TOWARDS AN EXPLICIT LOCAL JACQUET–LANGLANDS CORRESPONDENCE BEYOND THE CUSPIDAL CASE

by

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Abstract. — We show how the modular representation theory of inner forms of general linear groups over a non-Archimedean local field can be brought to bear on the complex theory in a remarkable way. Let F be a non-Archimedean locally compact field of residue characteristic p, and let G be an inner form of the general linear group $\mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$. We consider the problem of describing explicitly the local Jacquet–Langlands correspondence $\pi \mapsto {}_{\mathrm{JL}}\pi$ between the complex discrete series representations of G and $\mathrm{GL}_n(\mathrm{F})$, in terms of type theory. We show that the congruence properties of the local Jacquet–Langlands correspondence exhibited by A. Mínguez and the first named author give information about the explicit description of this correspondence. We prove that the problem of the invariance of the endo-class by the Jacquet–Langlands correspondence can be reduced to the case where the representations π and ${}_{\mathrm{JL}}\pi$ are both cuspidal with torsion number 1. We also give an explicit description of the Jacquet–Langlands correspondence for all essentially tame discrete series representations of G, up to an unramified twist, in terms of admissible pairs, generalizing previous results by Bushnell and Henniart. In positive depth, our results are the first beyond the case where π and ${}_{\mathrm{JL}}\pi$ are both cuspidal.

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1. Introduction

1.1.

Let F be a non-Archimedean locally compact field of residue characteristic p, let H be the general linear group $\mathrm{GL}_n(\mathrm{F})$, $n \geqslant 1$, and let G be an inner form of H. This is a group of the form $\mathrm{GL}_m(\mathrm{D})$, where m divides n and D is a central division F-algebra whose reduced degree is denoted d, with $n = md$. Let $\mathcal{D}(\mathrm{G}, \mathbf{C})$ denote the set of all isomorphism classes of essentially square integrable, irreducible complex smooth representations of G. The local Jacquet–Langlands correspondence [**24, 29, 17, 1**] is a bijection

$$\begin{array}{ccc} \mathcal{D}(\mathrm{G}, \mathbf{C}) & \to & \mathcal{D}(\mathrm{H}, \mathbf{C}) \\ \pi & \mapsto & {}_{\mathrm{JL}}\pi \end{array}$$

specified by a character relation on elliptic regular conjugacy classes. Bushnell and Henniart have elaborated a vast programme aiming at giving an explicit description of this correspondence [**21, 7, 10, 12**]. The present article is a contribution to this programme.

We first have to explain what we mean by an explicit description of the Jacquet–Langlands correspondence. Essentially square integrable representations of G can be described in terms of parabolic induction. Given such a representation π, there are a unique integer r dividing m and a cuspidal irreducible representation ρ of $\mathrm{GL}_{m/r}(\mathrm{D})$, unique up to isomorphism, such that π is isomorphic to the unique irreducible quotient of the parabolically induced representation

$$\rho \times \rho\nu^{s(\rho)} \times \cdots \times \rho\nu^{s(\rho)(r-1)}$$

where ν is the unramified character "absolute value of the reduced norm" and $s(\rho)$ is a positive integer dividing d, associated to ρ in [**39**]. The essentially square integrable representation π is entirely characterized by the pair (ρ, r); this goes back to Bernstein–Zelevinski [**42**] when D is equal to F, and Tadić [**39**] in the general case (see also Badulescu [**2**] when F has positive characteristic). In particular, we may write $s(\pi) = s(\rho)$. Similarly, associated with the Jacquet-Langlands transfer ${}_{\mathrm{JL}}\pi$, there are an integer u dividing n and a cuspidal irreducible representation σ of $\mathrm{GL}_{n/u}(\mathrm{F})$. The integers r, u are related by the identity $u = rs(\pi)$. It remains to understand how the cuspidal representations ρ, σ are related.

Thanks to the theory of simple types, developed by Bushnell and Kutzko [**15**] for the general linear group $\mathrm{GL}_n(\mathrm{F})$ and by Broussous [**3**] and the authors [**30, 31, 32, 33**] for its inner forms, the cuspidal representation ρ is compactly induced from a compact mod centre, open subgroup. More precisely, there are an *extended maximal simple type*, made of a compact mod centre subgroup $\mathbf{J}$ of $\mathrm{GL}_{m/r}(\mathrm{D})$ and an irreducible representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, both constructed in a very specific way, such that the compact induction of $\boldsymbol{\lambda}$ to $\mathrm{GL}_{m/r}(\mathrm{D})$ is irreducible and isomorphic to ρ. Such a type is uniquely determined up to conjugacy. Giving an explicit description of the local Jacquet–Langlands correspondence will thus consist of describing the extended maximal simple type associated with the representation σ in terms of that of ρ.

This programme was first carried out for essentially square integrable representations of depth zero, by Silberger–Zink [**37, 38**] and Bushnell–Henniart [**13**]. Before explaining the other cases which have already been dealt with, we need to introduce two numerical invariants associated to an essentially square integrable, irreducible representation of G. Such a representation π has: a *torsion number* $t(\pi)$, the number of unramified characters χ of G such that the twisted representation $\pi\chi$ is isomorphic to π; and a *parametric degree* $\delta(\pi)$, defined in [**12**] via the theory of simple types, which is a multiple of $t(\pi)$ and divides n. Both of these integers are invariant under the Jacquet–Langlands correspondence [**12**]. It is interesting to note that the invariance of the parametric degree implies that $\delta(\pi)s(\pi) = n/r$. Consequently, the representation ${}_{\mathrm{JL}}\pi$ is cuspidal if and only if the parametric degree of π is equal to n.

In [**12**], Bushnell and Henniart treat the case where the cuspidal representation π is *essentially tame* (that is, $\delta(\pi)/t(\pi)$ is prime to p) and of parametric degree n. In that case, they explicitly describe the Jacquet–Langlands correspondence by parametrizing the conjugacy classes of extended maximal simple types in G and H by objects called *admissible pairs* [**22**]. (We will see these objects in Section 9.)

In [**10**], they also treat the case which is in some sense at the opposite extreme to the essentially tame case, where n is of the form p^k, with $k \geqslant 1$ and $p \neq 2$, and where π is a cuspidal representation of $\mathrm{D}^\times$ which is *maximal totally ramified* (that is, $\delta(\pi) = n$ and $t(\pi) = 1$).

In [**23**], Imai and Tsushima treat the case where π is an epipelagic cuspidal representation of G, that is, of depth $1/n$. Such representations are maximal totally ramified.

With the exception of [**37, 38**] and [**13**], these results all concern cases where the representations π and ${}_{\mathrm{JL}}\pi$ are both cuspidal, that is, when π is of parametric degree n. In such cases, since the cuspidal representation π can be expressed as the compact induction of an extended maximal simple type $(\mathbf{J}, \boldsymbol{\lambda})$, there is a relatively straightforward formula giving the trace of π at an elliptic regular element in terms of the trace of $\boldsymbol{\lambda}$ (see [**8**, Theorem A.14] and [**12**, (1.2.2)]). The strategies followed in [**12, 10**] and [**23**] depend crucially on such a formula. When considering a non-cuspidal essentially square integrable representation, we are in a much less favourable situation. For the group $\mathrm{GL}_n(\mathrm{F})$, Broussous [**4**] and Broussous–Schneider [**5**] have obtained formulae expressing the trace of such a representation at an elliptic regular element by bringing in the theory of simple types. However, in this article, we follow a different route.

1.2.

An important first step towards the general case is to look at the behavior of the local Jacquet–Langlands correspondence with respect to *endo-classes*. An endo-class is a type-theoretic invariant associated to any essentially square integrable representation of G, whose construction requires a considerable machinery [**8, 6**]. However, for cuspidal representations of H, it turns out to have a rather simple arithmetical interpretation through the local Langlands correspondence [**9**]. Indeed, two cuspidal irreducible representations of general linear groups over F have the same endo-class if and only if the irreducible representations of the absolute Weil group $\mathcal{W}_{\mathrm{F}}$ associated to them by the local Langlands correspondence share an irreducible component when restricted to the wild inertia subgroup $\mathcal{P}_{\mathrm{F}}$. The local Langlands correspondence thus induces a bijection between the set of $\mathcal{W}_{\mathrm{F}}$-conjugacy classes of irreducible representations of $\mathcal{P}_{\mathrm{F}}$ and the set $\mathcal{E}(\mathrm{F})$ of endo-classes over F.

It is expected that the local Jacquet–Langlands correspondence preserves endo-classes. More precisely, there is the following conjecture.

Endo-class Invariance Conjecture. *For any essentially square integrable, irreducible complex representation π of* G*, the endo-classes of π and ${}_{\mathrm{JL}}\pi$ are the same.*

Our first main result is the following (see Theorem 7.1), which reduces this conjecture to the case of maximal totally ramified cuspidal representations.

