frac{1}{1+\omega(x)} \leq B(x) \leq A_1(x) + \beta(x) < 1.$$

We deduce that equality holds and that $\mathrm{VDir}(x) =< U_1, U_2 >$. Since $\omega(x) \geq p \geq 2$, we obtain $\omega(x') < \omega(x)$ after blowing up along x, so x is resolved.

If $E = \mathrm{div}(u_1)$ and $\kappa(x) = 3$, we blow up along x. Note that $\beta(x) > 0$ since $A_1(x) < 1$. Let

$$x' := (Z' := \frac{Z}{u_2}, u_1' = \frac{u_1}{u_2}, u_2, v' = \frac{v}{u_2})),\ E' = \mathrm{div}(u_1' u_2).$$

If $x_1 \neq x'$, proposition 9.16(iv) gives

$$A_1(x') = B(x) - 1 \leq A_1(x) + \beta(x) - 1 < A_1(x) \text{ and } \beta(x') \leq \beta(x).$$

Therefore x_1 satisfies again assumption (i) of the proposition together with (9.28) and $c(x') < c(x)$.

If $x_1 = x'$, proposition 9.16(iii) gives

$$A_1(x') = A_1(x),\ C(x') < \frac{1}{2},\ \beta(x') = \beta(x) + A_1(x) - 1 < \beta(x).$$

Therefore x' satisfies again assumption (ii) of the proposition together with (9.28) and $c(x') < c(x)$.

If $E = \mathrm{div}(u_1)$ and $\kappa(x) = 4$, x is resolved by propositions 9.14(1)(2) and 9.15.

Therefore the proposition holds by induction on $c(x)$ under the extra assumption (9.28).

Assume now that x satisfies assumption (i) with $A_1(x) \geq 1$. In particular $\epsilon(x) = 1 + \omega(x)$ and $V \in \mathrm{Vdir}(x)$ by proposition 9.13. Furthermore,

$$d_1 + \frac{1 + \omega(x)}{p} > 1. \tag{9.29}$$

We have $m(y) = m(x)$, $\epsilon(y) = \epsilon(x)$ where $\mathcal{Y} = V(Z, u_1, v)$ with generic point y, so $\mathcal{Y}$ is permissible of first kind. Let us blow up along $\mathcal{Y}$.

We are done by theorem 3.6 if $\mathrm{Vdir}(x) =< V, U_1 >$. Otherwise we have $A_1(x) > 1$ or $\beta(x) > 0$. Since $V \in \mathrm{Vdir}(x)$, the only point which may be ω-near x is the point

$$x' := (Z', u_1', u_2', v') = (Z/u_1, u_1, u_2, v/u_1),\ E' = \mathrm{div}(u_1). \tag{9.30}$$

These are well adapted coordinates. If $A_1(x) > 1$, we have

$$\beta(x') = \beta(x),\ A_1(x') = A_1(x) - 1 > 0,\ d_1' = d_1 + \frac{1 + \omega(x)}{p} - 1.$$

Then x' satisfies again conditions (**) and **(E)'** by (9.29). By induction on $A_1(x)$, we reduce to $A_1(x) = 1$, since $A_1(x) < 1$ is (9.28).

If $A_1(x) = 1$, expand

$$f_{p,Z} = u_1^{pd_1}(\gamma v^{1+\omega(x)} + \sum_{1\le i\le 1+\omega(x)} \gamma_i v^{1+\omega(x)-i} u_1^i u_2^{a_2(i)} + f_1),$$

with $f_1 \in (v, u_1)^{2+\omega(x)}$, $\gamma \in S$ invertible, $\gamma_i \in S$ invertible or zero, γ_{i_0} invertible for some i_0 with $a_2(i_0) = i_0\beta(x) < i_0$. We get

$$f_{p,Z'} = {u_1'}^{pd_1+1+\omega(x)-p}(\gamma {v'}^{1+\omega(x)} + \sum_{1\le i\le 1+\omega(x)} \gamma_j {v'}^{1+\omega(x)-i} {u_2'}^{a_2(i)} + u_1' f_1'),\ f_1' \in S'.$$

Clearly $\iota(x') \le (p, \omega(x), 2)$ and x is resolved for $(p, \omega(x), 3)$.

There remains to prove the proposition in case (iii) with $A_i(x) \ge 1$, $i = 1$ or 2. See [26] **II.6.2** and **II.6.3** on pp. 1950-1951. The argument is similar to the one used in the proof of proposition 6.9(b)(c).

