7 Appendix

This appendix contains various results : some of them are used repeatedly in the proof of the main result (in particular Proposition 4, Lemmas 7, and 10, and to a lesser extent Lemmas 9 and 8), the other ones concern parts of the main proof which are postponed to the appendix for better clarity of the main flow of the proof (Lemmas 11, 12 and 13).

Definition 1 *An ultimately positive function $f : \mathbb{R}^+ \to \mathbb{R}$ is regularly varying (at infinity) with index $\alpha \in \mathbb{R}$, if*

$$\lim_{t\to+\infty} \frac{f(tx)}{f(t)} = x^\alpha \quad (\forall x > 0).$$

This is noted $f \in RV_\alpha$. If $\alpha = 0$, f is said to be slowly varying.

Proposition 4 *(See de Haan and Ferreira (2006) Proposition B.1.9)*
Suppose $f \in RV_\alpha$. If $x < 1$ and $\epsilon > 0$, then there exists $t_0 = t_0(\epsilon)$ such that for every $t \geqslant t_0$,

$$(1-\epsilon)x^{\alpha+\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha-\epsilon}$$

and if $x \geqslant 1$,

$$(1-\epsilon)x^{\alpha-\epsilon} < \frac{f(tx)}{f(t)} < (1+\epsilon)x^{\alpha+\epsilon}. \tag{44}$$

Lemma 7 *Let $x \in \mathbb{R}_+^*$, $\alpha \in \mathbb{R}_+$, $\beta > -1$, and for a and b real numbers, f and g are two regular varying functions at infinity, with index, respectively, a and b. Then, as $t \to +\infty$,*

$$(i)\ \ J_\beta(x) = \int_1^{+\infty} \log^\beta(y)\ y^{-x-1}dy = \frac{\Gamma(\beta+1)}{x^{\beta+1}}.$$

$$(ii)\ \ I_{\alpha,a,b} = \int_1^{+\infty} \log^\alpha(y)\ \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b\Gamma(\alpha+1)}{(-a-b)^{\alpha+1}},\ \text{if } a+b<0$$

$$(iii)\ \ J_{a,b} = \int_0^1 \frac{f(yt)}{f(t)}\ \frac{dg(yt)}{g(t)} \to \frac{b}{a+b},\ \text{if } a+b>0$$

Proof :

- (i) A simple change of variable and the definition of the Γ function yields the result.
- (ii) For the sake of simplicity, we are going to treat the case $a < 0$ and $b < 0$. The only difference for the other cases is the sign in front of the ϵ or ϵ' appearing below (coming from the application of (44) several times), which can depend on the sign of a, b or another constant, but does not affect the result. Using Potter-bounds (44) for f yields, for n sufficiently large and $\epsilon > 0$,

$$(1+\epsilon)\int_1^{+\infty} \log^\alpha(y)\ y^{a+\epsilon}\ \frac{dg(yt_n)}{g(t_n)} \leqslant I_{\alpha,a,b} \leqslant (1-\epsilon)\int_1^{+\infty} \log^\alpha(y)\ y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)}.$$

Let us treat only the upper bound and the case $\alpha \neq 0$ (the other cases being similar). By integration by parts, with $a + b < 0$, we have

$$\int_1^{+\infty} \log^\alpha(y)\, y^{a-\epsilon}\ \frac{dg(yt_n)}{g(t_n)} = -\alpha\int_1^{+\infty} \log^{\alpha-1}(y)\, y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy - (a-\epsilon)\int_1^{+\infty} \log^\alpha(y)\, y^{a-1-\epsilon}\ \frac{g(yt_n)}{g(t_n)}dy.$$

Using Potter-bounds (44) for g yields, for n sufficiently large and $\epsilon' > 0$

$$\int_1^{+\infty} \log^\alpha(y)\, y^{a-\epsilon}\, \frac{dg(yt_n)}{g(t_n)} \leqslant -\alpha(1-\epsilon')J_{\alpha-1}(-a-b+\epsilon+\epsilon') - (a-\epsilon)(1+\epsilon')J_\alpha(-a-b+\epsilon-\epsilon').$$

Doing the same with the lower bound and making ϵ and ϵ' tend to 0, yields the result after simplifications.

(*iii*) As in (*ii*), using Potter-bounds (44) for f, integration by parts and then again (44) for g yields the result.