Theorem A. *Assume that, for all* F *and n, and all cuspidal irreducible complex representations π of* G *such that $\delta(\pi) = n$ and $t(\pi) = 1$, the cuspidal representations π and ${}_{\mathrm{JL}}\pi$ have the same endo-class. Then the Endo-class Invariance Conjecture is true.*

Before explaining our strategy, we must first make a detour through the modular representation theory of G and explain recent developments concerning the modular properties of the Jacquet–Langlands correspondence. Fix a prime number ℓ different from p, and consider the smooth

ℓ-adic representations of G, that is, with coefficients in the algebraic closure $\overline{\mathbf{Q}}_\ell$ of the field of ℓ-adic numbers. There is then the notion of *integral* irreducible representation of G: containing a G-stable $\overline{\mathbf{Z}}_\ell$-lattice (where $\overline{\mathbf{Z}}_\ell$ is the ring of integers of $\overline{\mathbf{Q}}_\ell$), which can then be reduced modulo ℓ. More precisely, given such a representation π containing a stable $\overline{\mathbf{Z}}_\ell$-lattice Λ, Vignéras [**40, 41**] showed that the representation $\Lambda \otimes_{\overline{\mathbf{Z}}_\ell} \overline{\mathbf{F}}_\ell$ is smooth of finite length (where $\overline{\mathbf{F}}_\ell$ is the residue field of $\overline{\mathbf{Z}}_\ell$), and its semisimplification is independent of the choice of Λ; we call this semisimplification the *reduction mod ℓ* of π. Thus we can say that two integral irreducible ℓ-adic representations of G are *congruent mod ℓ* if their reductions mod ℓ are isomorphic.

To relate this to the local Jacquet–Langlands correspondence, we fix an isomorphism of fields between $\mathbf{C}$ and $\overline{\mathbf{Q}}_\ell$; replacing one by the other via this isomorphism, we get an ℓ-adic Jacquet–Langlands correspondence

$$\mathcal{D}(\mathrm{G}, \overline{\mathbf{Q}}_\ell) \xrightarrow{\simeq} \mathcal{D}(\mathrm{H}, \overline{\mathbf{Q}}_\ell)$$

which is independent of the choice of isomorphism. Thus one can study the compatibility of this correspondence with the relation of congruence mod ℓ, which was done by Dat [**16**] and then in full generality by Mínguez and the first author [**28**]: two integral representations of $\mathcal{D}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ are congruent mod ℓ if and only their images under the ℓ-adic Jacquet–Langlands correspondence are congruent mod ℓ ([**28**, Théorème 1.1]).

We now need to explain how modular representation theory can give us information on the complex representation theory. The starting point for our strategy to prove Theorem A using modular methods is the fact that two representations of $\mathcal{D}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ which are congruent mod ℓ have the same endo-class. The converse is, of course, not true but we will see that one can nevertheless link two essentially square integrable representations with the same endo-class by a chain of congruence relations. Let us explain this in more detail.

Firstly, for any irreducible ℓ-adic representation of G, we have a notion of *mod-ℓ inertial cuspidal support* (Definition 4.1, and also [**20**] in the split case), coming from the notion of supercuspidal support for irreducible representations of G with coefficients in $\overline{\mathbf{F}}_\ell$, defined in [**25**]. Two irreducible complex representations of G are said to be *ℓ-linked* (Definitions 5.1 and 4.2) if there is a field isomorphism $\mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$ such that the resulting irreducible ℓ-adic representations have the same mod-ℓ inertial cuspidal support. This is independent of the choice of field isomorphism and it is not hard, using the work done in [**28**], to show that the Jacquet–Langlands correspondence preserves the relation of being ℓ-linked for essentially square-integrable representations (Proposition 6.1 and Corollary 6.3). We can now introduce the following definition (Definition 5.6).

Definition. *Two irreducible complex representations π, π' of* G *are said to be* linked *if there are a finite sequence of prime numbers $\ell_1, \dots, \ell_r$, all different from p, and a finite sequence of irreducible complex representations $\pi = \pi_0, \pi_1, \dots, \pi_r = \pi'$ such that, for each $i \in \{1, \dots, r\}$, the representations π_{i-1} and π_i are ℓ_i-linked.*

Two essentially square integrable complex representations which are linked have the same endo-class. More generally, if we define the *semi-simple endo-class* of an irreducible representation to be the weighted formal sum of the endo-classes of the cuspidal representations in its cuspidal support (with multiplicities determined by the sizes of the groups – see (5.2)), then two irreducible representations which are linked have the same semi-simple endo-class. The interest of the definition is apparent from the following theorem (see Theorem 5.10), which says that the converse is also true.

Theorem B. *Two irreducible complex representations of* G *are linked if and only if they have the same semi-simple endo-class.*

In particular, two essentially square integrable complex representations have the same endo-class if and only if they are linked; moreover, one can then link them by a sequence of essentially square integrable representations (Remark 5.9).

Theorem B gives a remarkable reinterpretation of what it means for two irreducible complex representations to have the same semi-simple endo-class. Beyond the intrinsic interest in explicating the notion of endo-class and its relation with modular representation theory, the main interest in this reformulation comes from the fact that, applying results from [**28**], we are able to prove the following (Theorem 6.4).

Theorem C. *Two essentially square integrable complex representations of* G *are linked if and only if their transfers to* H *are linked.*

It follows from Theorems B and C that two essentially square integrable complex representations of G have the same endo-class if and only if their transfers to H have the same endo-class. Thus, denoting by $\mathcal{E}_n(\mathrm{F})$ the set of endo-classes over F of degree dividing n, the Jacquet–Langlands correspondence induces a bijection

$$\boldsymbol{\pi}_1 : \mathcal{E}_n(\mathrm{F}) \to \mathcal{E}_n(\mathrm{F}).$$

We now observe the following fact (Proposition 6.5).

Proposition. *For every essentially square integrable complex representation of* G*, there is a cuspidal complex representation of* G *with the same endo-class and with parametric degree n.*

To prove the conjecture – that is, to prove that $\boldsymbol{\pi}_1$ is the identity map – it is therefore sufficient to prove that, for every cuspidal complex representation π of G of parametric degree n, the representations π and ${}_{\mathrm{JL}}\pi$ have the same endo-class. Using techniques developed in [**12**, Section 6], we can go further and show that one need only consider cuspidal representations of parametric degree n and torsion number 1, thus obtaining Theorem A. Therefore, to prove the Endo-class Invariance Conjecture, it remains only to prove the following conjecture. Say that an endo-class is *totally ramified* if it has residual degree 1, that is, if its tame parameter field (in the sense of [**14**, Section 2]) is totally ramified.

Conjecture. *For all* F *and n, and for every totally ramified* F*-endo-class $\boldsymbol{\Theta}$ of degree n, there is a cuspidal complex representation π of* G *with endo-class $\boldsymbol{\Theta}$ such that ${}_{\mathrm{JL}}\pi$ has endo-class $\boldsymbol{\Theta}$.*

1.3.

We now leave to one side the preservation of endo-classes and pass to the next step towards and explicit description of the Jacquet–Langlands correspondence. We will see that the modular methods described in the previous paragraphs can be pushed further to yield additional information. Let $\boldsymbol{\Theta}$ be an endo-class of degree dividing n and suppose it is invariant under the Jacquet–Langlands correspondence, i.e. $\boldsymbol{\pi}_1(\boldsymbol{\Theta}) = \boldsymbol{\Theta}$. (This is the case, for example, if $\boldsymbol{\Theta}$ is essentially tame or epipelagic.) The correspondence thus induces a bijection between isomorphism classes of essentially square integrable complex representations of G with endo-class $\boldsymbol{\Theta}$, and those of H. Since the correspondence is also compatible with unramified twisting, we get a bijection

$$\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta}) \xrightarrow{\simeq} \mathcal{D}_0(\mathrm{H}, \boldsymbol{\Theta}),$$

where $\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta})$ denotes the set of inertia classes of essentially square integrable complex representations of G with endo-class $\boldsymbol{\Theta}$. The theory of simple types [**15, 32, 33, 34**] gives us a canonical bijection between $\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta})$ and the set $\mathcal{T}(\mathrm{G}, \boldsymbol{\Theta})$ of G-conjugacy classes of simple types for G with endo-class $\boldsymbol{\Theta}$. More precisely, the inertia class of an essentially square integrable complex representation π corresponds to the conjugacy class of a simple type (J, λ), formed of a compact open subgroup J of G and an irreducible representation λ of J, if and only if λ is an irreducible component of the restriction of π to J. Thus we get a bijection

$$\mathcal{T}(\mathrm{G}, \boldsymbol{\Theta}) \xrightarrow{\simeq} \mathcal{T}(\mathrm{H}, \boldsymbol{\Theta}). \tag{1.1}$$

To go further, we need to enter into the detail of the structure of simple types (Paragraph 3.3).

Given a simple type (J, λ) of G with endo-class $\boldsymbol{\Theta}$, the group J has a unique normal pro-p subgroup, denoted J^1. The restriction of λ to J^1 is isotypic, that is, it is a direct sum of copies of a single irreducible representation η. This representation η can be extended to a representation of J with the same intertwining set as η. If we fix such an extension κ, then the representation λ can be expressed in the form $\kappa \otimes \sigma$, where σ is an irreducible representation of J, trivial on J^1. Moreover, the quotient group J/J^1 is (non-canonically) isomorphic to a product of copies of a single finite general linear groups and σ, viewed as a representation of such a product of groups, is the tensor product of copies of a single cuspidal representation. A theorem of Green [**19**] allows us to parametrize σ by a character of $\boldsymbol{k}^\times$, where $\boldsymbol{k}$ is a certain finite extension of the residue field of F, which depends only on the endo-class $\boldsymbol{\Theta}$ and on n. This character is determined up to conjugation by the Galois group $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{e})$ of $\boldsymbol{k}$ over a subfield $\boldsymbol{e}$ which depends only on $\boldsymbol{\Theta}$.

