Collision of two solitons for the quartic gKdV eq.

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Introduction

We call **soliton** a solution $u(t,x) = Q_c(x - ct)$ $(c > 0)$ of

$$(gKdV) \quad \partial_t u + \partial_x(\partial_x^2 u + f(u)) = 0 \quad t, x \in \mathbb{R}$$

General questions about the collision of two solitons

Let $u(t)$ be a solution such that

$$u(t) \sim Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t) \quad \text{as } t \to -\infty,$$

where $Q_{c_1}(x - c_1 t)$, $Q_{c_2}(x - c_2 t)$ are two solitons $(0 < c_2 < c_1)$

- What is the behavior of $u(t)$ during and after the collision?
- Do the solitons survive the collision at the principal order?
- If yes, are their speeds (size) and trajectories (shift) modified?
- Is the collision **elastic** or **inelastic**?

Previous results concerning the collision of solitons

- Integrable cases ($f(u) = u^2,\ u^3,\ u^2 + au^3$)

  There exist explicit multi-solitons solutions describing for all time the interaction of solitons.

  The collision is elastic

  [Fermi, Pasta and Ulam], [Zabusky and Kruskal], [Lax], [GGKM], [Hirota], [Miura et al.], etc.

- Numerical simulations and experiments

  For several non integrable models, the collision seems inelastic but almost elastic (small dispersive trail)

  [Eilbeck and McGuire], [Bona et al.] (BBM), [Shih] (gKdV), [Craig et al.] and references therein (Euler, KdV, experiments) etc.

For non integrable models, there is no general rigorous argument saying that solitons survive a collision.

We report on recent works describing the collision of two solitons for the quartic (non integrable) gKdV

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0 \quad t, x \in \mathbb{R}$$

in two different regimes:

- Very different sizes: $0 < c_2 \ll c_1$
  [Martel, Merle: Annals of Math. (2011)] (arXiv 2007)
- Almost equal sizes: $c_1 \sim c_2$.
  [Martel, Merle: Inventiones Math. (2011)] (arXiv 2009)

Asymptotic results in the energy space - One soliton

Solitons are $u(t,x) = Q_c(x - ct - x_0)$, $c > 0$,

$$Q_c(x) = c^{\frac{1}{3}} Q(\sqrt{c}x)$$

$$Q'' + Q^4 = Q, \quad Q(x) = \left(\tfrac{5}{2}\right)^{\frac{1}{3}} \cosh^{-\frac{2}{3}}\left(\tfrac{3}{2}x\right)$$

Orbital stability by conservation laws [Weinstein, 1986]

$$\|u(0) - Q_c\|_{H^1} \textit{ is small } \Rightarrow \forall t, \ \|u(t) - Q_c(. - \rho(t))\|_{H^1} \textit{ is small}$$

Asymptotic stability [Martel and Merle, 2001-2007]

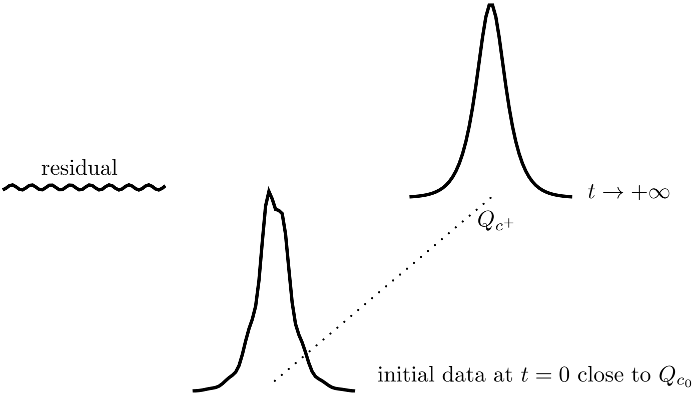
Under the same assumptions, there exists $c^+ \sim c$ such that

$$u(t) - Q_{c^+}(. - y(t)) \to 0 \quad \textit{for } x > \tfrac{c}{10} t$$

$$y'(t) \to c^+ \quad \text{as } t \to +\infty$$

Earlier result by [Pego and Weinstein, 1994].

Schematic representation of asymptotic stability: a solution initially close to a soliton simplifies to a soliton plus a small residue.