Lemma 8 *For any $\delta > 0$, let C_δ denote the function*

$$C_\delta(t) = \int_0^t \frac{dG(v)}{\overline{G}(v)\overline{H}^\delta(v)}.$$

Under condition (1)*, this function is regularly varying of order δ/γ and $C_\delta(t) \sim (\gamma/\gamma_C)/(\delta \overline{H}^\delta(t))$, as $t \to +\infty$.*

Proof : by writing $\overline{H}^\delta(t)C_\delta(t) = -\int_0^1 \frac{\overline{H}^\delta(t)}{\overline{H}^\delta(tu)} \frac{\overline{G}(t)}{\overline{G}(tu)} \frac{d\overline{G}(tu)}{\overline{G}(t)}$, the lemma is an immediate consequence of part (*iii*) of Lemma 7, with $a+b = (\delta/\gamma+1/\gamma_C)+(-1/\gamma_C) = \delta/\gamma > 0$ and $-b/(a+b) = (\gamma/\gamma_C)/\delta$.

Remark 3 *In the Lemma above, C_1 is the important function C introduced at the beginning of Section 6, and thus $C(t) \sim (\gamma/\gamma_C)/\overline{H}(t) = (1-\gamma/\gamma_F)/\overline{H}(t)$, as $t \to +\infty$. Hence, C is regularly varying at infinity with index $1/\gamma$, a property which proves useful several times in the main proofs.*

Lemma 9 *Let $\psi(\phi_n, u) = \int_u^{+\infty} \phi_n(s) dF^{(k)}(x)$, for $u \geqslant 0$ and $\phi_n(u) = \frac{1}{\overline{F}^{(k)}(t_n)} \log(u/t_n)\mathbb{I}_{u>t_n}$. Under condition* (1)*, we have*

$$\begin{aligned}\psi(\phi_n, u) &= \gamma_{n,k}, \ \text{if } u \leqslant t_n \\ &= \log\left(\frac{u}{t_n}\right) \frac{\overline{F}^{(k)}(u)}{\overline{F}^{(k)}(t_n)} + \gamma_k \left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \epsilon_n(u)\left(\frac{u}{t_n}\right)^{-1/\gamma_k+\delta} \quad \text{if } u > t_n,\end{aligned}$$

where $\epsilon_n(u)$ is a sequence tending to 0 uniformly in u, as $n \to \infty$, and δ a positive real number such that $-\frac{1}{\gamma_k} + \delta < 0$.

Proof : We only consider the second situation where $u > t_n$ (the first one is straightforward) :

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = -\int_{\frac{u}{t_n}}^{+\infty} \log(y) \frac{d\overline{F}^{(k)}(yt_n)}{\overline{F}^{(k)}(t_n)}$$

An integration by part and the fact that $\overline{F}^{(k)}$ is regularly varying at infinity with index $-1/\gamma_k$, yields

$$\int_u^{+\infty} \phi_n(s) dF^{(k)}(x) = \log\left(\frac{u}{t_n}\right) \frac{\overline{F}^{(k)}(u)}{\overline{F}^{(k)}(t_n)} + \gamma_k \left(\frac{u}{t_n}\right)^{-1/\gamma_k} + \Delta_n(u),$$

where

$$\Delta_n(u) = \int_{\frac{u}{t_n}}^{+\infty} \left(\frac{\overline{F}^{(k)}(yt_n)}{\overline{F}^{(k)}(t_n)} - y^{-1/\gamma_k} \right) \frac{dy}{y}$$

Let δ be a positive real number. Then

$$|\Delta_n(u)| = \left| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} \left(y^{1/\gamma_k - \delta} \frac{\overline{F}^{(k)}(yt_n)}{\overline{F}^{(k)}(t_n)} - y^{-\delta} \right) dy \right|$$
$$\leqslant \sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\overline{F}^{(k)}(yt_n)}{\overline{F}^{(k)}(t_n)} - y^{-\delta} \right| \int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy,$$

where the function $y \to y^{1/\gamma_k - \delta} \overline{F}^{(k)}(y)$ is regularly varying with index $-\delta$. Then since

$$\sup_{y \geqslant 1} \left| y^{1/\gamma_k - \delta} \frac{\overline{F}^{(k)}(yt_n)}{\overline{F}^{(k)}(t_n)} - y^{-\delta} \right| \overset{n \to \infty}{\longrightarrow} 0$$

and, when $-\frac{1}{\gamma_k} + \delta < 0$, we have $\int_{\frac{u}{t_n}}^{+\infty} y^{-1/\gamma_k - 1 + \delta} dy = cst\, (u/t_n)^{-1/\gamma_k + \delta}$, this concludes the proof.