We denote by X the group of characters of $\boldsymbol{k}^\times$ and by Γ the Galois group $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{e})$. Fixing once and for all a choice of representation κ for a *maximal* simple type in G with endo-class $\boldsymbol{\Theta}$, we get a bijection from $\Gamma\backslash\mathrm{X}$ to $\mathcal{T}(\mathrm{G}, \boldsymbol{\Theta})$ (see Paragraph 3.3 for details). Making a similar choice for the group H, we also obtain a bijection from $\Gamma\backslash\mathrm{X}$ to $\mathcal{T}(\mathrm{H}, \boldsymbol{\Theta})$. Composing with the bijection (1.1), we thus obtain a permutation

$$\Upsilon : \Gamma\backslash\mathrm{X} \to \Gamma\backslash\mathrm{X}$$

which depends on the choice of κ and of its analogue for H.

We write $[\alpha]$ for the Γ-orbit of a character $\alpha \in \mathrm{X}$. The following result (see Proposition 8.8), which again is proved via modular methods, suggests that, in order to determine the permutation Υ it is sufficient to compute the value of $\Upsilon([\alpha])$ for certain characters α only.

Proposition. *Let $\alpha \in \mathrm{X}$ and let $\boldsymbol{l}$ be the unique subfield of $\boldsymbol{k}$ such that the stabilizer of α in Γ is $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{l})$. Suppose there are an $\boldsymbol{e}$-regular character $\beta \in \mathrm{X}$ and a prime number $\ell \neq p$ prime to the order of $\boldsymbol{l}^\times$ such that the order of $\beta\alpha^{-1}$ is a power of ℓ. Suppose further that $\Upsilon([\beta]) = [\beta\mu]$, for some character $\mu \in \mathrm{X}$. Then $\Upsilon([\alpha]) = [\alpha\nu]$, where $\nu \in \mathrm{X}$ is the unique character of order prime to ℓ such that $\mu\nu^{-1}$ has order a power of ℓ.*

In fact we need a more powerful version of this result, which we do not explain here, which requires being able to pass from G to a bigger group $\mathrm{GL}_{m'}(\mathrm{D})$, with $m' > m$. (See Section 8, in particular Paragraph 8.3.)

To conclude, in the final section of the paper, we illustrate this principle in the essentially tame case where, for an appropriate choice of κ and of its analogue for H, the value of $\Upsilon([\alpha])$ is known for all $\boldsymbol{e}$-regular characters $\alpha \in \mathrm{X}$ (see [**12**]). We end with the following result (Theorem 9.1).

Theorem D. *Let $\boldsymbol{\Theta}$ be an essentially tame* F*-endo-class of degree dividing n. There is a canonically determined character μ of $\boldsymbol{k}^\times$, depending only on m, d and $\boldsymbol{\Theta}$, such that $\mu^2 = 1$ and*

$$\Upsilon([\alpha]) = [\alpha\mu],$$

for all characters $\alpha \in \mathrm{X}$. More precisely, μ is the "rectifier" given by Bushnell–Henniart's First Comparison Theorem [**12**] 6.1.

See also Theorem 9.11, which is a reformulation of Theorem D in terms of admissible pairs.

2. Notation

We fix a non-Archimedean locally compact field F with residual characteristic p.

Given D a finite dimensional central division F-algebra and a positive integer $m \geqslant 1$, we write $\mathrm{M}_m(\mathrm{D})$ for the algebra of $m \times m$ matrices with coefficients in D and $\mathrm{GL}_m(\mathrm{D})$ for the group of its invertible elements. Choose an $m \geqslant 1$ and write $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$.

Given an algebraically closed field R of characteristic different from p, we will consider smooth representations of the locally profinite group G with coefficients in R. We write $\mathrm{Irr}(\mathrm{G}, \mathrm{R})$ for the set of isomorphism classes of irreducible representations of G and $\mathcal{R}(\mathrm{G}, \mathrm{R})$ for the Grothendieck group of its finite length representations, identified with the free abelian group with basis $\mathrm{Irr}(\mathrm{G}, \mathrm{R})$. If π is a representation of G, the integer m is called its *degree*.

Given $\alpha = (m_1, \dots, m_r)$ a family of positive integers of sum m, we write $\boldsymbol{i}_\alpha$ for the functor of standard parabolic induction associated with α, normalized with respect to the choice of a square root in the field R of the cardinality q of the residual field of F. Given, for each $i \in \{1, \dots, r\}$, a representation π_i of $\mathrm{GL}_{m_i}(\mathrm{D})$, we write

$$\pi_1 \times \cdots \times \pi_r = \boldsymbol{i}_\alpha(\pi_1 \otimes \cdots \otimes \pi_r).$$

Given a representation π and a character χ of G, we write $\pi\chi$ for the twisted representation defined by $g \mapsto \chi(g)\pi(g)$.

We fix once and for all a smooth additive character $\psi : \mathrm{F} \to \mathrm{R}^\times$, trivial on the maximal ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}$ of F but not trivial on $\mathcal{O}$.

3. Preliminaries

In this section we let R be an algebraically closed field of characteristic different from p. Write $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$ for some positive integer $m \geqslant 1$. Write d for the reduced degree of D over F, and define $n = md$.

3.1.

Let ρ be a cuspidal irreducible R-representation of G. Associated with ρ there is an unramified character ν_ρ of G (see [**26**, Section 4]). In [**25**] we attach to ρ and any integer $r \geqslant 1$ an irreducible subrepresentation $\mathrm{Z}(\rho, r)$ and an irreducible quotient $\mathrm{L}(\rho, r)$ of the induced representation

$$\rho \times \rho\nu_\rho \times \cdots \times \rho\nu_\rho^{r-1} \tag{3.1}$$

(see [**25**, Paragraph 7.2 and Définition 7.5]).

When R is the field of complex numbers, $\mathrm{Z}(\rho, r)$ and $\mathrm{L}(\rho, r)$ are uniquely determined in this way, and all essentially square integrable representations of G are isomorphic to a representation of the form $\mathrm{L}(\rho, r)$ for a unique pair (ρ, r).

For an arbitrary R, the representation $\mathrm{L}(\rho, r)$ is called a *discrete series* R-representation of G and $\mathrm{Z}(\rho, r)$ is called a *Speh* R-representation. If ρ is supercuspidal, $\mathrm{Z}(\rho, r)$ is called a super-Speh representation.

Associated to ρ there is also its parametric degree $\delta(\rho)$ (see [**12**, Section 2]). The parametric degree is related to the integer $s(\rho)$ defined in [**26**, Paragraph 3.4] by the formula $s(\rho)\delta(\rho) = n$, and the character ν_ρ is equal to $\nu^{s(\rho)}$.

According to [**25**, Paragraph 8.1], where the notion of residually nondegenerate representation is defined, the induced representation (3.1) contains a unique residually nondegenerate irreducible subquotient, denoted

$$\mathrm{Sp}(\rho, r).$$

When R has characteristic 0, this is equal to $\mathrm{L}(\rho, r)$. When R has characteristic $\ell > 0$ however, it may differ from $\mathrm{L}(\rho, r)$ (see [**25**, Remark 8.14]).

Assume R has characteristic $\ell > 0$, and let us write $\omega(\rho)$ for the smallest positive integer $i \geqslant 1$ such that $\rho\nu_\rho^i$ is isomorphic to ρ. Then the irreducible representation

$$\mathrm{Sp}(\rho, \omega(\rho)\ell^v) \tag{3.2}$$

is cuspidal for any integer $v \geqslant 0$. Moreover, any cuspidal non-supercuspidal irreducible representation is of the form (3.2) for a *supercuspidal* irreducible representation ρ and a unique integer $v \geqslant 0$ (see [**25**, Théorème 6.14]). We record this latter fact for future reference.

Proposition 3.1. — *Assume* R *has positive characteristic ℓ, and let ρ be a cuspidal irreducible representation of* G. *There are a unique positive integer $k = k(\rho)$ and a supercuspidal irreducible representation τ of degree m/k such that ρ is isomorphic to* $\mathrm{Sp}(\tau, k)$.

3.2.

In this paragraph, we assume that R is an algebraic closure $\overline{\mathbf{Q}}_\ell$ of the field of ℓ-adic numbers. Let $\widetilde{\rho}$ be an ℓ-adic cuspidal irreducible representation of G. Assume $\widetilde{\rho}$ is integral [**40**] and write $a = a(\widetilde{\rho})$ for the length of its reduction mod ℓ, denoted $\mathsf{r}_\ell(\widetilde{\rho})$.

Proposition 3.2 **(**[**26**, Theorem 3.15]**)**. — *Let ρ be an irreducible factor of $\mathsf{r}_\ell(\widetilde{\rho})$. Then*

$$\mathsf{r}_\ell(\widetilde{\rho}) = \rho + \rho\nu + \cdots + \rho\nu^{a-1},$$

where ν denotes the unramified mod ℓ character "absolute value of the reduced norm".