Asymptotic results in the energy space - Multi–soliton

Existence of asymptotic multi-solitons [Martel, 2005]

Let $c_1^- \neq c_2^-$. There exists a **unique** *solution in H^1 such that*

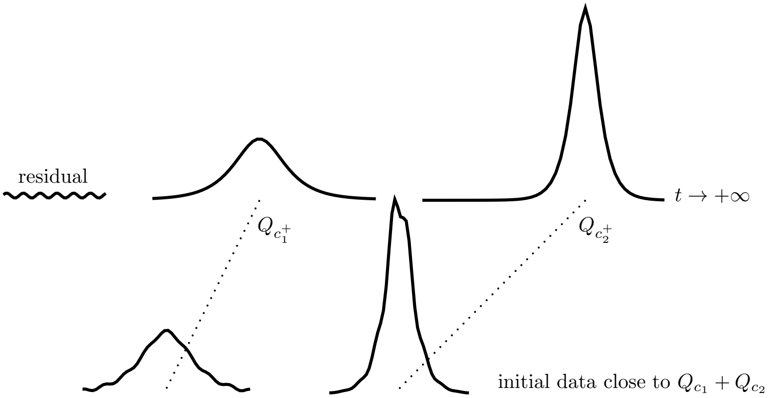
$$\lim_{t\to -\infty} U(t) - (Q_{c_1^-}(. - c_1^- t) + Q_{c_2^-}(. - c_2^- t)) = 0$$

Stability of multi-solitons in H^1 [Martel, Merle and Tsai, 2002]

For T large:

$$\left\| u(T) - (Q_{c_1}(. - c_1 T) + Q_{c_2}(. - c_2 T)) \right\|_{H^1} \text{ is small} \quad \Rightarrow$$
$$\forall t > T,\ \left\| u(t) - (Q_{c_1}(. - y_1(t)) + Q_{c_2}(. - y_2(t))) \right\|_{H^1} \text{ is small}$$

Schematic representation of stability of two decoupled solitons



I. Interaction of two solitons with very different sizes

Assume $0 < c_2^- \ll c_1^-$. Let $U(t)$ be such that

$$\lim_{t\to-\infty} U(t) - (Q_{c_1^-}(. - c_1^- t) + Q_{c_2^-}(. - c_2^- t)) = 0$$

THM 1 [Martel and Merle, 2007]

There exist $c_1^+ \underset{c_2^-\sim 0}{\sim} c_1^-$, $c_2^+ \underset{c_2^-\sim 0}{\sim} c_2^-$ *such that*

$$U(t,x) = Q_{c_1^+}(x - y_1(t)) + Q_{c_2^+}(x - y_2(t)) + w^+(t,x),$$

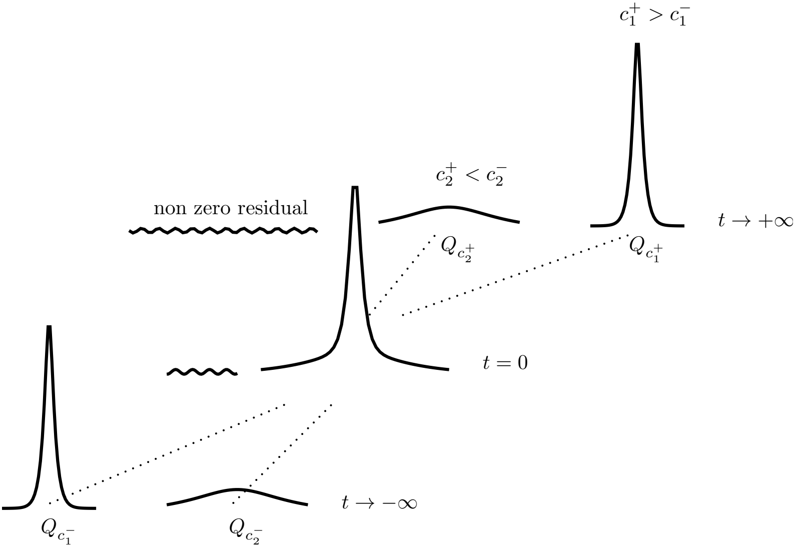
- **Stability:**

$$\sup_{t\in\mathbb{R}} \|w^+(t)\|_{H^1} \leq K(c_2^-)^{\frac{1}{3}}$$

- **Inelasticity:**

$$\frac{c_1^+}{c_1^-} - 1 \gtrsim (c_2^-)^{\frac{17}{6}}, \quad \frac{c_2^+}{c_2^-} - 1 \lesssim -(c_2^-)^{\frac{8}{3}}, \quad \lim_{t\to+\infty} \|w^+(t)\|_{H^1} \neq 0.$$