Lemma 10 *Recalling that H is a distribution function with infinite right endpoint, we have :*

(i) $\sup_{0 \leqslant x < Z^{(n)}} \overline{H}(x) / \overline{H}_n(x) = O_{\mathbb{P}}(1)$

(ii) for any $a < 1/2$,

$$\sqrt{n} \sup_{t \geqslant 0} \frac{|\overline{H}_n(t) - \overline{H}(t)|}{(\overline{H}(t))^a} = O_{\mathbb{P}}(1) \quad \textit{and} \quad \sqrt{n} \sup_{t \geqslant 0} \frac{|\overline{H}_n^{(0)}(t) - \overline{H}^{(0)}(t)|}{(\overline{H}^{(0)}(t))^a} = O_{\mathbb{P}}(1).$$

Proof : part (i) is well known (see for instance section 3 of chapter 10 of Shorack and Wellner (1986)), while the two statements in (ii) are proved by usual empirical processes techniques, showing that the family of functions $(f_t)_{t < \infty}$ defined in one case by $f_t(z) = \mathbb{I}_{z > t} / (\overline{H}(t))^a$, and in the other case by $f_t(\delta, z) = (1 - \delta) \mathbb{I}_{z > t} / (\overline{H}^{(0)}(t))^a$ are Donsker whenever $a < 1/2$ (using respective square integrable envelope functions $f^*(z) = 1/(\overline{H}(z))^a$ and $f^*(\delta, z) = (1 - \delta)/(\overline{H}^{(0)}(z))^a$, which bound from above the functions f_t uniformly in t) .

Lemma 11 *Under conditions (1) and (2), suppose that $\alpha \geqslant 0$ and $d \geqslant 1$ are real numbers. If $\gamma_k < \gamma_C$ and*

$$X_{i,n} = \frac{\sqrt{v_n}}{n^{1+d}} \frac{\phi_n(Z_i)}{\overline{G}(Z_i)(\overline{H}^{(0)}(Z_i))^{d+\alpha}} \mathbb{I}_{\xi_i = k},$$

then we have $\sum_{i=1}^n X_{i,n} \overset{\mathbb{P}}{\longrightarrow} 0$, as n tends to infinity, if α is 0 or sufficiently close to it.

Proof :

According to the LLN for triangular arrays, we need to prove the following three statements :

$$(i)\ \forall \epsilon > 0, \quad \sum_{i=1}^n \mathbb{P}(|X_{i,n}| > \epsilon) \overset{n \to \infty}{\longrightarrow} 0$$
$$(ii) \quad \sum_{i=1}^n \mathbb{E}((X_{i,n})^2 \mathbb{I}_{|X_{i,n}| \leqslant 1}) \overset{n \to \infty}{\longrightarrow} 0$$
$$(iii) \quad \sum_{i=1}^n \mathbb{E}(X_{i,n} \mathbb{I}_{|X_{i,n}| \leqslant 1}) \overset{n \to \infty}{\longrightarrow} 0$$

But, $X_{i,n}$ being positive, (iii) clearly implies (ii). We thus need to prove that (i) and (iii) hold.

Let us start with assertion (i). If $\epsilon > 0$ is given, then

$$X_{i,n} = \frac{v_n^{1/2}}{n^{1+d}} \frac{\log(Z_i/t_n)}{\overline{F}^{(k)}(t_n) \overline{G}(t_n) (\overline{H}^{(0)}(t_n))^{d+\alpha}} \mathbb{I}_{Z_i > t_n} \mathbb{I}_{\xi_i = k} \frac{\overline{G}(t_n)}{\overline{G}(Z_i)} \left(\frac{\overline{H}^{(0)}(t_n)}{\overline{H}^{(0)}(Z_i)} \right)^{d+\alpha}.$$

Now, put $a = \frac{1}{\gamma_C} + \frac{d+\alpha}{\gamma}$ (> 0); since, for a given $\epsilon' > 0$, there exists $c > 0$ such that $\forall x \geqslant 1$, $\log(x) \leqslant cx^{\epsilon'}$, and using Potter-bounds (44) for $\overline{G}^{-1}(\overline{H}^{(0)})^{-(d+\alpha)} \in RV_{-a}$, we can write (using the definition of v_n)

$$\{\, |X_{i,n}| > \epsilon \,\} \subset \left\{ v_n^{-1/2} n^{-d} \left(\overline{H}^{(0)}(t_n)\right)^{-(d+\alpha)} c(1+\epsilon') \left(\frac{Z_i}{t_n}\right)^{a+2\epsilon'} > \epsilon \right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\}$$
$$\subset \left\{\, Z_i > c(\epsilon,\epsilon')\, t_n w_n \,\right\} \cap \{\xi_i = k \text{ and } Z_i > t_n\}\,,$$

where $w_n = \left(v_n^{1/2} n^d \left(\overline{H}^{(0)}(t_n)\right)^{d+\alpha} \right)^{1/(a+2\epsilon')}$ and $c(\epsilon,\epsilon')$ is a constant depending on ϵ and ϵ' only. Consequently, if w_n tends to infinity,