Now write $k(\rho)$ for the positive integer associated with ρ by Proposition 3.1. Let us write

$$w(\widetilde{\rho}) = a(\widetilde{\rho})k(\rho).$$

This integer has proved to have important properties with respect to the local Jacquet–Langlands correspondence [**28**].

3.3.

We assume the reader is familiar with the language of simple types. For a detailed presentation, see [**32, 26**]. For simple strata, we will use the simplified notation of [**14**, Chapter 2].

Let $[\mathfrak{a}, \beta]$ be a simple stratum in the simple central F-algebra $\mathrm{M}_m(\mathrm{D})$. The centralizer of β in it, denoted B, is a simple central $\mathrm{F}[\beta]$-algebra. There are an integer $m' \geqslant 1$ and an $\mathrm{F}[\beta]$-division algebra D' such that

$$\mathrm{B} \simeq \mathrm{M}_{m'}(\mathrm{D}'). \tag{3.3}$$

Assume that $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is a maximal order in B, and let us fix an isomorphism (3.3) such that the image of $\mathfrak{b}$ is the maximal order made of all matrices with integer entries.

Let (J, κ) be a β-extension with coefficients in R associated with the simple stratum $[\mathfrak{a}, \beta]$, that is, we have $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$ and the restriction of κ to $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$ is the Heisenberg representation of a simple R-character associated with $[\mathfrak{a}, \beta]$. Moreover, any $y \in \mathrm{B}^\times$ intertwines κ. (A β-extension associated with such a simple stratum, that is, such that $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is maximal in B, is said to be *maximal*.) We write $\boldsymbol{\Theta}$ for the endo-class defined by this simple character [**8, 6**]. Thanks to the isomorphism (3.3) we identify the two groups

$$\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_{m'}(\boldsymbol{d}),$$

where $\boldsymbol{d}$ is the residue field of D'. We write $\mathcal{G}$ for the group on the right hand side.

Recall [**30, 26**] that the set $\mathcal{C}(\mathfrak{a}, \beta)$ of simple characters associated with $[\mathfrak{a}, \beta]$ depends on the choice of the smooth additive character ψ fixed in Section 2.

We fix a finite extension $\boldsymbol{k}$ of $\boldsymbol{d}$ of degree m'. We write Σ for the Galois group of this extension and X for the group of R-characters of $\boldsymbol{k}^\times$. Given $\alpha \in \mathrm{X}$, there is a unique subfield $\boldsymbol{d} \subseteq \boldsymbol{d}[\alpha] \subseteq \boldsymbol{k}$ such that the Σ-stabilizer of α is $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{d}[\alpha])$ and then a character α_0 of $\boldsymbol{d}[\alpha]^\times$ such that α is equal to α_0 composed with the norm of $\boldsymbol{k}$ over $\boldsymbol{d}[\alpha]$. If we write u for the degree of $\boldsymbol{d}[\alpha]/\boldsymbol{d}$, then α_0 defines a supercuspidal irreducible R-representation σ_0 of $\mathrm{GL}_u(\boldsymbol{d})$ — see [**19**] if R has characteristic 0 and [**18**] or [**27**] otherwise.

Remark 3.3. — More precisely, if R has characteristic 0, fix an embedding of $\boldsymbol{d}[\alpha]$ in $\mathrm{M}_u(\boldsymbol{d})$. Then σ_0 is the unique (up to isomorphism) irreducible representation of $\mathrm{GL}_u(\boldsymbol{d})$ such that

$$\mathrm{tr}\ \sigma_0(g) = (-1)^{u-1} \cdot \sum_{\gamma} \alpha_0^{\gamma}(g),$$

for all $g \in \boldsymbol{d}[\alpha]^{\times}$ of degree u over $\boldsymbol{d}$, where γ runs over $\mathrm{Gal}(\boldsymbol{d}[\alpha]/\boldsymbol{d})$.

The character $\alpha \in \mathrm{X}$ thus defines a supercuspidal R-representation

$$\boldsymbol{\sigma}(\alpha) = \sigma_0 \otimes \cdots \otimes \sigma_0$$

of the Levi subgroup $\mathrm{GL}_u(\boldsymbol{d}) \times \cdots \times \mathrm{GL}_u(\boldsymbol{d})$ in $\mathcal{G}$. Moreover, the fibers of the map $\alpha \mapsto \boldsymbol{\sigma}(\alpha)$ are the Σ-orbits of X. Now write r for the integer defined by $ru = m'$. The maximal order $\mathfrak{b}$ contains a unique principal order $\mathfrak{b}_r$ of period r, whose image by (3.3) is standard. Write $\mathfrak{a}_r$ for the unique order normalized by $\mathrm{F}[\beta]^{\times}$ such that $\mathfrak{a}_r \cap \mathrm{B} = \mathfrak{b}_r$, and κ_r for the transfer of κ with respect to the simple stratum $[\mathfrak{a}_r, \beta]$ in the sense of [**26**, Proposition 2.3]. Considering $\boldsymbol{\sigma}(\alpha)$ as a representation of the group $\mathrm{J}_r = \mathrm{J}(\mathfrak{a}_r, \beta)$ trivial on $\mathrm{J}^1(\mathfrak{a}_r, \beta)$, we define

$$\boldsymbol{\lambda}(\alpha) = \kappa_r \otimes \boldsymbol{\sigma}(\alpha),$$

which is a simple supertype in G defined on J_r in the sense of [**35**]. Write Γ for the Galois group of $\boldsymbol{k}$ over $\boldsymbol{e}$, where $\boldsymbol{e}$ denotes the residue field of $\mathrm{F}[\beta]$.

Proposition 3.4. — *The map*

$$\alpha \mapsto \boldsymbol{\lambda}(\alpha) \tag{3.4}$$

induces a surjection from X *onto the set of* G*-conjugacy classes of simple supertypes in* G *with endo-class* $\boldsymbol{\Theta}$*. Its fibers are the* Γ*-orbits of* X*.*

Proof. — Surjectivity follows from the definition of simple supertype ([**35**, Paragraph 2.2]) and from the fact that any supercuspidal irreducible R-representation of $\mathcal{G}$ is of the form $\boldsymbol{\sigma}(\alpha)$ for some $\alpha \in \mathrm{X}$ with trivial Σ-stabilizer.

The description of the fibers follows from [**34**, Theorem 7.2] together with the fact that the map $\alpha \mapsto \boldsymbol{\sigma}(\alpha)$ is Γ-equivariant, with fibers the Σ-orbits of X. □

Write $\mathcal{T}(\boldsymbol{\Theta}, \mathrm{R})$ for the set of isomorphism classes of simple R-supertypes of endo-class $\boldsymbol{\Theta}$.

Proposition 3.5. — *The bijection*

$$\{\Gamma\text{-}orbits\ of\ \mathrm{X}\} \leftrightarrow \{\mathrm{G}\text{-}conjugacy\ classes\ of\ \mathcal{T}(\boldsymbol{\Theta}, \mathrm{R})\} \tag{3.5}$$

depends only on the choice of κ*, not on that of the isomorphism* (3.3)*.*

Proof. — Choosing another isomorphism $\mathrm{B} \simeq \mathrm{M}_{m'}(\mathrm{D}')$ such that the image of $\mathfrak{b}$ is the maximal order made of all matrices with integer entries has the effect – according to the Skolem–Noether theorem – of conjugating by an element $g \in \mathrm{GL}_{m'}(\mathrm{D}')$ normalizing this standard maximal order.

Consequently, if $\boldsymbol{\sigma}'(\alpha)$ is the representation of J_r trivial on $\mathrm{J}^1(\mathfrak{a}_r, \beta)$ corresponding to α with respect to that choice of isomorphism, it differs from $\boldsymbol{\sigma}(\alpha)$ by conjugating by g. □

3.4.

We call an inertial class of supercuspidal pairs of G *simple* if it contains a pair of the form

$$(\mathrm{GL}_{m/r}(\mathrm{D})^r, \rho \otimes \cdots \otimes \rho) \tag{3.6}$$

for some integer r dividing m and some supercuspidal R-representation ρ of $\mathrm{GL}_{m/r}(\mathrm{D})$, and we define the endo-class of such an inertial class to be the endo-class of ρ. By [**35**, Section 8], there is a bijective correspondence between simple inertial classes of supercuspidal pairs of the group G and G-conjugacy classes of simple supertypes of G, that preserves endo-classes. More precisely, the inertial class of (3.6), denoted Ω, corresponds to the G-conjugacy class of a simple supertype (J, λ) if and only if the irreducible representations of G occurring as a subquotient of the compact induction of λ to G are exactly those irreducible representations of G occurring as a subquotient of the parabolic induction to G of an element of Ω.

From the previous paragraph, we have an endo-class $\boldsymbol{\Theta}$ and a maximal β-extension κ. Combining the map (3.4) with the correspondence between simple inertial classes of supercuspidal pairs and conjugacy classes of simple supertypes, we get a surjective map

$$\alpha \mapsto \boldsymbol{\Omega}(\alpha) \tag{3.7}$$

from X onto the set of simple inertial classes of supercuspidal pairs of G with associated endo-class $\boldsymbol{\Theta}$. Its fibers are the Γ-orbits of X.