Inelastic interaction of a large soliton and a small soliton



Comments on THM 1

- **The two solitons are preserved through the collision**

$$\sup_t \|w^+(t)\|_{H^1} \leq K(c_2^-)^{\frac{1}{3}} \quad \text{and} \quad \|Q_{c_2^-}\|_{H^1} \sim K(c_2^-)^{\frac{1}{12}}$$

- The collision is almost elastic

$$\|w^+(t)\|_{H^1} \ll \|Q_{c_2^-}\|_{H^1}$$

- **Speed change and dispersive residue**

$$\|w^+(t)\|_{H^1} \not\to 0 \text{ as } t \to +\infty \textbf{ and } c_1^+ > c_1^- \text{ and } c_2^+ < c_2^-$$

- **Nonexistence of a pure** 2**-soliton solution in this regime**

- THM 1 is the first rigorous result describing an inelastic collision between two nonlinear objects
- Extension by Claudio Muñoz : the same result holds for any $f(u)$ instead of u^4, for small solitons ($0 < c_2^- \ll c_1^- \ll 1$), provided the Taylor expansion of f near zero is

$$f(u) = u^p + au^q + u^q o(1) \quad \text{for } p = 2, 3 \text{ or } 4,\ q > p,$$

in the non integrable case. Therefore, **all collisions are elastic if and only if the equation is integrable.**

II. Interaction of two solitons with almost equal sizes

Now, we assume

$$0 < \mu_0 = \frac{c_2^- - c_1^-}{c_1^- + c_2^-} \ll 1.$$

• **In the integrable case,** let $U_{c_1^-,c_2^-}$ be a KdV 2–soliton. Using explicit formulas, [LeVeque, 1987] proved

$$\sup_{t,x} \left| U_{c_1^-,c_2^-}(t,x) - Q_{c_1(t)}(x - y_1(t)) - Q_{c_2(t)}(x - y_2(t)) \right| \leq C\mu_0^2.$$

Moreover,

$$\min_t (y_1(t) - y_2(t)) = 2|\ln \mu_0| + O(1).$$

Two solitons with almost equal speeds do not cross and remain at a large distance for all time.

• **In the quartic (non integrable) case, there are no explicit 2-solitons, but $\mu_0 \ll 1$ allows use of perturbation theory.**

THM [Mizumachi, 2003]
Assume

$$u(0) \sim Q(x) + Q(x + Y_0) \quad \textit{where } Y_0 > 0 \textit{ is large}$$

Then, for some $c_1^+ > c_2^+$ close to 1, for large time,

$$u(t,x) = Q_{c_1^+}(x - c_1^+ t - y_1^+) + Q_{c_2^+}(x - c_2^+ t - y_2^+) + w(t,x),$$

where w is small.

Using time reversibility ($t \to -t$, $x \to -x$), Mizumachi's result constructs a 2-**soliton like solution for all time:**

At $+\infty$,

$$u(t,x) = Q_{c_1^+}(x - c_1^+ t - y_1^+) + Q_{c_2^+}(x - c_2^+ t - y_2^+) + w(t,x),$$

and at $-\infty$,

$$u(t,x) = Q_{c_1^+}(x - c_1^+ t - y_1^-) + Q_{c_2^+}(x - c_2^+ t - y_2^-) + w(t,x).$$

For all time, the two solitons are separated at least by $\frac{1}{2} Y_0$.

Our result improves on Mizumachi's ansatz to give more information on such solutions.

The principal question concerns the **elastic** or **inelastic** character of the collision.

Assume $c_1^- \sim c_2^- \sim 1$.
We change variables, so that the problem is to study solutions of

$$U_t + (U_{xx} - U + U^4)_x = 0, \quad t, x \in \mathbb{R}.$$

behaving as a sum $Q_{1+\mu_1(t)}(x - y_1(t)) + Q_{1+\mu_2(t)}(x - y_2(t))$ for μ_1, μ_2 small.