$$\sum_{i=1}^{n} \mathbb{P}(|X_{i,n}| > \epsilon) \leqslant n\mathbb{E}(\mathbb{I}_{Z_i > c(\epsilon,\epsilon')\, t_n w_n} \mathbb{I}_{\xi_i = k}) = n \int_{c(\epsilon,\epsilon')\, t_n w_n}^{\infty} \overline{G}(x) dF^k(x)$$
$$\leqslant v_n \frac{(\overline{F}^{(k)}\overline{G})(c(\epsilon,\epsilon')\, t_n w_n)}{(\overline{F}^{(k)}\overline{G})(t_n)}$$
$$\leqslant cst\ v_n w_n^{-\beta},$$

where $\beta = \frac{1}{\gamma_C} + \frac{1}{\gamma_k} - \epsilon'$ and the last inequality is due to Potter-bounds (44) applied to $\overline{F}^{(k)}\overline{G} \in RV_{-\frac{1}{\gamma_C} - \frac{1}{\gamma_k}}$. Then, assertion (i) above will be true as soon as we prove that $w_n \to \infty$ and $v_n w_n^{-\beta} \to 0$, as $n \to \infty$.
Since $\overline{H}^{(0)}(t)$ is equivalent to a positive constant times $\overline{H}(t)$ when $t \to +\infty$, and $\overline{H}(t_n) \geqslant v_v/n$, then $w_n^{a+2\epsilon'} \geqslant cst\ (n^{-\eta} v_n)^r$, for $r = \frac{1}{2} + d + \alpha > 0$ and $\eta = \frac{\alpha}{r} \geqslant 0$. Assumption (2) finally yields that w_n tends to $+\infty$, since $0 \leqslant \eta \leqslant \eta_0$ for α sufficiently close to 0.

Now, proving that $v_n w_n^{-\beta}$ tends to 0 is equivalent to proving that $v_n^{-(a+2\epsilon')/\beta} v_n^{1/2} n^d \left(\overline{H}^{(0)}(t_n)\right)^{d+\alpha}$ tends to $+\infty$. The same arguments as in the previous paragraph yield that it is sufficient to prove that $v_n^A n^{-\alpha} = (n^{-\eta} v_n)^A$ tends to $+\infty$, for $A = -(a+2\epsilon')/\beta + 1/2 + d + \alpha$ and $\eta = \frac{\alpha}{A}$. This is a consequence of hypothesis (2), since $A > 0$ and $\alpha \leqslant \eta_0 A$, for α sufficiently close to 0. This ends the proof of (i).

Let us now start the proof of assertion (iii). If $\epsilon > 0$ is given, using Potter-Bounds (44) for $\overline{G}^{-1}(\overline{H}^{(0)})^{-(d+\alpha)}$ which belongs to RV_{-a}, and introducing $h(x) = \log(x) x^{a-\epsilon}$, we find that (for some positive constant c)

$$\mathbb{I}_{|X_{i,n}| \leqslant 1} \mathbb{I}_{Z_i > t_n} \leqslant \mathbb{I}_{h(Z_i/t_n) \leqslant c w_n} \mathbb{I}_{Z_i > t_n}$$

where we set $w_n = v_n^{1/2} n^d \left(\overline{H}^{(0)}(t_n)\right)^{d+\alpha}$. Hence, denoting by h^{-1} the inverse function of h,

$$\mathbb{I}_{|X_{i,n}| \leqslant 1} \mathbb{I}_{Z_i > t_n} \mathbb{I}_{\xi_i = k} \leqslant \mathbb{I}_{t_n < Z_i < t_n h^{-1}(c w_n)} \mathbb{I}_{\xi_i = k}.$$

Consequently, using once again Potter-Bounds (44) and bounding the log with a constant times a power of z/t_n, we get

$$n\mathbb{E}(X_{1,n} \mathbb{I}_{|X_{1,n}| \leqslant 1}) \leqslant \frac{v_n^{1/2}}{n^d} \int_{t_n}^{t_n h^{-1}(c w_n)} \frac{\log(z/t_n)}{\overline{F}^{(k)}(t_n) \overline{G}(z) (\overline{H}^{(0)}(z))^{d+\alpha}} dH^{(1,k)}(z)$$
$$\leqslant cst\ \frac{v_n}{w_n} \int_1^{h^{-1}(c w_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(s t_n)}{\overline{F}^{(k)}(t_n)},$$

where $b = \frac{d+\alpha}{\gamma}$ and $\epsilon' > 0$ is some given positive value (the inequality $\log(s) \leqslant cst\, s^{\epsilon'}$, $\forall s \geqslant 1$, was used). But, by integration by parts and (44) applied to $\overline{F}^{(k)}$, setting $h_n = h^{-1}(cw_n)$, we have

$$\frac{v_n}{w_n}\int_1^{h^{-1}(cw_n)} s^{b+2\epsilon'}\ \frac{dF^{(k)}(st_n)}{\overline{F}^{(k)}(t_n)} \leqslant cst\frac{v_n}{w_n}\left(1+h_n^{b-1/\gamma_k+3\epsilon'}\right).$$