Let us recall the following important result from [**25**, Théorème 8.16]: given an irreducible representation π of G, there are integers $m_1, \dots, m_r \geqslant 1$ such that $m_1 + \cdots + m_r = m$, and supercuspidal irreducible representations $\rho_1, \dots, \rho_r$ of $\mathrm{GL}_{m_1}(\mathrm{D}), \dots, \mathrm{GL}_{m_r}(\mathrm{D})$ respectively, such that π occurs as a subquotient of the induced representation $\rho_1 \times \cdots \times \rho_r$. Moreover, up to renumbering, the supercuspidal representations $\rho_1, \dots, \rho_r$ are unique. The conjugacy class of the supercuspidal pair $(\mathrm{GL}_{m_1}(\mathrm{D}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{D}), \rho_1 \otimes \cdots \otimes \rho_r)$ is called the *supercuspidal support* of π.

Let us call an irreducible R-representation of G *simple* if the inertial class of its supercuspidal support is simple. For instance, any discrete series R-representation of G is simple.

Definition 3.6. — Let π be a simple irreducible representation of G with endo-class $\boldsymbol{\Theta}$. The *parametrizing class* of π is the Γ-orbit of a character $\alpha \in \mathrm{X}$ such that the two following equivalent conditions hold:

(1) the supercuspidal support of π belongs to the inertial class $\boldsymbol{\Omega}(\alpha)$;
(2) the representation π occurs as a subquotient of the compact induction of $\boldsymbol{\lambda}(\alpha)$ to G.

The parametrizing class of π is denoted $\mathrm{X}(\kappa, \pi)$, or simply $\mathrm{X}(\pi)$ if there is no ambiguity on the maximal β-extension κ.

Remark 3.7. — Let κ' be another maximal β-extension in G. By [**31**, Théorème 2.28] there is a character χ of $\boldsymbol{e}^\times$ such that $\kappa' = \kappa\xi$, where ξ is the character of J trivial on J^1 that corresponds to the character $\chi \circ \mathrm{N}_{\boldsymbol{d}/\boldsymbol{e}} \circ \det$ of $\mathcal{G}$, where $\mathrm{N}_{\boldsymbol{d}/\boldsymbol{e}}$ is the norm map with respect to $\boldsymbol{d}/\boldsymbol{e}$. Then we have $\alpha' \in \mathrm{X}(\kappa', \pi)$ if and only if $\alpha'\mu \in \mathrm{X}(\kappa, \pi)$, where μ is the character $\chi \circ \mathrm{N}_{\boldsymbol{k}/\boldsymbol{e}}$ of $\boldsymbol{k}^\times$.

Remark 3.8. — When R has characteristic 0, the two equivalent conditions of Definition 3.6 are also equivalent to:

(3) the representation π occurs as a quotient of the compact induction of $\boldsymbol{\lambda}(\alpha)$ to G.

Equivalently, the restriction of π to J_r contains $\boldsymbol{\lambda}(\alpha)$ as a subrepresentation.

4. Linked ℓ-adic representations

In this section, we fix a prime number ℓ different from p and write $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$.

4.1.

Let $\widetilde{\pi}$ be an irreducible ℓ-adic representation of G. Fix a representative $(\mathrm{M}, \widetilde{\rho})$ in the inertial class of its cuspidal support, with M a standard Levi subgroup $\mathrm{GL}_{m_1}(\mathrm{D}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{D})$ and $\widetilde{\rho}$ of the form $\widetilde{\rho}_1 \otimes \cdots \otimes \widetilde{\rho}_r$, where $\widetilde{\rho}_i$ is an ℓ-adic cuspidal irreducible representation of $\mathrm{GL}_{m_i}(\mathrm{D})$, for $i \in \{1, \ldots, r\}$, and with $m_1 + \cdots + m_r = m$. Since $\widetilde{\rho}_i$ is determined up to an unramified twist, we may assume it is integral, and fix an irreducible subquotient ρ_i of its reduction modulo ℓ. By the classification of mod ℓ irreducible cuspidal representations in terms of supercuspidal representations [**25**, Théorème 6.14], there are a unique integer $u_i \geqslant 1$ dividing m_i and a supercuspidal irreducible representation τ_i of degree u_i such that the supercuspidal support of ρ_i is inertially equivalent to

$$(\mathrm{GL}_{u_i}(\mathrm{D}) \times \cdots \times \mathrm{GL}_{u_i}(\mathrm{D}), \tau_i \otimes \cdots \otimes \tau_i)$$

where the factors are repeated k_i times, with $m_i = k_i u_i$.

Definition 4.1. — Let $\widetilde{\pi}$ be an irreducible ℓ-adic representation of G as above. Let us write

$$\mathrm{L} = \mathrm{GL}_{u_1}(\mathrm{D})^{k_1} \times \cdots \times \mathrm{GL}_{u_r}(\mathrm{D})^{k_r}, \quad \tau = \underbrace{\tau_1 \otimes \cdots \otimes \tau_1}_{k_1 \text{ times}} \otimes \cdots \otimes \underbrace{\tau_r \otimes \cdots \otimes \tau_r}_{k_r \text{ times}}.$$

The inertial class in G of the supercuspidal pair (L, τ), denoted $\mathbf{i}_\ell(\widetilde{\pi})$, is uniquely determined by the irreducible representation $\widetilde{\pi}$. It is called the *mod ℓ inertial supercuspidal support* of $\widetilde{\pi}$.

Definition 4.2. — Two irreducible ℓ-adic representations $\widetilde{\pi}_1, \widetilde{\pi}_2$ of G are said to *belong to the same ℓ-block* if $\mathbf{i}_\ell(\widetilde{\pi}_1) = \mathbf{i}_\ell(\widetilde{\pi}_2)$.

An *ℓ-block* in the set $\mathrm{Irr}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ of all isomorphism classes of irreducible ℓ-adic representations of G is an equivalence class for the equivalence relation defined by $\mathbf{i}_\ell$.

Let $\widetilde{\pi}$ be an irreducible ℓ-adic representation of G as above. By definition, $\mathbf{i}_\ell(\widetilde{\pi})$ depends only on the inertial class of the supercuspidal support of $\widetilde{\pi}$. Assume now $\widetilde{\pi}$ is integral.

Lemma 4.3. — *All irreducible subquotients occurring in $\mathbf{r}_\ell(\widetilde{\pi})$, the reduction mod ℓ of $\widetilde{\pi}$, have their supercuspidal support in $\mathbf{i}_\ell(\widetilde{\pi})$.*

Proof. — The representation $\widetilde{\pi}$ is a subquotient of $\widetilde{\rho}_1 \times \cdots \times \widetilde{\rho}_r$. Since $\widetilde{\pi}$ is integral, all the $\widetilde{\rho}_i$'s are integral and, by Proposition 3.2, for each i there is an integer $a_i \geqslant 1$ such that

$$\mathbf{r}_\ell(\widetilde{\rho}_i) = \rho_i + \rho_i\nu + \cdots + \rho_i\nu^{a_i - 1},$$

where ν denotes the unramified mod ℓ character "absolute value of the reduced norm". Thus any irreducible subquotient of $\mathbf{r}_\ell(\widetilde{\pi})$ occurs as a subquotient of $\rho_1\nu^{i_1} \times \cdots \times \rho_r\nu^{i_r}$ for some integers $i_1, \dots, i_r \in \mathbf{N}$. The result now follows by looking at the supercuspidal support of each ρ_i. □

Corollary 4.4. — *Any two integral irreducible ℓ-adic representation of* G *whose reductions mod ℓ share a common irreducible component belong to the same ℓ-block.*

4.2.

Let $\widetilde{\pi}$ be a simple irreducible ℓ-adic representation of G. There are an integer $r \geqslant 1$ dividing m and a cuspidal irreducible representation $\widetilde{\rho}$ of $\mathrm{G}_{m/r}$ such that the inertial class of its cuspidal support contains

$$(\mathrm{GL}_{m/r}(\mathrm{D})^r, \widetilde{\rho} \otimes \cdots \otimes \widetilde{\rho}). \tag{4.1}$$

We may assume $\widetilde{\rho}$ is integral. We fix an irreducible subquotient ρ of its reduction modulo ℓ. As in Paragraph 4.1, there are a unique integer $u \geqslant 1$ dividing m/r and a supercuspidal irreducible representation τ of degree u such that the supercuspidal support of ρ is inertially equivalent to $(\mathrm{GL}_u(\mathrm{D}) \times \cdots \times \mathrm{GL}_u(\mathrm{D}), \tau \otimes \cdots \otimes \tau)$, with $m = kur$. Therefore, the mod ℓ inertial supercuspidal support $\mathbf{i}_\ell(\widetilde{\pi})$ of the ℓ-adic discrete series representation $\widetilde{\pi}$ is the inertial class of the pair

$$(\mathrm{GL}_u(\mathrm{D})^{kr}, \tau \otimes \cdots \otimes \tau). \tag{4.2}$$

Recall that, according to [**25**, Théorème 6.11], any supercuspidal irreducible mod ℓ representation can be lifted to an ℓ-adic irreducible representation. The following lemma is an immediate consequence of the definition of the mod ℓ inertial supercuspidal support.

