Eigenvalues of a nonlinear ground state in the Thomas-Fermi approximation.

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(GP) $$iu_t + \varepsilon^2 \Delta u + (1 - |x|^2)u - |u|^2 u = 0,$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, $d = 1, 2, 3$ and ε is a small parameter.

Existence of the ground state η_ε

Theorem (Ignat-Millot, 2006). *For ε sufficiently small, there exists a positive minimizer η_ε of the Gross–Pitaevskii energy*

$$E_\varepsilon(u) = \int_{\mathbb{R}} \left(\varepsilon^2 |\nabla u|^2 + (|x|^2 - 1)|u|^2 + \frac{1}{2}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}^d) : \ |x|u \in L^2(\mathbb{R}^d) \right\}.$$

η_ε is a stationnary solution of (GP):

(SGP) $$\varepsilon^2 \Delta \eta_\varepsilon + (1 - |x|^2)\eta_\varepsilon - \eta_\varepsilon^3 = 0.$$

Convergence of η_ε

As $\varepsilon \to 0$, $\eta_\varepsilon(x)$ converges to

$$\eta_0(x) = \begin{cases} \sqrt{1-|x|^2} & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

More precisely,

$$\begin{cases} (1 - C\varepsilon^{1/3}) \leqslant \frac{\eta_\varepsilon(x)}{(1-x^2)^{1/2}} \leqslant 1 & \text{for } \ |x| \leqslant 1 - \varepsilon^{2/3} \\ 0 \leqslant \eta_\varepsilon(x) \leqslant C\varepsilon^{1/3} \exp\left(\frac{1-x^2}{4\varepsilon^{2/3}}\right) & \text{for } \ |x| \geqslant 1 - \varepsilon^{2/3} \end{cases}$$

(Aftalion-Alama-Bronsard, 2005; Ignat-Millot, 2006)

Goal

Understand better the behaviour of η_ε as $\varepsilon \to 0$.

Motivation: behaviour of the eigenvalues of the linearized operator of (GP) at η_ε as $\varepsilon \to 0$, for $d = 1$:

$$u = \eta_\varepsilon + v + iw + \mathcal{O}(\|v\|^2 + \|w\|^2)$$

$$\partial_t \begin{pmatrix} v \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \varepsilon^2 L_-^\varepsilon \\ -\varepsilon^2 L_+^\varepsilon & 0 \end{pmatrix}}_{\mathcal{L}_\varepsilon} \begin{pmatrix} v \\ w \end{pmatrix},$$

where $L_-^\varepsilon = -\partial_x^2 + \frac{x^2-1+\eta_\varepsilon^2}{\varepsilon^2}$, $L_+^\varepsilon = -\partial_x^2 + \frac{x^2-1+3\eta_\varepsilon^2}{\varepsilon^2}$.

λ is an eigenvalue of $\mathcal{L}_\varepsilon$ iff $\gamma = -\lambda^2/\varepsilon^2$ is an eigenvalue of

($\mathcal{L}$) $$\varepsilon^2 L_+^\varepsilon L_-^\varepsilon w = \gamma w.$$

Formal convergence in a simplified problem

In the original operators $L_{\pm}^{\varepsilon}$ we replace η_{ε}^2 by η_0^2. We obtain new operators $L_{\pm}^{\varepsilon}$:

$$L_{-}^{\varepsilon} = -\partial_x^2 + \frac{x^2-1+\eta_0{}^2}{\varepsilon^2}, \qquad L_{+}^{\varepsilon} = -\partial_x^2 + \frac{x^2-1+3\eta_0{}^2}{\varepsilon^2}.$$

Then, formally,

$$\begin{aligned} \varepsilon^2 L_{+}^{\varepsilon} L_{-}^{\varepsilon} &= \left(-\varepsilon^2\partial_x^2 + x^2 - 1 + 3\eta_0^2\right)\left(-\partial_x^2 + \frac{x^2-1+\eta_0^2}{\varepsilon^2}\right) \\ &\to \left(x^2 - 1 + 3\eta_0^2\right)\left(-\partial_x^2 + \infty\cdot\mathbf{1}_{\{|x|>1\}}\right). \end{aligned}$$

Theorem. *The eigenvalues of $(\mathcal{L})$ converge as $\varepsilon \to 0$ to the eigenvalues $\gamma_n = 2n(n+1)$, $n \geqslant 1$ of*

$$2(1-x^2)(-\partial_x^2)w = \gamma w, \qquad x \in (-1,1)$$

with Dirichlet boundary conditions $w(\pm 1) = 0$.