Proceeding similarly as in the previous paragraphs, we find that $w_n/v_n \to \infty$ (and thus w_n and h_n as well) thanks to assumption (2), for α close to 0. We are thus left to prove that $(v_n/w_n)\times h_n^{b'}$ tends to 0, where $b' = b - 1/\gamma_k + 3\epsilon'$. If $b - 1/\gamma_k$ is negative, this is immediate. We thus suppose that $b - 1/\gamma_k \geqslant 0$ and, after some simple computations, we find out that $(v_n/w_n)h_n^{b'}$ tends to 0 if $v_n^{-a+\epsilon'} w_n^{a-b'-\epsilon'}$ tends to ∞, a property which holds true thanks to assumption (2), for α close to 0 (we omit the details).

Lemma 12 *Suppose that V_1 and W_2 are independent improper random variables of respective subdistribution functions $H^{(0)}$ and $H^{(1,k)}$, and Z_3 is independent of V_1 and W_2 and has distribution H. Consider h, $\underline{h}$, $\mathscr{H}$ and $\underline{\mathscr{H}}$ the functions defined in (31) and (40).*

(*i*) *For any $d \geqslant 1$, there exist some positive constants c and c' such that*

$$\mathbb{E}\left(\,|\mathscr{H}^d(V_1,W_2)|\,\right) \leqslant c\,\mathbb{E}\left(\,h^d(V_1,W_2)\,\right) \quad \text{and} \quad \mathbb{E}\left(\,|\underline{\mathscr{H}}^d(Z_3,V_1,W_2)|\,\right) \leqslant c'\,\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right).$$

(*ii*) *For any $d \in]1, 1+(1+2\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,h^d(V_1,W_2)\,\right) = O\left((\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{2(1-d)}\right).$$

In particular, if $\gamma_k < \gamma_C$, then $\mathbb{E}(h^{4/3}(V_1,W_2))$ is of the order of $(\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{-2/3}$ and $\mathbb{E}(h^d(V_1,W_2))$ is finite whenever d is (greater than but) sufficiently close to $4/3$.

(*iii*) *For any $d \in]1, 1+(1+3\gamma_k/\gamma_C)^{-1}[$, we have*

$$\mathbb{E}\left(\,\underline{h}^d(Z_3,V_1,W_2)\,\right) = O\left((\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{3(1-d)}\right).$$

In particular, if $\gamma_k < \gamma_C$, then $\mathbb{E}(\underline{h}^{6/5}(Z_3,V_1,W_2))$ is of the order of $(\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{-3/5}$ and $\mathbb{E}(\underline{h}^d(Z_3,V_1,W_2))$ is finite whenever d is (greater than but) sufficiently close to $6/5$.

(*iv*) *For any $d \in]1/2, (2\gamma_C^{-1}+\gamma_F^{-1}+\gamma_k^{-1})/(3\gamma_C^{-1}+2\gamma_F^{-1})[$, we have $\mathbb{E}\left(h^d(V_1,W_2)/\overline{H}^d(V_1)\right) = O\left((\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{2-3d}\right)$. In particular, if $\gamma_k < \gamma_C$ then taking δ (greater than but) sufficiently close to $4/5$ is permitted, otherwise it is $2/3$ instead of $4/5$.*

(*v*) *The integral $\theta_n = \iint h(v,w)dH^{(0)}(v)dH^{(1,k)}(w)$ is equivalent, as $n \to \infty$, to $\gamma_k(-\log \overline{G}(t_n))$.*

Proof :

(*i*) Let $d \geqslant 1$, and remind that h is a non-negative function. Using several times the inequality $|a+b|^d \leqslant 2^{d-1}(|a|^d + |b|^d)$, we can write

$$\mathbb{E}(|\mathscr{H}^d(V_1,W_2)|) \leqslant cst\left\{\mathbb{E}(h^d(V_1,W_2)) + \mathbb{E}[(h_{1\bullet}(V_1))^d] + \mathbb{E}[(h_{\bullet 1}(W_2))^d] + (\mathbb{E}(h(V_1,W_2)))^d\right\}.$$