Lemma 4.5. — *Let $\widetilde{\tau}$ be an ℓ-adic lift of τ. Any simple irreducible ℓ-adic representation whose cuspidal support is inertially equivalent to*

$$(\mathrm{GL}_u(\mathrm{D})^{kr}, \widetilde{\tau} \otimes \cdots \otimes \widetilde{\tau})$$

is in the same ℓ-block as $\widetilde{\pi}$. In particular, the ℓ-adic discrete series representation $\mathrm{L}(\widetilde{\tau}, kr)$ *is in the same ℓ-block as $\widetilde{\pi}$.*

4.3.

Recall that we have fixed in Section 2 a smooth additive character $\psi_\ell : \mathrm{F} \to \overline{\mathbf{Q}}_\ell^\times$, trivial on $\mathfrak{p}$ but not on $\mathcal{O}$. We may and will assume that it has values in $\overline{\mathbf{Z}}_\ell^\times$. For any simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_m(\mathrm{D})$, the set of simple ℓ-adic characters associated with $[\mathfrak{a}, \beta]$ will be defined with respect to this choice (see Paragraph 3.3), whereas the set of ℓ-modular simple characters associated with $[\mathfrak{a}, \beta]$ will be defined with respect to the reduction mod ℓ of ψ_ℓ. Reduction mod ℓ thus induces a bijection between ℓ-adic and ℓ-modular simple characters associated with $[\mathfrak{a}, \beta]$. It also induces a bijection between endo-classes of ℓ-adic and ℓ-modular simple characters. Thus we will speak of endo-classes of simple characters, without referring to the coefficient field.

Write X_ℓ for the group of $\overline{\mathbf{Q}}_\ell$-characters of $\boldsymbol{k}^\times$, and fix a maximal ℓ-adic β-extension $\widetilde{\kappa}$ of G. The map (3.4) gives us a bijection $\widetilde{\boldsymbol{\lambda}}_\ell$ from $\Gamma\backslash\mathrm{X}_\ell$ onto the set of G-conjugacy classes of simple ℓ-adic supertypes of G with endo-class $\boldsymbol{\Theta}$. Also write Y_ℓ for the group of $\overline{\mathbf{F}}_\ell$-characters of $\boldsymbol{k}^\times$, and κ for the reduction mod ℓ of $\widetilde{\kappa}$. This gives us a bijection $\boldsymbol{\lambda}_\ell$ from $\Gamma\backslash\mathrm{Y}_\ell$ onto the set of

G-conjugacy classes of simple mod ℓ supertypes of G with endo-class $\mathbf{\Theta}$. These two bijections are compatible in the following sense.

Proposition 4.6. — *Let $\widetilde{\pi}$ be a simple irreducible ℓ-adic representation of* G *with endo-class $\mathbf{\Theta}$, let $\alpha \in \mathrm{X}_\ell(\widetilde{\pi})$ and let $\phi \in \mathrm{Y}_\ell$ be the reduction mod ℓ of α. Then the inertial class $\mathbf{i}_\ell(\widetilde{\pi})$ corresponds through* (3.5) *and* (3.7) *to the* G*-conjugacy class of the simple supertype $\boldsymbol{\lambda}_\ell(\phi)$.*

Proof. — Write the inertial class of the cuspidal support of $\widetilde{\pi}$ as in (4.1). Let r be the degree of $\boldsymbol{k}$ over $\boldsymbol{d}[\alpha]$ and $\widetilde{\sigma}_0$ be the ℓ-adic supercuspidal representation of $\mathrm{GL}_u(\boldsymbol{k})$ associated with α, where $m' = ru$. There is a maximal β-extension $\widetilde{\kappa}_0$ of $\mathrm{GL}_{m/r}(\mathrm{D})$ such that $\widetilde{\kappa}_0 \otimes \widetilde{\sigma}_0$ is a maximal simple type contained in $\widetilde{\rho}$. Let ρ be an irreducible factor of the reduction mod ℓ of $\widetilde{\rho}$. Then ρ contains the maximal simple type $\kappa_0 \otimes \sigma_0$, where κ_0 is the reduction mod ℓ of κ_0 and σ_0 is that of $\widetilde{\sigma}_0$.

Let t be the degree of $\boldsymbol{k}$ over $\boldsymbol{d}[\phi]$. By [**28**, Lemme 3.2], if we write ρ in the form $\mathrm{Sp}(\tau, k)$, with τ supercuspidal (see Proposition 3.1), then $kr = t$ and σ_0 is the unique nondegenerate irreducible subquotient of the induced representation $\sigma_1 \times \cdots \times \sigma_1$, where σ_1 is the supercuspidal mod ℓ representation of $\mathrm{GL}_{m'/t}(\boldsymbol{d})$ corresponding to ϕ. Moreover, there exists a maximal β-extension κ_1 of $\mathrm{GL}_{m/t}(\mathrm{D})$ such that $\kappa_1 \otimes \sigma_1$ is a maximal simple type contained in τ. □

Given $\alpha \in \mathrm{X}$, write $[\alpha]$ for its Γ-orbit and ϕ for its reduction mod ℓ. The orbit $[\phi]$ depends only on $[\alpha]$, and is called the reduction mod ℓ of $[\alpha]$.

Proposition 4.7. — *Two simple irreducible ℓ-adic representations of* G *of endo-class $\mathbf{\Theta}$ are in the same ℓ-block if and only if their parametrizing classes have the same reduction mod ℓ.*

Proof. — This follows from Propositions 3.4 and 4.6. □

We keep in mind the straightforward but important following fact.

Remark 4.8. — Two simple irreducible ℓ-adic representations of G in the same ℓ-block have the same endo-class.

5. Linked complex representations

In this section, we write $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$, $m \geqslant 1$.

5.1.

We fix a prime number ℓ different from p and an isomorphism of fields $\iota_\ell : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$. If π is a complex representation of G, write $\iota_\ell^* \pi$ for the ℓ-adic representation of G obtained by extending scalars from $\mathbf{C}$ to $\overline{\mathbf{Q}}_\ell$ along ι_ℓ.

Definition 5.1. — Two irreducible complex representations π_1, π_2 of G are said to be *ℓ-linked* if the irreducible ℓ-adic representations $\iota_\ell^* \pi_1$ and $\iota_\ell^* \pi_2$ are in the same ℓ-block.

Lemma 5.2. — *This definition does not depend on the choice of ι_ℓ.*

Proof. — It is enough to prove that, for any field automorphism $\theta \in \mathrm{Aut}(\overline{\mathbf{Q}}_\ell)$, two simple ℓ-adic representations $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ of G are in the same ℓ-block if and only if $\widetilde{\pi}_1^\theta$ and $\widetilde{\pi}_2^\theta$ are in the same ℓ-block. By twisting by an unramified character of G if necessary, we may assume that $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are integral and that their central characters have finite order. Thus the central characters of $\widetilde{\pi}_1^\theta$ and $\widetilde{\pi}_2^\theta$ also have finite order, which implies that they are integral too.

Given a simple ℓ-adic representation $\widetilde{\pi}$, write an element of its mod ℓ inertial supercuspidal support as in (4.2). Then the mod ℓ inertial supercuspidal support of $\widetilde{\pi}^\theta$ is the inertial class of the pair

$$(\mathrm{GL}_u(\mathrm{D})^{kr}, \tau^\theta \otimes \cdots \otimes \tau^\theta).$$

The result follows. □

5.2.

Recall that we have fixed in Section 2 a smooth additive character $\psi : \mathrm{F} \to \mathbf{C}^\times$, trivial on $\mathfrak{p}$ but not on $\mathcal{O}$. For any simple stratum $[\mathfrak{a}, \beta]$, the set of simple complex characters associated with $[\mathfrak{a}, \beta]$ will be defined with respect to this choice (see Paragraphs 3.3 and 4.3). Moreover, we assume the character $\iota_\ell \circ \psi$ is the character ψ_ℓ used in Paragraph 4.3. This choice gives us a bijection between endo-classes of complex and ℓ-adic simple characters of G. Again, we will speak of endo-classes of simple characters, without referring to the coefficient field.

Let κ be a maximal complex β-extension of G with endo-class $\boldsymbol{\Theta}$. Write X for the group of complex characters of $\boldsymbol{k}^\times$.

Lemma 5.3. — *Let π be a simple irreducible complex representation of* G *with endo-class* $\boldsymbol{\Theta}$. *Then we have*

$$\alpha \in \mathrm{X}(\kappa, \pi) \quad \Leftrightarrow \quad \iota_\ell \circ \alpha \in \mathrm{X}_\ell(\iota_\ell^*\kappa, \iota_\ell^*\pi).$$

Proof. — We have $\alpha \in \mathrm{X}(\kappa, \pi)$ if and only if π contains the simple type $\boldsymbol{\lambda}(\alpha) = \boldsymbol{\kappa}(\alpha) \otimes \boldsymbol{\sigma}(\alpha)$, which occurs if and only if $\iota_\ell^*\pi$ contains the ℓ-adic simple type $\iota_\ell^*\boldsymbol{\lambda}(\alpha)$. Thus it suffices to prove that $\iota_\ell^*\boldsymbol{\lambda}(\alpha)$ is equal to $\widetilde{\boldsymbol{\lambda}}_\ell(\iota_\ell \circ \alpha)$, where $\widetilde{\boldsymbol{\lambda}}_\ell$ is the map as in Paragraph 4.3 defined with respect to the maximal β-extension $\iota_\ell^*\kappa$.