Formal convergence in the original problem

With the original operators $L_{\pm}^{\varepsilon}$:

$$L_{-}^{\varepsilon} = -\partial_x^2 + \tfrac{x^2-1+\eta_\varepsilon{}^2}{\varepsilon^2}, \qquad L_{+}^{\varepsilon} = -\partial_x^2 + \tfrac{x^2-1+3\eta_\varepsilon{}^2}{\varepsilon^2},$$

we have formally from the expansion of η_ε

$$\begin{aligned}
\varepsilon^2 L_{+}^{\varepsilon} L_{-}^{\varepsilon} &= \left(-\varepsilon^2\partial_x^2 + x^2 - 1 + 3\eta_\varepsilon{}^2\right)\left(-\partial_x^2 + \frac{x^2-1+\eta_\varepsilon{}^2}{\varepsilon^2}\right) \\
&\to \left(x^2 - 1 + 3\eta_0^2\right)\left(-\partial_x^2 - \frac{1}{(1-x^2)^2}\mathbf{1}_{\{|x|<1\}} + \infty\cdot\mathbf{1}_{\{|x|>1\}}\right).
\end{aligned}$$

Conjecture. *The eigenvalues of $(\mathcal{L})$ converge as $\varepsilon \to 0$ to the eigenvalues $\gamma_n = 2n(n+1)$, $n \geqslant 0$ of*

$$2(1-x^2)\left(-\partial_x^2 - \tfrac{1}{(1-x^2)^2}\right) w = \gamma w, \qquad x \in (-1,1)$$

with Dirichlet boundary conditions $w(\pm 1) = 0$.

A change of variable

Write

$$\eta_\varepsilon(x) = \varepsilon^{1/3}\nu_\varepsilon(y), \quad y = \frac{1-|x|^2}{\varepsilon^{2/3}}.$$

Then, ν_ε solves

(Eqν_ε)

$$4(1-\varepsilon^{2/3}y)\nu_\varepsilon'' - 2\varepsilon^{2/3}d\nu_\varepsilon' + y\nu_\varepsilon - \nu_\varepsilon^3 = 0, \quad -\infty < y \leqslant \varepsilon^{-2/3}.$$

In the limit $\varepsilon \to 0$, we formally obtain the Painlevé II equation,

(PII) $$4\nu_0'' + y\nu_0 - \nu_0^3 = 0, \quad y \in \mathbb{R}.$$

Moreover, as $\varepsilon \to 0$,

$$\varepsilon^{1/3}\nu_\varepsilon\left(\tfrac{1-|x|^2}{\varepsilon^{2/3}}\right) \to \begin{cases} \sqrt{1-|x|^2}, & |x|<1 \quad \rightsquigarrow \quad \nu_0(y) \underset{y\to+\infty}{\sim} \sqrt{y}, \\ 0, & |x|>1 \quad \rightsquigarrow \quad \nu_0(y) \underset{y\to-\infty}{\to} 0. \end{cases}$$

The Hastings McLeod solution of (PII)

Proposition. (Hastings-McLeod, 1980 ; Fokas et al., 2006)
(PII) has a unique solution $\nu_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\nu_0(y) \sim y^{1/2} \quad \text{as} \quad y \to +\infty \quad \text{and} \quad \nu_0(y) \to 0 \quad \text{as} \quad y \to -\infty.$$

Moreover, ν_0 is strictly increasing on $\mathbb{R}$. As $y \to -\infty$ we have

$$\nu_0(y) = \frac{1}{2\sqrt{\pi}}(-2y)^{-1/4} e^{-\frac{2}{3}(-2y)^{3/2}} \left(1 + \mathcal{O}(|y|^{-3/4})\right) \underset{y \to -\infty}{\approx} 0,$$

whereas as $y \to +\infty$,

$$\nu_0(y) \underset{y \to +\infty}{\approx} y^{1/2} \sum_{n=0}^{\infty} \frac{b_n}{(2y)^{3n/2}},$$

where $b_0 = 1$, $b_1 = 0$, and for $n \geqslant 0$,

$$b_{n+2} = 4(9n^2 - 1)b_n - \frac{1}{2}\sum_{m=1}^{n+1} b_m b_{n+2-m} - \frac{1}{2}\sum_{l=1}^{n+1}\sum_{m=1}^{n+2-l} b_l b_m b_{n+2-l-m}.$$