But using the fact that the L^1 norm is bounded by the L^d norm whenever $d \geqslant 1$, we have $(\mathbb{E}(h(V_1,W_2)))^d \leqslant \mathbb{E}(h^d(V_1,W_2))$ and it is quite simple to prove (by independency of V_1 and W_2) that it is also the case of $\mathbb{E}[(h_{1\bullet}(V_1))^d] = \mathbb{E}[(\mathbb{E}(h(V_1,W_2)|V_1))^d] \leqslant \mathbb{E}[\mathbb{E}(h^d(V_1,W_2)|V_1)] = \mathbb{E}(h^d(V_1,W_2))$, as well as for $\mathbb{E}[(h_{\bullet 1}(W_2))^d]$. The inequality is thus proved. The other one (concerning $\underline{\mathscr{H}}$ and $\underline{h}$) is proved similarly.

(*ii*) Let $d > 1$. Since $h(v,\infty) = h(\infty,w) = 0$ ($\forall v, w$), we have

$$\begin{aligned}\mathbb{E}(h^d(V_1,W_2)) &= (\overline{F}^{(k)}(t_n))^{-d}\iint \log^d(w/t_n)(\overline{H}(v)\overline{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w)\\ &= (\overline{F}^{(k)}(t_n))^{1-d}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\int_0^w \frac{dG(v)}{\overline{G}(v)\overline{H}^{d-1}(v)}\right)\overline{G}^{1-d}(w)\frac{dF^{(k)}(w)}{\overline{F}^{(k)}(t_n)}\\ &= \frac{C_{d-1}(t_n)}{(\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{d-1}}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\frac{\overline{G}(t_n)}{\overline{G}(w)}\right)^{d-1}\frac{C_{d-1}(w)}{C_{d-1}(t_n)}\frac{dF^{(k)}(w)}{\overline{F}^{(k)}(t_n)}\end{aligned}$$

where the function C_{d-1} was defined in the statement of Lemma 8. This lemma and Lemma 7, applied with $\alpha = d$, $a = (d-1)/\gamma_C + (d-1)/\gamma$ and $b = -1/\gamma_k$ (the constraint specified on d certifies that $a + b < 0$), imply that the integral in the previous line converges to a constant. And Lemma 8 also implies that the ratio in front of this integral is equivalent, as $n \to \infty$, to a positive constant times $\left(\overline{H}(t_n)\overline{F}^{(k)}(t_n)\overline{G}(t_n)\right)^{1-d}$, which is itself lower than $\left(\overline{F}^{(k)}(t_n)\overline{G}(t_n)\right)^{2(1-d)}$, as desired.

(*iii*) Let $d > 1$. By definition of $\underline{h}$ in (31), and proceeding as in the previous item, $\mathbb{E}(\underline{h}^d(Z_3,V_1,W_2))$ equals

$$\begin{aligned}&(\overline{F}^{(k)}(t_n))^{-d}\iiint \log^d\left(\tfrac{w}{t_n}\right)(\overline{H}(v))^{-2d}(\overline{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}\mathbb{I}_{u>v}dH(u)dH^{(0)}(v)dH^{(1,k)}(w)\\ &= \frac{C_{2d-2}(t_n)}{(\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{d-1}}\int_{t_n}^{\infty}\log^d(w/t_n)\left(\frac{\overline{G}(t_n)}{\overline{G}(w)}\right)^{d-1}\frac{C_{2d-2}(w)}{C_{2d-2}(t_n)}\frac{dF^{(k)}(w)}{\overline{F}^{(k)}(t_n)},\end{aligned}$$

which is equivalent to $O\left((\overline{H}(t_n))^{2-2d}(\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{1-d}\right) = O\left((\overline{F}^{(k)}(t_n)\overline{G}(t_n))^{3(1-d)}\right)$ as soon as, thanks to Lemma 7, the sum $((d-1)/\gamma_C + (2d-2)/\gamma) - 1/\gamma_k$ is negative, which turns out to be true whenever $d < 1 + (2 + 3\gamma_k/\gamma_C)^{-1}$, as specified.

(*iv*) The proof is very similar to the previous ones, starting from

$$\mathbb{E}\left(h^d(V_1,W_2)/\overline{H}^d(V_1)\right) = (\overline{F}^{(k)}(t_n))^{-d}\iint \log^d(w/t_n)(\overline{H}(v))^{-2d}(\overline{G}(w))^{-d}\mathbb{I}_{w>t_n}\mathbb{I}_{w>v}dH^{(0)}(v)dH^{(1,k)}(w)$$

so we omit the details.