If ($\omega(x) \ge p$ and $A_1(x) \ge 1$), then $\mathcal{Y} := (Z, u_1, v)$ is permissible of the first kind. Blowing up along $\mathcal{Y}$, the only point which may be ω-near x is the point x' as in (9.30). We have

$$A_1(x') = A_1(x) - 1,\ A_2(x') = A_2(x'),\ C(x') = 0,\ d_1' = d_1 + \frac{1+\omega(x)}{p} - 1 \ge 1.$$

Then x' satisfies again conditions (**) and **(E)'**. A descending induction on $\max\{A_1(x), A_2(x)\}$ reduces to $A_1(x), A_2(x) < 1$ which is (9.28) and the proof is complete.

If $1 + \omega(x) < p$, we argue by induction on

$$c'(x) := (\max\{A_1(x), A_2(x)\}, \max\{d_1, d_2\}, n)$$

where $n := 2$ if $(A_1(x) = A_2(x), d_1 = d_2)$, $n := 1$ otherwise.

Suppose that $A_1(x) \ge 1$, $d_1 + \frac{1+\omega(x)}{p} \ge 1$. Up to renumbering u_1, u_2, it can be assumed that $c'(x) = (A_1(x), d_1, n)$ or $(c'(x) = (A_2(x), d_2, 1)$ with $d_2 + \frac{1+\omega(x)}{p} < 1$. Blowing up along $\mathcal{Y} := (Z, u_1, v)$, the only point which may be ω-near x is the point x' as in (9.30). If $(m(x'), \omega(x')) = (p, \omega(x))$, x' is in case (**) and we have

$$A_1(x') = A_1(x) - 1,\ C(x') = 0,\ d_1' = d_1 + \frac{1+\omega(x)}{p} - 1 < d_1.$$

It is easily seen that $c'(x') < c'(x)$.

The remaining case: up to renumbering u_1, u_2, we have

$$A_1(x) < 1 \leq A_2(x),\ d_2 + \frac{1+\omega(x)}{p} < 1 \leq d_1 + \frac{1+\omega(x)}{p}.$$

We then blow up along x. As case (i) is resolved, we have just to look at the origins of both charts. Let us look at the first chart, of origin the point x' as above. If $(m(x'), \omega(x')) = (p, \omega(x))$, x' is in case (**) and we have $A_2(x') = A_2(x)$, $d_2' = d_2$ and

$$A_1(x') = A_1(x) + A_2(x) - 1 < A_2(x),\ C(x') = 0,\ d_1' = d_1 + \frac{1+\omega(x)}{p} - 1 < d_1.$$

Therefore $c'(x') < c'(x)$. The last point to look at is the point

$$x'' = (\frac{Z}{u_2}, \frac{u_1}{u_2}, u_2, \frac{v}{u_2}).$$

If $(m(x''), \omega(x'')) = (p, \omega(x))$, x'' is in case (**), and we have $A_1(x'') = A_1(x)$ and

$$A_2(x'') = A_1(x) + A_2(x) - 1 < A_2(x),\ C(x'') = 0.$$

Therefore $c'(x'') < c'(x)$. This concludes the proof. $\square$

References

[1] Abhyankar S., Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$, *Ann. of Math.* **63** (1956), 491-526.

[2] Abhyankar S., On the valuations centered in a domain, *Amer. J. Math.* **78** (1956), 321-348.

[3] Abhyankar S., Ramification theoretic methods in algebraic geometry, *Ann. of Math. Studies* **43**, Princeton University Press (1959).

[4] Abhyankar S., Resolution of singularities of arithmetical surfaces, *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, Harper and Row (1965), 111-152.

[5] Abhyankar S., Resolution of singularities of embedded algebraic surfaces, second edition, *Springer Monographs in Math.*, Springer Verlag (1998).

[6] Abramovich D., de Jong A.J., Smoothness, semistablility, and toroidal geometry, *J. Alg. Geom.* **6** (1997), 789-801.

[7] Benito A., Villamayor O., Monoidal transforms and invariants of singularities in positive characteristic, *Compos. Math.* **149** (2013), no. 8, 1267-1311.

[8] Benito A., Villamayor O., On elimination of variables in the study of singularities in positive characteristic, *preprint* arXiv:1103.3462, 1-35.

[9] Berthomieu J., Hivert P., Mourtada H., Computing Hironaka's invariants: ridge and directrix, *in* Arithmetic, geometry, cryptography and coding theory 2009, *Contemp. Math.* **521**, Amer. Math. Soc., Providence RI (2010), 9-20.

[10] Bierstone E., Milman, P., Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* **128** (1997), 207-302.

[11] Bravo A., Villamayor O., Singularities in positive characteristic, stratification and simplification of the singular locus, *Adv. Math.* **224** (2010), 1349-1418.

[12] Cano F, Desingularization strategies for three-dimensional vector fields, *Lect. Notes in Math.* **1259**, Springer-Verlag (1987).