Assume $0 < \mu_0 \ll 1$. Let $U(t)$ be such that

$$\lim_{t \to -\infty} U(t) - Q_{1-\mu_0}(. + \mu_0 t) - Q_{1+\mu_0}(. - \mu_0 t) = 0,$$

THM 2 [Martel and Merle, 2010]
There exist $\mu_1(t)$, $\mu_2(t)$, $y_1(t)$, $y_2(t)$ such that

$$U(t,x) = Q_{1+\mu_1(t)}(x - y_1(t)) + Q_{1+\mu_2(t)}(x - y_2(t)) + w(t,x)$$

- **Sharp stability:**

$$\sup_{t \in \mathbb{R}} \|w(t)\|_{H^1} \le \mu_0^{2^-},$$

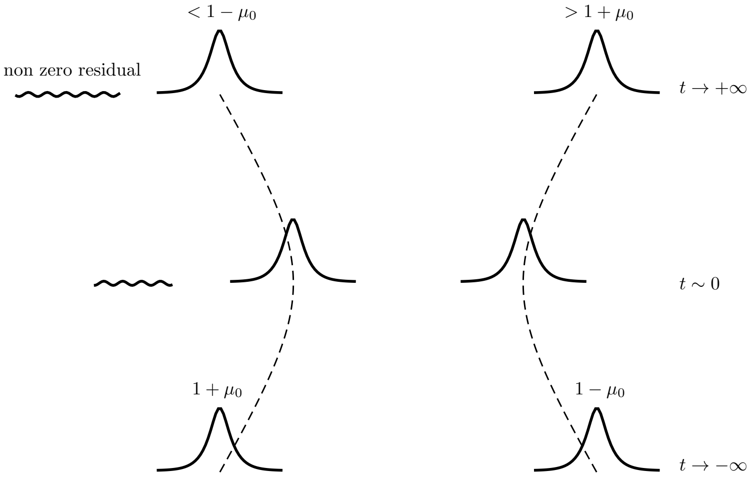
$$\min_t (y_1(t) - y_2(t)) = 2|\ln \mu_0| + O(1)$$

$$\lim_{t \to +\infty} \mu_1(t) = \mu_1^+ \sim \mu_0, \qquad \lim_{t \to +\infty} \mu_2(t) = \mu_2^+ \sim -\mu_0$$

- **Inelasticity:** *for $K > 0$,*

$$\mu_1^+ - \mu_0 \gtrsim \mu_0^5, \quad \mu_2^+ + \mu_0 \lesssim -\mu_0^5, \quad \liminf_{t \to +\infty} \|w(t)\|_{H^1} \neq 0$$

Interaction of two solitons of almost equal size for quartic gKdV



Sketch of the proof of THM 2

(1) Main step: construction of an approximate solution

We use separation of variables, solitons spatial decoupling and the special form of the nonlinear interactions of the two solitons

(2) Stability proof

We use refinements of stability and asymptotic stability techniques. Such arguments are needed after the collision but also during the collision to justify the approximate solution on a large time interval $(T \gg |\ln \mu_0|)$.

(3) Inelasticity

Qualitative version, using a contradiction argument

(1) Main step: construction of an approximate solution

We look for an approximate solution of $V_t + (V_{xx} - V + V^4)_x = 0$,

$$V_{\mu_1(t),\mu_2(t),y_1(t),y_2(t)}(x) = R_1(t,x) + R_2(t,x) + w(t,x),$$

where

$$R_j(t,x) = Q_{1+\mu_j(t)}(x - y_j(t)).$$

Recall

$$Q''_{1+\mu_j} + Q^4_{1+\mu_j} = (1+\mu_j)Q_{1+\mu_j},$$

$$\left(\frac{d}{d\mu}Q_{1+\mu}\right)_{|\mu=\mu_j} = \frac{1}{1+\mu_j}\left(\frac{1}{3}Q_{1+\mu_j} + \frac{1}{2}xQ'_{1+\mu_j}\right) = \Lambda Q_{1+\mu_j},$$

and set $\Lambda R_j(t,x) = \Lambda Q_{1+\mu_j}(x - y_j(t))$, Then,

$$V_t + (V_{xx} - V + V^4)_x = E + F + G(w) + H(w).$$

$$E = \sum_{j=1,2} \dot{\mu}_j \Lambda R_j + \sum_{j=1,2} (\mu_j - \dot{y}_j)(R_j)_x,$$

$$F = \left((R_1 + R_2)^4 - R_1^4 - R_2^4\right)_x,$$

$$G(w) = \left(w_{xx} - w + 4\left(R_1^3 + R_2^3\right) w\right)_x,$$

$$H(w) = w_t + \Big((R_1 + R_2 + w)^4 - \left((R_1 + R_2)^4 + 4\left(R_1^3 + R_2^3\right) w\right)\Big)_x$$