Main Result

Theorem. *For every $N \geqslant 0$, there is $\varepsilon_N > 0$ and $C_N > 0$ such that for $0 < \varepsilon < \varepsilon_N$, there exists $R_{N,\varepsilon} \in \mathcal{C}^\infty \cap L^\infty(-\infty, \varepsilon^{-2/3})$ such that*

$$\nu_\varepsilon = \sum_{n=0}^{N} \varepsilon^{2n/3} \nu_n + \varepsilon^{2(N+1)/3} R_{N,\varepsilon}, \quad -\infty < y \leqslant \varepsilon^{-2/3}$$

is a positive solution of ($Eq\nu_\varepsilon$), where ν_0 is the Hastings-McLeod solution of the Painlevé II equation and the $(\nu_n)_{n\geqslant 1} \subset H^\infty(\mathbb{R})$ are defined recursively. Moreover,

$$\|R_{N,\varepsilon}\|_{L^\infty(-\infty,\varepsilon^{-2/3})} \leqslant C_N \varepsilon^{-(d-1)/3}$$

and

$$x \mapsto R_{N,\varepsilon}\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right) \in H^2(\mathbb{R}^d).$$

Proof of the main result: Multiple scale analysis

We plug $\nu_\varepsilon = \sum_{n=0}^{N} \varepsilon^{2n/3}\nu_n + \varepsilon^{2(N+1)/3} R_{N,\varepsilon}$ into

$$4(1-\varepsilon^{2/3}y)\nu_\varepsilon'' - 2\varepsilon^{2/3} d\nu_\varepsilon' + y\nu_\varepsilon - \nu_\varepsilon^3 = 0, \quad -\infty < y \leqslant \varepsilon^{-2/3}.$$

We get

- ν_0 solves (PII) $\rightsquigarrow$ we choose ν_0 to be the Hastings McLeod solution of (PII)
- for $n \geqslant 1$, ν_n solves
  $$-4\nu_n'' + W_0\nu_n = F_n, \qquad y \in \mathbb{R},$$
  where
  $$W_0(y) = 3\nu_0^2(y) - y,$$
  $$F_n(y) = -\sum_{\substack{n_1,\, n_2,\, n_3 \,<\, n \\ n_1 + n_2 + n_3 = n}} \nu_{n_1}\nu_{n_2}\nu_{n_3} - 2d\nu_{n-1}' - 4y\nu_{n-1}''.$$

  Since $W_0 > 0$ and $F_n \in L^2(\mathbb{R})$ (if $(n,d) \neq (1,2),(1,3)$), we choose
  $$\nu_n = \left(-\partial_y^2 + W_0\right)^{-1} F_n \in H^2(\mathbb{R}).$$

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Proof of the main result: Equation of $R_{N,\varepsilon}$

- $R_{N,\varepsilon}$ solves, for $y \in (-\infty, \varepsilon^{-2/3})$,

$$-4(1-\varepsilon^{2/3}y)R''_{N,\varepsilon} + 2\varepsilon^{2/3}dR'_{N,\varepsilon} + W_0 R_{N,\varepsilon} = F_{N,\varepsilon}(y, R_{N,\varepsilon}),$$

where

$$\begin{aligned} F_{N,\varepsilon}(y,R) \;=\; & -(4y\nu''_N + 2d\nu'_N) \\ & - \sum_{n=0}^{2N-1} \varepsilon^{2n/3} \sum_{\substack{n_1+n_2+n_3=n+N+1 \\ 0\leqslant n_1,n_2,n_3\leqslant N}} \nu_{n_1}\nu_{n_2}\nu_{n_3} \\ & - \left(3\sum_{n=1}^{2N} \varepsilon^{2n/3} \sum_{\substack{n_1+n_2=n \\ 0\leqslant n_1,n_2\leqslant N}} \nu_{n_1}\nu_{n_2} \right) R \\ & - \left(3 \sum_{n=N+1}^{2N+1} \varepsilon^{2n/3}\nu_{n-(N+1)} \right) R^2 - \varepsilon^{4(N+1)/3} R^3. \end{aligned}$$