[13] Cano F, Reduction of the singularities of codimension one singular foliations in dimension three, *Ann. of Math.* (2) **160** (2004), no. 3, 907-1011.

[14] Cano F, Roche C., Spivakovsky M., Reduction of singularities of three-dimensional line foliations, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **108** (2014), no. 1, 221-258.

[15] Cossart V., Sur le polyèdre caractéristique d'une singularité, *Bull. Soc. math. France* **103** (1975), 13-19.

[16] Cossart V., Desingularization of embedded excellent surfaces, *Tohoku Math. J.*, II. Ser. **33** (1981), 25-33.

[17] Cossart V., Resolution of surface singularities, in *Lect. Notes in Math.* **1101**, Springer-Verlag (1984), 79-98.

[18] Cossart V., Forme normale d'une fonction sur un k-schéma de dimension 3 and de caractéristique positive, *in* Géométrie algébrique et applications, C. R. 2ème Conf. int., La Rabida/Espagne 1984, I: Géométrie and calcul algébrique, *Trav. Cours* **22** (1987), 1-21.

[19] Cossart V., Polyèdre caractéristique d'une singularité, *Thesis*, Université de Paris-Sud, Centre d'Orsay (1987), 1-424.

[20] Cossart V., Polyèdre caractéristique et éclatements combinatoires, *Rev. Mat. Iberoam.* **5** No.1/2 (1989), 67-95.

[21] Cossart V., Contact maximal en caractéristique positive et petite multiplicité, *Duke Math. J.* **63** (1991), no.1, 57-64.

[22] Cossart V., Modèle projectif régulier et désingularisation, *Math. Ann.* **293** (1992), no.1, 115-122.

[23] Cossart V., Désingularisation en dimension 3 et caractéristique p, Proc. La Rabida. *Progress in Math.* **134**, Birkhauser (1996) 1-7.

[24] Cossart V., Uniformisation et désingularisation des surfaces, dédié à O. Zariski (Hauser, Lipman, Oort, Quiros Éd.), *Progress in Math.* **181**, Birkhauser (2000), 239-258.

[25] Cossart V., Jannsen U., Saito S., Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes, *preprint* arXiv:0905.2191 (2009), 1-169.

[26] Cossart V., Piltant O., Resolution of singularities of threefolds in positive characteristic I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings, *J. Algebra* **320** (2008), no. 3, 1051-1082.

[27] Cossart V., Piltant O., Resolution of singularities of threefolds in positive characteristic II, *J. Algebra* **321** (2009), no. 7, 1836-1976.

[28] Cossart V., Piltant O., Characteristic polyhedra of singularities without completion, *Math. Ann.* (2014), 1-11, DOI 10.1007/s00208-014-1064-0.

[29] Cossart V., Piltant O., Resolution of Singularities of Threefolds in Mixed Characteristics. Case of small multiplicity, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **108** (2014), no. 1, 113-151.

[30] Cutkosky S.D., Resolution of singularities, *Grad. Stud. in Math.* **63**, American Mathematical Society, Providence, RI (2004).

[31] Cutkosky S.D., Toroidalization of dominant morphisms of 3-folds, *Mem. Amer. Math. Soc.* **190** (2007).

[32] Cutkosky S.D., Resolution of singularities for 3-folds in positive characteristic, *Amer. J. Math.* **131** (2009), no. 1, 59-127.

[33] Cutkosky S.D., A skeleton key to Abhyankar's proof of embedded resolution of characteristic p surfaces, *Asian J. Math.* **15** (2011), no. 3, 369-416.

[34] Cutkosky S.D., A simpler proof of toroidalization of morphisms from 3-folds to surfaces, *Ann. Inst. Fourier 63* (2013), no. 3, 865-922.

[35] Giraud J., Étude locale des singularités, Cours de 3ème cycle, *Publ. Math. d'Orsay* **26**, Univ. Paris XI, Orsay (1972).

[36] Giraud J., Contact maximal en caractéristique positive, *Ann. Sci. Ec. Norm. Sup.* 4ème série **8** (1975), no. 2, 201-234.

[37] Grothendieck A., Dieudonne J., Éléments de géométrie algébrique IV-2, *Publ. Math. I.H.E.S.* **24** (1965).

[38] Grothendieck A., Dieudonne J., Éléments de géométrie algébrique IV-4, *Publ. Math. I.H.E.S.* **32** (1967).

[39] Hartshorne R., Algebraic Geometry, *Graduate Texts in Mathematics* **52**, Springer-Verlag (1977).

[40] Hironaka H., Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math* **79** (1964), 109-326.

[41] Hironaka H., Desingularization of excellent surfaces, Advanced Science Bowdoin College, Brunswick, Maine (1967).