(*v*) Noting that $-\log\overline{G}$ is slowly varying at infinity null at 0, we have

$$\theta_n = \int_{t_n}^{\infty}\log(w/t_n)\left(\int_0^w dG(v)/\overline{G}(v)\right)\frac{dF^{(k)}(w)}{\overline{F}^{(k)}(t_n)} = (-\log\overline{G}(t_n))\left(-\int_1^{\infty}\log(u)\frac{-\log\overline{G}(ut_n)}{-\log\overline{G}(t_n)}\frac{d\overline{F}^{(k)}(ut_n)}{\overline{F}^{(k)}(t_n)}\right)$$

which can be dealt with using part (*ii*) of Lemma 7 with $\alpha = 1$, $a = 0$ and $b = -1/\gamma_k$: the obtained constant is indeed equal to γ_k.

Lemma 13 *In this Lemma, various notations defined in sections 6.2.2 to 6.2.4 are used.*

(*i*) *The variables* $\mathscr{H}_I^{**}$ *for* $I \in \{(i,j)\,;\, 1 \leqslant i < j \leqslant n\}$ *are centred and uncorrelated . This is also true for the variables* $\underline{\mathscr{H}}_I^{**}$ *for* $I \in \{(i,j,l)\,;\, 1 \leqslant i < j < l \leqslant n\}$.

(*ii*) *We have* $\mathbb{E}\left[(\mathscr{H}^{**}(V_1,W_2))^2\right] \leqslant 48\mathbb{E}[\mathscr{H}_1^2\mathbb{I}_{|\mathscr{H}_1|\leqslant M_n}]$.

(*iii*) *We have* $\mathbb{E}\left(\,|\mathscr{H}_1 - \mathscr{H}^*(V_1,W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\,\right) \leqslant 4\mathbb{E}\left(|\mathscr{H}_1|\mathbb{I}_{|\mathscr{H}_1|>M_n}\right)$

Proof :

(i) Let us consider the first situation, where $\mathscr{I} = \{(i,j)\,;\, 1 \leqslant i < j \leqslant n\}$. First, if $I = (i,j) \in \mathscr{I}$, then $\mathbb{E}(\mathscr{H}_I^{**}) = 0 - \mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) - \mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j))$; but, by definition of $\mathscr{H}_{1\bullet}^*$ and independency of V_i and W_j, we have $\mathbb{E}(\mathscr{H}_{1\bullet}^*(V_i)) = \mathbb{E}(\mathscr{H}^*(V_i, W_j)) = 0$, and $\mathbb{E}(\mathscr{H}_{\bullet 1}^*(W_j)) = 0$ is obtained similarly, so we proved that $\mathbb{E}(\mathscr{H}_I^{**}) = 0$. Note that we can prove (with similar arguments) that $\mathscr{H}_{1\bullet}^{**}(v) = \mathscr{H}_{\bullet 1}^{**}(w) = 0$ for every v, w in $[0,\infty]$, a property which is repeatedly used below . Let us now deal with the non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$, by considering the various cases where $I \neq I'$ with $I = (i,j)$ and $I' = (k,l)$ are in $\mathscr{I}$.

If all four indices i,j,k,l are distinct, then non-correlation of $\mathscr{H}_I^{**}$ and $\mathscr{H}_{I'}^{**}$ is immediate by mutual independence of the variables $Z_1, \ldots, Z_n$.
If $i = k$ but $j \neq l$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i))$ where $\psi(v) = \mathbb{E}(\mathscr{H}^{**}(v, W_j)\mathscr{H}^{**}(v, W_l)) = (\mathscr{H}_{1\bullet}^{**}(v))^2 = 0$, by independence of V_i with (W_j, W_l), and of W_j and W_l.
The case $i \neq k$ and $j = l$ is similar using $\mathscr{H}_{\bullet 1}^{**}(\cdot) \equiv 0$.
If $i = l$ but $j \neq k$, then $\mathbb{E}(\mathscr{H}_I^{**}\mathscr{H}_{I'}^{**}) = \mathbb{E}(\psi(V_i, W_i))$ where $\psi(v,w) = \mathbb{E}(\mathscr{H}^{**}(v, W_j)\mathscr{H}^{**}(V_k, w)) = \mathscr{H}_{1\bullet}^{**}(v)\mathscr{H}_{\bullet 1}^{**}(w) = 0 \times 0 = 0$; the case $j = k$ and $i \neq l$ is treated similarly.
Note that the case $i = l$ and $j = k$ (*i.e.* $\mathscr{H}_I^{**} = \mathscr{H}^{**}(V_i, W_j)$, $\mathscr{H}_{I'} = \mathscr{H}^{**}(V_j, W_i)$) is not permitted (it would lead to dependency) since we cannot have simultaneously $i < j$ and $j < i$; this is the reason why, in the beginning of section 6.2.3, we restricted the study of the sum $\mathscr{U}_n$ to that of the sum S_N having terms $\mathscr{H}(V_i, W_j)$ satisfying $i < j$.