Proof of the main result: Construction of $R_{N,\varepsilon}$

We consider the map T_ε defined by

$$(T_\varepsilon u)(z) = u(\varepsilon^{-2/3} - \varepsilon^{2/3}|z|^2), \quad z \in \mathbb{R}^d.$$

The equation satisfied by $R_{N,\varepsilon}$ can be rewritten as

$$T_\varepsilon R_{N,\varepsilon} = (K_\varepsilon)^{-1} T_\varepsilon F_{N,\varepsilon}(\varepsilon^{-2/3} - \varepsilon^{2/3}|z|^2, R_{N,\varepsilon}),$$

where

$$K_\varepsilon = -\Delta + W_0(\varepsilon^{-2/3} - \varepsilon^{2/3}|z|^2)$$

Then, $T_\varepsilon R_{N,\varepsilon}$ is obtained by a fixed point argument in $H^1(\mathbb{R}^d)$.

- $S \mapsto T_\varepsilon F_{N,\varepsilon}(\varepsilon^{-2/3} - \varepsilon^{2/3}|z|^2, S)$ is continuous from $H^1_{rad}(\mathbb{R}^d)$ into $L^2_{rad}(\mathbb{R}^d)$ (Sobolev embeddings)
- $W_0 \geqslant C > 0$, thus $(K_\varepsilon)^{-1} : L^2_{rad}(\mathbb{R}^d) \mapsto H^2_{rad}(\mathbb{R}^d)$ is continuous and uniformly bounded in ε.

Proof of the main result: $\nu_\varepsilon > 0$

From the asymptotic expansion of ν_ε, we deduce

$$\nu_\varepsilon(y) - \nu_0(y) \geqslant -C\varepsilon^{2/3}, \quad y \in (-\infty, \varepsilon^{-2/3}).$$

Since $\nu_0(y)$ increases from 0 to $+\infty$ as y goes from $-\infty$ to $+\infty$, we deduce that for $\varepsilon \ll 1$,

$$\nu_\varepsilon(y) \geqslant \nu_0(-1) - C\varepsilon^{2/3} > 0, \quad y \in [-1, \varepsilon^{-2/3}],$$

which means

$$\tilde{\eta}_\varepsilon(x) := \varepsilon^{1/3}\nu_\varepsilon\left(\frac{1-|x|^2}{\varepsilon^{2/3}}\right) > 0, \quad |x| \leqslant (1+\varepsilon^{2/3})^{1/2}.$$

It remains to prove that $\tilde{\eta}_\varepsilon(x) > 0$ for all $|x| > (1+\varepsilon^{2/3})^{1/2}$.

Assume by contradiction that there is $r_\varepsilon > (1+\varepsilon^{2/3})^{1/2}$ such that

$$\tilde{\eta}_\varepsilon(r_\varepsilon) = 0, \quad \tilde{\eta}'_\varepsilon(r_\varepsilon) < 0,$$

where $\tilde{\eta}_\varepsilon(|x|) = \tilde{\eta}_\varepsilon(x)$. Then, as long as $r > r_\varepsilon$ and $\tilde{\eta}_\varepsilon(r) < 0$,

$$\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\tilde{\eta}_\varepsilon\right)(r) = \frac{r^{d-1}}{\varepsilon^2}(r^2 - 1 + \tilde{\eta}_\varepsilon(r)^2)\tilde{\eta}_\varepsilon(r) \leqslant 0,$$

thus, by integration,

$$r^{d-1}\tilde{\eta}'_\varepsilon(r) \leqslant r_\varepsilon^{d-1}\tilde{\eta}'_\varepsilon(r_\varepsilon) < 0,$$

and

$$\tilde{\eta}_\varepsilon(r) \leqslant r_\varepsilon^{d-1}\tilde{\eta}'_\varepsilon(r_\varepsilon)\int_{r_\varepsilon}^{r} s^{1-d}\,ds \searrow C \in [-\infty, 0),$$

which is a contradiction with the fact that $\tilde{\eta}_\varepsilon(r) \to 0$ as $r \to +\infty$. Therefore $\tilde{\eta}_\varepsilon(r) > 0$ for all $r \in \mathbb{R}_+$.