[42] Hironaka H., Characteristic polyhedra of singularities, *J. Math. Kyoto Univ.* **7** 1967, no.3, 251-293.

[43] Hironaka H., Additive groups associated with points of a projective space, *Ann. of Math.* **92** (1970), 327-334.

[44] Hironaka H., Idealistic exponents of singularity, (J.J. Sylvester symposium, John Hopkins Univ., Baltimore 1976), John Hopkins Univ. Press (1977), 52-125.

[45] Hironaka H., Theory of infinitely near singular points, *J. Korean Math. Soc.* **40** (2003), no.5, 901-920.

[46] Hironaka H., Three key theorems on infinitely near singularities, Singularités Franco-Japonaises Paris 2005, in *Sémin. Congr. SMF* **10**, 87-126.

[47] Illusie L., Laszlo Y., Orgogozo F., Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents. Seminaire a l'École Polytechnique 2006-2008, *preprint* arXiv:1207.3648 (2007), 1-416.

[48] de Jong A.J., Smoothness, semistability and Alterations, *Publ. Math. I.H.E.S.* **83** (1996), 51-93.

[49] Kawanoue H., Toward resolution of singularities over a field of positive characteristic I. Foundation; the language of the idealistic filtration, *Publ. RIMS* **43** (2007), no. 3, 819-909.

[50] Kawanoue H., Matsuki K., Toward resolution of singularities over a field of positive characteristic (the idealistic filtration program) Part II. Basic invariants associated to the idealistic filtration and their properties, *Publ. RIMS* **46** (2010), no. 2, 359-422.

[51] Knaf H., Kuhlmann F.V., Every place admits local uniformization in a finite extension of the function field, *Adv. Math.* **221** (2009), 428-453.

[52] Lipman J., Desingularization of two-dimensional schemes, *Ann. Math.* **107** (1978), 151-207.

[53] Mc Quillan M., Panazzolo D., Almost étale resolution of foliations, *J. Diff. Geom.* **95** (2013), no. 2, 279-319.

[54] Matsumura H., Commutative ring theory, 3rd edition, *Cambridge studies in advanced mathematics* **8**, Cambridge Univ. Press (1986).

[55] Moh T.T., On a Newton polygon approach to the uniformization of singularities of characteristic p, *in* Algebraic geometry and singularities (La Rábida, 1991), *Progr. Math.* **134** (1996), Birkhuser, 49-93.

[56] Moh T.T., On the bound of d_2, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **108** (2014), no. 1, 211-220.

[57] Novacoski J., Spivakovsky M., Reduction of Local Uniformization to the rank one case, *preprint* arXiv:1204.4751 (2012), 1-22.

[58] Panazzolo D., Resolution of singularities of real-analytic vector fields in dimension three, *Acta Math.* **197** (2006), no. 2, 167-289.

[59] Piltant O., An axiomatic version of Zariski's patching theorem, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **107** (2013), no. 1, 91-121.

[60] Pomerol, M.J., Sur la strate de Samuel du sommet d'un cĥe en caractŕistique positive, *Bull. Sci. Math.*(2) **98** (1974), no. 3, 173-182.

[61] Seidenberg A., Reduction of singularities of the differential equation $Ady = Bdx$, *Amer. J. Math.* **90** (1968), 248-269.

[62] Spivakovsky, M., Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990), 107-156.

[63] Teissier B., Valuations, Deformations and Toric Geometry, *in* Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), *Fields Inst. Commun.* **33** (2003), 361-459.

[64] Teissier B., Overweight deformations of affine toric varieties and local uniformization, *preprint* arXiv:1401.5204 (2014).

[65] Temkin M., Inseparable local uniformization, *J. Algebra* **373** (2013), 65-119.

[66] Villamayor O., Patching local uniformizations, *Ann. Sci. École Norm. Sup.* **25** (1992), no. 6, 629-677.

[67] Wlodarczyk J., Simple Hironaka resolution in characteristic zero, *J. Amer. Math. Soc.* **18** (2005), no. 4, 779-822.

[68] Zariski O., Local uniformization of algebraic varieties, *Ann. of Math.* **41** (1940), 852-896.

[69] Zariski O., Reduction of the singularities of algebraic three dimensional varieties, *Ann. of Math.* **45** (1944), 472-542.

[70] Zariski O., The fundamental ideas of abstract algebraic geometry, *Proc. Int. Congress Math.* Vol. II (1950), 78-89, *in* Oscar Zariski: Collected Papers, Vol. 3, MIT Press (1978), 363-375.

[71] Zariski O. and Samuel P., Commutative Algebra I, 2nd edition, *Graduate Texts in Mathematics* **28**, Springer-Verlag (1979).

[72] Zariski O. and Samuel P., Commutative Algebra II, Van Nostrand, Princeton (1960).