The second situation, for $\underline{\mathscr{H}}_I^{**}$ and $\underline{\mathscr{H}}_{I'}^{**}$ with $I \neq I'$ in $\mathscr{I} = \{I = (i,j,l)\,;\, 1 \leqslant i < j < l \leqslant n\}$, is a bit more tedious (with more cases to detail) but very similar, so we omit its proof.

(ii) We start by the trivial bound

$$\mathbb{E}[(\mathscr{H}^{**}(V_1, W_2))^2] \leqslant 4\left\{\mathbb{E}[(\mathscr{H}^*(V_1, W_2))^2] + \mathbb{E}[(\mathscr{H}_{1\bullet}^*(V_1))^2] + \mathbb{E}[(\mathscr{H}_{\bullet 1}^*(W_2))^2]\right\}.$$

Noting $\mathscr{H}_1^- = \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| \leqslant M_n}$, we can write, on one hand, by definition of $\mathscr{H}^*$, $\mathbb{E}[(\mathscr{H}^*(V_1, W_2))^2] \leqslant 2\left\{\mathbb{E}[(\mathscr{H}_1^-)^2] + (\mathbb{E}[\mathscr{H}_1^-])^2\right\} \leqslant 4\mathbb{E}[(\mathscr{H}_1^-)^2]$. On the other hand, if W is independent of V_1, we have $\mathbb{E}[(\mathscr{H}_{1\bullet}^*(V_1))^2] = \mathbb{E}[(\mathbb{E}[\mathscr{H}^*(V_1, W)|V_1])^2] \leqslant \mathbb{E}[\mathbb{E}[(\mathscr{H}^*(V_1, W))^2|V_1]] = \mathbb{E}[(\mathscr{H}^*(V_1, W_2))^2]$, which is the same term as the first one, and is thus lower than $4\mathbb{E}[(\mathscr{H}_1^-)^2]$. The same is true of $\mathbb{E}[(\mathscr{H}_{\bullet 1}^*(W_2))^2]$, so the desired inequality is proved.

(iii) First recall that $\mathscr{H}_1$ denotes $\mathscr{H}(V_1, W_2)$. Now, since $\mathscr{H}_1$ is centred and we trivially have $\mathscr{H}_1 = \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| \leqslant M_n} + \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| > M_n}$, noting $\mathscr{H}_1^+ = \mathscr{H}_1 \mathbb{I}_{|\mathscr{H}_1| > M_n}$ yields

$$\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) = \mathscr{H}_1^+ - \mathbb{E}(\mathscr{H}_1^+).$$

Secondly, using the fact that $\mathscr{H}_{1\bullet}(\cdot) \equiv 0$ (simple to prove), we can write

$$\mathscr{H}_{1\bullet}^*(v) = \mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| \leqslant M_n}) - \mathbb{E}(\mathscr{H}_1 \mathbb{I}_{|\mathscr{H}1| \leqslant M_n}) = -\mathscr{H}_{1\bullet}^+(v) + \mathbb{E}(\mathscr{H}_1^+),$$

where $\mathscr{H}_{1\bullet}^+(v)$ denotes $\mathbb{E}(\mathscr{H}(v, W)\mathbb{I}_{|\mathscr{H}(v,W)| > M_n})$ and satisfies $\mathbb{E}(\mathscr{H}_{1\bullet}^+(V_1)) = \mathbb{E}(\mathscr{H}_1^+)$, and similarly

$$\mathscr{H}_{\bullet 1}^*(w) = -\mathscr{H}_{\bullet 1}^+(w) + \mathbb{E}(\mathscr{H}_1^+)$$

with $\mathscr{H}_{\bullet 1}^+(w) = \mathbb{E}(\mathscr{H}(V, w)\mathbb{I}_{|\mathscr{H}(V,w)| > M_n})$ and $\mathbb{E}(\mathscr{H}_{\bullet 1}^+(W_2)) = \mathbb{E}(\mathscr{H}_1^+)$. Summing these three terms finally leads to

$$\mathbb{E}\left(|\mathscr{H}_1 - \mathscr{H}^*(V_1, W_2) + \mathscr{H}_{1\bullet}^*(V_1) + \mathscr{H}_{\bullet 1}^*(W_2)|\right) = \mathbb{E}\left(|\mathscr{H}_1^+ - \mathscr{H}_{1\bullet}^+(V_1) - \mathscr{H}_{\bullet 1}^+(W_2) + \mathbb{E}(\mathscr{H}_1^+)|\right)$$

which is lower than $4\mathbb{E}(|\mathscr{H}_1^+|)$, as announced.