Application to the spectrum of L_+^ε for $d = 1$

The Schrödinger operator

$$L_+^\varepsilon = -\partial_x^2 + \frac{x^2 - 1 + 3\eta_\varepsilon(x)^2}{\varepsilon^2}$$

is a positive self-adjoint operator on $L^2(\mathbb{R})$ and has a compact resolvent. The eigenfunctions corresponding to the (simple) eigenvalues λ_n^ε sorted in increasing order are even (resp. odd) in x if n is odd (resp. even).

If λ is an eigenvalue of L_+^ε and $\varphi \in L^2(\mathbb{R})$ is a corresponding eigenfunction, we define a function

$$v \in L_\varepsilon^2 = \left\{ u \in L_{\text{loc}}^1(-\infty, \varepsilon^{-2/3}) : \quad (1 - \varepsilon^{2/3} y)^{-1/4} u \in L^2 \right\}$$

by

$$\varphi(x) = v\left(\tfrac{1-x^2}{\varepsilon^{2/3}}\right), \quad x \in \mathbb{R}_+.$$

$\varphi \in L^2(\mathbb{R})$ is an even (resp. odd) eigenfunction of L_+^ε corresponding to the eigenvalue λ if and only if $v \in L^2_\varepsilon$ solves for $y \in (-\infty, \varepsilon^{-2/3})$

$$\left(-4(1-\varepsilon^{2/3}y)^{1/2}\partial_y(1-\varepsilon^{2/3}y)^{1/2}\partial_y + W_\varepsilon\right) v = \varepsilon^{4/3}\lambda v,$$

where $W_\varepsilon(y) = 3\nu_\varepsilon^2(y) - y$, and the Neumann boundary condition

$$\varphi'(0) = -2\varepsilon^{-2/3}\left((1-\varepsilon^{2/3}y)^{1/2}v'(y)\right)\Big|_{y=\varepsilon^{-2/3}} = 0 \qquad \text{(NC)}$$

(resp. the Dirichlet boundary condition

$$\varphi(0) = v(\varepsilon^{-2/3}) = 0). \qquad \text{(DC)}$$

Let $\check{M}^{\varepsilon}$ (resp. $\tilde{M}^{\varepsilon}$) be the Neumann (resp. Dirichlet) realization on L^2_{ε} of

$$-4(1-\varepsilon^{2/3}y)^{1/2}\partial_y(1-\varepsilon^{2/3}y)^{1/2}\partial_y + W_{\varepsilon}(y)$$

The eigenvalues of L^{ε}_{+} are related to the eigenvalues $\{\check{\mu}^{\varepsilon}_n\}_{n\geqslant 1}$ and $\{\tilde{\mu}^{\varepsilon}_n\}_{n\geqslant 1}$ of $\check{M}^{\varepsilon}$ and $\tilde{M}^{\varepsilon}$ by

$$\check{\mu}^{\varepsilon}_n = \varepsilon^{4/3}\lambda^{\varepsilon}_{2n-1} \quad \text{and} \quad \tilde{\mu}^{\varepsilon}_n = \varepsilon^{4/3}\lambda^{\varepsilon}_{2n}.$$

As $\varepsilon \to 0$, the eigenvalues of $\check{M}^{\varepsilon}$ and $\tilde{M}^{\varepsilon}$ converge to the eigenvalues of the operator on $L^2(\mathbb{R})$,

$$M^0 = -4\partial_y^2 + W_0.$$

Theorem *The spectrum of* L_+^ε *consists of an increasing sequence of positive eigenvalues* $\{\lambda_n^\varepsilon\}_{n \geqslant 1}$ *such that for each* $n \geqslant 1$*,*

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda_{2n-1}^\varepsilon}{\varepsilon^{2/3}} = \lim_{\varepsilon \downarrow 0} \frac{\lambda_{2n}^\varepsilon}{\varepsilon^{2/3}} = \mu_n,$$

where μ_n *is the* n^{th} *eigenvalue of* M^0*.*

Open Questions

- generalize the analysis to the original problem, with the *true* operators $L^{\varepsilon}_{\pm}$.
- consider the case when the potential is not radially symmetric.
- uniqueness of the ground state for $d = 3$