E: time derivative of $R_1(t)$ and $R_2(t)$

F: nonlinear interaction terms between R_1 and R_2

G: linear terms of order 1 in w

H: higher order terms

Using $Q(x) \underset{x\to\infty}{\sim} (10)^{1/3} e^{-|x|}$,

$$\begin{aligned} F &= \left(4R_2R_1^3 + 6R_1^2R_2^2 + 4R_1R_2^3\right)_x \\ &= 4(10)^{1/3}\left(e^{-x+y_2}Q^3(x-y_1) + 4(10)^{1/3}e^{x-y_1}Q^3(x-y_2)\right)_x + \dots \\ &= 4(10)^{1/3}e^{-y}\left(e^{-x+y_1}Q^3(x-y_1) + e^{x-y_2}Q^3(x-y_2)\right)_x + \dots \end{aligned}$$

where $y = y_1 - y_2$ is the distance between the two solitons.

Three independent variables : $x_1 = x - y_1$, $x_2 = x - y_2$ and e^{-y}.

Let

$$w_A(t,x) = e^{-y(t)}\left(A_1(x - y_1(t)) + A_2(x - y_2(t))\right).$$

For a suitable choice of $A_1(x)$, $A_2(x)$, a and α, we find

$$F_A + G(w_A) + \alpha e^{-y}(\Lambda R_1 - \Lambda R_2) + ae^{-y}((R_1)_x + (R_2)_x) = O(e^{-\frac{3}{2}y})$$

The values of a and α are unique to solve the problem

Returning to the equation of V and the expression of E, we get

$$\dot{\mu}_1 \sim \alpha e^{-y}, \qquad \dot{\mu}_2 \sim -\alpha e^{y},$$
$$\dot{y}_1 \sim \mu_1 - a e^{-y}, \qquad \dot{y}_2 \sim \mu_2 - a e^{-y}.$$

In particular, $\ddot{y} \sim 2\alpha e^{-y}$

For the proof of THM 2, we need to compute an approximate solution V up to order $e^{-2y(t)}$. Then, the system becomes

$$\begin{aligned}
\dot{\mu}_1 &\sim \alpha\, e^{-y} + \beta\, \mu_1\, y\, e^{-y}, \\
\dot{\mu}_2 &\sim -\alpha\, e^{-y} - \beta\, \mu_2\, y\, e^{-y}, \\
\dot{y}_1 &\sim \mu_1 - a\, e^{-y} - b_1\, \mu_1\, y\, e^{-y}, \\
\dot{y}_2 &\sim \mu_2 - a\, e^{-y} - b_2\, \mu_2\, y\, e^{-y}.
\end{aligned}$$

(2) Stability proof

Write the solution $U(t)$ as

$$U(t,x) = V_{\mu_1(t),\mu_2(t),y_1(t),y_2(t)}(x) + \varepsilon(t,x),$$

where $\varepsilon(t)$ is a rest term.

The control of the rest term $\varepsilon(t)$ uses variants of techniques used for large time stability and asymptotic stability of solitons and multi-solitons. Note that the solitons are decoupled for all time since $y(t) = y_1(t) - y_2(t)$ is large as μ_0 is small.

We obtain $\|\varepsilon(t)\|_{H^1} \leq \mu_0^{\frac{5}{2}^-}$ for all t.

(3) Inelasticity

Assume $U(t)$ is a pure 2-soliton solution.

A contradiction follows from:

- On the one hand, by uniqueness properties, $U(t)$ satisfies $U(t,x) = U(-t,-x)$ up to translation in space and time. Thus, $\mu_j(t)$, $y_j(t)$ which are related to $U(t)$ also have symmetry properties under the transformation $t \to -t$.

- On the other hand, the dynamical system satisfied by $\mu_j(t)$, $y_j(t)$ is not symmetric by the transformation $x \to -x$, $t \to -t$ at order $e^{-\frac{3}{2}y(t)}$. Indeed, from the algebra in the quartic case, we have $b_1 \neq b_2$. (In the integrable case, $f(u) = u^2$, we have checked $b_1 = b_2$.)