Firstly, the ℓ-adic β-extension $\widetilde{\boldsymbol{\kappa}}_\ell(\iota_\ell \circ \alpha)$ associated with $\iota_\ell \circ \alpha$ with respect to $\iota_\ell^*\kappa$ is equal to $\iota_\ell^*\boldsymbol{\kappa}(\alpha)$. Secondly, the ℓ-adic supercuspidal representation $\widetilde{\boldsymbol{\sigma}}_\ell(\iota_\ell \circ \alpha)$ associated with $\iota_\ell \circ \alpha$ (with respect to the choice of an isomorphism (3.3)) is equal to $\iota_\ell^*\boldsymbol{\sigma}(\alpha)$, since it is characterized by a trace formula (see Remark 3.3). The result follows. □

Definition 5.4. — Let $\alpha \in \mathrm{X}$. The *ℓ-regular part* of α is the unique complex character $\alpha_\ell \in \mathrm{X}$ whose order is prime to ℓ and such that $\alpha\alpha_\ell^{-1}$ has order a power of ℓ.

Given $\alpha \in \mathrm{X}$, the orbit $[\alpha_\ell]$ depends only on $[\alpha]$. It is called the ℓ-regular part of $[\alpha]$, denoted $[\alpha]_\ell$.

Proposition 5.5. — *Two simple irreducible complex representations of* G *with endo-class* $\boldsymbol{\Theta}$ *are ℓ-linked if and only if the ℓ-regular parts of their parametrizing classes are equal.*

Proof. — Let π_1, π_2 be simple irreducible complex representations of G with endo-class $\boldsymbol{\Theta}$. We fix $\alpha_i \in \mathrm{X}(\kappa, \pi_i)$ for each $i = 1, 2$. By Lemma 5.3 and Proposition 4.7, the representations π_1, π_2 are ℓ-linked if and only if $[\iota_\ell \circ \alpha_1]$ and $[\iota_\ell \circ \alpha_2]$ have the same reduction mod ℓ. But the reduction mod ℓ of $[\iota_\ell \circ \alpha]$, for a character $\alpha \in \mathrm{X}$, is the same as that of $[\iota_\ell \circ \alpha_\ell]$. It follows that we have $[\iota_\ell \circ (\alpha_1)_\ell] = [\iota_\ell \circ (\alpha_2)_\ell]$, thus $[\alpha_1]_\ell = [\alpha_2]_\ell$. □

5.3.

Write q for the cardinality of the residue field of F. For each prime number ℓ dividing

$$(q^n - 1)(q^{n-1} - 1) \dots (q - 1) \tag{5.1}$$

we fix an isomorphism of fields $\iota_\ell : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$.

Definition 5.6. — Two irreducible complex representations π, π' of G are *linked* if there are a finite family $\ell_1, \dots, \ell_r$ of prime numbers dividing (5.1) and a finite family of irreducible complex representations $\pi = \pi_0, \pi_1, \dots, \pi_r = \pi'$ such that, for all integers $i \in \{1, \dots, r\}$, the representations π_{i-1} and π_i are ℓ_i-linked.

Remark 5.7. — By Lemma 5.2, this does not depend on the choice of the isomorphisms ι_ℓ for ℓ dividing (5.1).

Two linked simple complex representations of G have the same endo-class (see Remark 4.8). The converse is given by the following proposition.

Proposition 5.8. — *Two simple irreducible complex representations are linked if and only if they have the same endo-class.*

Proof. — Assume π and π' are simple irreducible complex representations with the same endo-class $\boldsymbol{\Theta}$. Let α and α' be characters in $\mathrm{X}(\pi)$ and $\mathrm{X}(\pi')$, respectively, and write $\xi = \alpha'\alpha^{-1}$. Let $\ell_1, \dots, \ell_r$ be the prime numbers dividing (5.1). The character ξ decomposes uniquely as

$$\xi = \xi_1 \dots \xi_r$$

where the order of ξ_i is a power of ℓ_i, for $i \in \{1, \dots, r\}$. Write $\alpha_0 = \alpha$ and define inductively

$$\alpha_i = \alpha_{i-1} \cdot \xi_i$$

for all $i \in \{1, \dots, r\}$. Let π_i be a simple irreducible complex representation of endo-class $\boldsymbol{\Theta}$ and parametrizing class $[\alpha_i]$. The result follows from Proposition 5.5. □

Remark 5.9. — Suppose that π and π' are discrete series representations with the same endo-class. The proof of Proposition 5.8 shows that the simple representations $\pi_1, \dots, \pi_{r-1}$ linking π to π' can be chosen to be discrete series representations as well.

5.4.

Let π be an irreducible complex representation of G. Fix a representative (M,ρ) in its cuspidal support, with $\mathrm{M} = \mathrm{GL}_{m_1}(\mathrm{D}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{D})$ and $\rho = \rho_1 \otimes \cdots \otimes \rho_r$, with $m_1 + \cdots + m_r = m$, and where ρ_i is a cuspidal irreducible representation of $\mathrm{GL}_{m_i}(\mathrm{D})$ for $i \in \{1,\dots,r\}$. Write $\boldsymbol{\Theta}_i$ for the endo-class of ρ_i and g_i for the degree of $\boldsymbol{\Theta}_i$. We define the *semi-simple endo-class* of π to be the formal sum

$$\boldsymbol{\Theta}(\pi) = \sum_{i=1}^{r} \frac{m_i d}{g_i} \cdot \boldsymbol{\Theta}_i \tag{5.2}$$

in the free abelian semigroup generated by all F-endo-classes. It depends only on the inertial class of the cuspidal support of π.

Note that, if π is a simple irreducible representation with endo-class $\boldsymbol{\Theta}$, then its semi-simple endo-class is $\boldsymbol{\Theta}(\pi) = ng^{-1} \cdot \boldsymbol{\Theta}$ where g is the degree of $\boldsymbol{\Theta}$.

The following theorem, which is our first main result, generalizes Proposition 5.8.

Theorem 5.10. — *Two irreducible complex representations are linked if and only if they have the same semi-simple endo-class.*

Proof. — Any two linked irreducible complex representations automatically have the same semi-simple endo-class. We thus start with two irreducible complex representations π, π' with the same semi-simple endo-class. By [**26**, Théorème 4.16], the representation π can be written

$$\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_k$$

where $\pi_1, \pi_2, \dots, \pi_k$ are simple irreducible representations whose inertial cuspidal supports are pairwise distinct, and this decomposition is unique up to renumbering. We have the following straightforward lemma.

Lemma 5.11. — *Let δ be an irreducible complex representation of $\mathrm{GL}_{m-k}(\mathrm{D})$ for some integer $k \in \{1,\dots,m-1\}$. Let σ, σ' be two irreducible complex representations of $\mathrm{GL}_k(\mathrm{D})$, and let π, π' be irreducible subquotients of $\sigma \times \delta$ and $\sigma' \times \delta$, respectively. If σ and σ' are linked, then π and π' are linked.*

For each $i \in \{1,\dots,k\}$, thanks to Lemma 5.11 and Proposition 5.8, we may and will assume that π_i is a discrete series representation of the form $\mathrm{L}(\rho_i, r_i)$ for some cuspidal representation ρ_i of $\mathrm{GL}_{m_i}(\mathrm{D})$ with same endo-class as π_i and some integer r_i, such that $m_1 r_1 + \cdots + m_k r_k = m$. We may even assume that ρ_i has minimal degree among all cuspidal irreducible representations of $\mathrm{GL}_a(\mathrm{D})$, $a \geqslant 1$, with the same endo-class as π_i. This amounts to saying that m_i is equal to $g_i/(g_i, d)$, where g_i is the degree of the endo-class of π_i.

Moreover, if ρ_i, ρ_j have the same endo-class for some $i, j \in \{1,\dots,k\}$, then they have the same degree, thus they are linked. We thus may assume $\rho_1,\dots,\rho_k$ have distinct endo-classes, denoted $\boldsymbol{\Theta}_1,\dots,\boldsymbol{\Theta}_k$, respectively.

Similarly, we may assume the representation π' decomposes as a product $\pi'_1 \times \pi'_2 \times \cdots \times \pi'_t$, where π'_j is a discrete series representation of the form $\mathrm{L}(\rho'_j, s_j)$ for some cuspidal representation ρ'_j of $\mathrm{GL}_{m'_j}(\mathrm{D})$ and some integer $s_j \geqslant 1$, and we may assume that the endo-classes $\boldsymbol{\Theta}'_1,\dots,\boldsymbol{\Theta}'_t$

of $\rho'_1, \dots, \rho'_t$ are distinct. It follows that $k = t$ and, up to renumbering, we may assume that we have $\boldsymbol{\Theta}'_i = \boldsymbol{\Theta}_i$ for each $i \in \{1, \dots, k\}$. It then follows that ρ'_i and ρ_i have the same degree, by the minimality of m_i.

Since π and π' have the same semi-simple endo-class, we have $s_i = r_i$ for all i, thus π_i and π'_i have the same degree. Proposition 5.8 then implies that π_i and π'_i are linked. Theorem 5.10 now follows from Lemma 5.11 again. □

6. Application to the local Jacquet–Langlands correspondence

We fix $n = md$ and write $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$ and $\mathrm{H} = \mathrm{GL}_n(\mathrm{F})$. As in the introduction, we write $\mathcal{D}(\mathrm{G}, \mathbf{C})$ for the set of all isomorphism classes of complex discrete series representations of G, and similarly for H. We write

$$\boldsymbol{\pi} : \mathcal{D}(\mathrm{G}, \mathbf{C}) \to \mathcal{D}(\mathrm{H}, \mathbf{C}) \tag{6.1}$$

for the local Jacquet–Langlands correspondence.

6.1.

We fix an isomorphism of fields $\iota_\ell : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$ and write (as in [**28**])

$$\widetilde{\boldsymbol{\pi}}_\ell : \mathcal{D}(\mathrm{G}, \overline{\mathbf{Q}}_\ell) \to \mathcal{D}(\mathrm{H}, \overline{\mathbf{Q}}_\ell) \tag{6.2}$$

for the ℓ-adic local Jacquet–Langlands correspondence between ℓ-adic discrete series representations of G and H. The correspondence (6.2) does not depends on the choice of ι_ℓ ([**28**, Remarque 10.1]). According to [**2**, Paragraph 3.1], there is a unique surjective group homomorphism

$$\widetilde{\mathbf{J}}_\ell : \mathcal{R}(\mathrm{H}, \overline{\mathbf{Q}}_\ell) \to \mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$$

where $\mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ is the Grothendieck group of finite length ℓ-adic representations of G, with the following property: given positive integers $n_1, \dots, n_r$ such that $n_1 + \dots + n_r = n$ and an ℓ-adic discrete series representation $\widetilde{\sigma}_i$ of $\mathrm{GL}_{n_i}(\mathrm{F})$ for each i, the image of the product $\widetilde{\sigma}_1 \times \dots \times \widetilde{\sigma}_r$ by $\widetilde{\mathbf{J}}_\ell$ is 0 if n_i is not divisible by d for at least one i, and is $\widetilde{\pi}_1 \times \dots \times \widetilde{\pi}_r$ otherwise, where $n_i = m_i d$ and $\widetilde{\pi}_i$ is the ℓ-adic discrete series representation of $\mathrm{GL}_{m_i}(\mathrm{D})$ whose Jacquet–Langlands transfer is $\widetilde{\sigma}_i$, for each i.

By [**28**, Théorème 12.4], there exists a unique surjective group homomorphism of Grothendieck groups $\mathbf{J}_\ell : \mathcal{R}(\mathrm{H}, \overline{\mathbf{F}}_\ell) \to \mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell)$ such that the diagram

$$\begin{array}{ccc} \mathcal{R}(\mathrm{H}, \overline{\mathbf{Q}}_\ell)^{\mathrm{e}} & \xrightarrow{\widetilde{\mathbf{J}}_\ell} & \mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)^{\mathrm{e}} \\ {\scriptstyle \mathbf{r}_\ell}\downarrow & & \downarrow{\scriptstyle \mathbf{r}_\ell} \\ \mathcal{R}(\mathrm{H}, \overline{\mathbf{F}}_\ell) & \xrightarrow[\mathbf{J}_\ell]{} & \mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell) \end{array}$$

is commutative, where $\mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)^{\mathrm{e}}$ is the subgroup of $\mathcal{R}(\mathrm{G}, \overline{\mathbf{Q}}_\ell)$ generated by integral irreducible representations, and $\mathcal{R}(\mathrm{G}, \overline{\mathbf{F}}_\ell)$ is the Grothendieck group of ℓ-modular representations of G.

Proposition 6.1. — *Let $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ be ℓ-adic discrete series representations of* G*, and write $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ for their Jacquet–Langlands transfers to* H*, respectively. If $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ are in the same ℓ-block of* H*, then $\widetilde{\pi}_1$, $\widetilde{\pi}_2$ are in the same ℓ-block of* G*.*

Proof. — Let us write $\widetilde{\sigma}_i = \mathrm{L}(\widetilde{\rho}_i, r_i)$ and $k_i = k(\widetilde{\rho}_i)$ for $i = 1, 2$. Then $k_1r_1 = k_2r_2$, which we denote by v, and the mod ℓ inertial supercuspidal support of $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ contains the supercuspidal pair

$$(\mathrm{GL}_u(\mathrm{F}) \times \cdots \times \mathrm{GL}_u(\mathrm{F}), \tau \otimes \cdots \otimes \tau),$$

with $uv = m$ and for some mod ℓ supercuspidal representation τ of $\mathrm{GL}_u(\mathrm{D})$. Fix an ℓ-adic lift $\widetilde{\tau}$ of τ and write $\widetilde{\sigma} = \mathrm{L}(\widetilde{\tau}, v)$. The representation $\widetilde{\sigma}$ is in the same ℓ-block as $\widetilde{\sigma}_1, \widetilde{\sigma}_2$, by Lemma 4.5. If we write $\widetilde{\pi}$ for the ℓ-adic discrete series representation of G whose transfer to H is $\widetilde{\sigma}$, then it is enough to prove that $\widetilde{\pi}$ is in the same ℓ-block as $\widetilde{\pi}_1$.

In the remainder of the proof, it will be more convenient for us to deal with Speh representations rather than discrete series representations, as in [**28**]. We thus apply the Zelevinski involution to $\widetilde{\pi}$, $\widetilde{\pi}_1$ and $\widetilde{\sigma}$, $\widetilde{\sigma}_1$ and thus get ℓ-adic Speh representations.

Let us write $\widetilde{\sigma}^*$ for the Zelevinski dual of $\widetilde{\sigma}$. Its reduction mod ℓ is the ℓ-modular super-Speh representation $\mathrm{Z}(\tau, v)$, by [**25**, Théorème 9.39]. If we write $\widetilde{\pi}^* = \mathrm{Z}(\widetilde{\alpha}, t)$ for the Zelevinski dual of $\widetilde{\pi}$, for some t dividing m and some cuspidal irreducible representation $\widetilde{\alpha}$ of $\mathrm{GL}_{m/t}(\mathrm{D})$, then its reduction mod ℓ contains the Speh representation $\mathrm{Z}(\alpha, t)$ where α is an irreducible component of the reduction mod ℓ of $\widetilde{\alpha}$ (see for instance [**28**, Proposition 1.10]). The cuspidal representation α need not be supercuspidal but, according to Proposition 3.1, it can be written as $\mathrm{Sp}(\beta, k)$ for $k = k(\alpha)$ and some supercuspidal irreducible representation β.

We now look at the reduction mod ℓ of the Zelevinski dual of $\widetilde{\sigma}_1$. It is $\mathrm{Z}(\rho_1, r_1)$ where ρ_1, the reduction mod ℓ of $\widetilde{\rho}_1$, can be written as $\mathrm{Sp}(\tau\chi, k_1)$ for some unramified character χ. By twisting $\widetilde{\pi}_1$ by an unramified character of G, we may assume that χ is trivial. According to [**25**, Lemme 9.41], the representation $\mathrm{Z}(\rho_1, k_1)$ decomposes as a $\mathbf{Z}$-linear combination of products of the form

$$\mathrm{Z}(\tau\nu^{i_1}, v_1) \times \cdots \times \mathrm{Z}(\tau\nu^{i_r}, v_r)$$

with $v_1+\cdots+v_r = v$ and $i_1, \ldots, i_r \in \mathbf{Z}$, where ν stands for the absolute value of the reduced norm, as usual. (For an explicit formula for this decomposition, see [**28**, Sections 11 and 12].) Thanks to the commutative diagram above, the reduction modulo ℓ of the Zelevinski dual of $\widetilde{\pi}_1$ will be made of products of the form

$$\mathrm{Z}(\alpha\nu^{i_1}, t_1) \times \cdots \times \mathrm{Z}(\alpha\nu^{i_r}, t_r)$$

with $t_1 + \cdots + t_r = t$ and $i_1, \ldots, i_r \in \mathbf{Z}$, all of whose irreducible subquotients have supercuspidal support inertially equivalent to $(\mathrm{GL}_w(\mathrm{D}) \times \cdots \times \mathrm{GL}_w(\mathrm{D}), \beta \otimes \cdots \otimes \beta)$, with $wkt = m$. The result follows from Corollary 4.4. □

6.2.

Proposition 6.1 implies that two complex discrete series representations π_1, π_2 of G are linked if their Jacquet–Langlands transfers are linked. Then Proposition 5.8 (together with Remark

5.9) induces a map

$$\boldsymbol{\pi}_1 : \mathcal{E}_n(\mathrm{F}) \to \mathcal{E}_n(\mathrm{F})$$

depending on G, where $\mathcal{E}_n(\mathrm{F})$ is the set of F-endo-classes of degree dividing n. More precisely, given an endo-class $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F})$ and a complex discrete series representation σ of H of endo-class $\boldsymbol{\Theta}$, the endo-class of the Jacquet–Langlands transfer of σ to G depends only on $\boldsymbol{\Theta}$: we denote it $\boldsymbol{\pi}_1(\boldsymbol{\Theta})$.

This map does not depend on the choice of the isomorphisms ι_ℓ for ℓ dividing (5.1).

Proposition 6.2. — *The map $\boldsymbol{\pi}_1$ is bijective.*

Proof. — This map preserves depth. Thus, for any rational number $r \in \mathbf{Q}_+$, we have a map

$$\boldsymbol{\pi}_{1,r} : \mathcal{E}_n(\mathrm{F}, r) \to \mathcal{E}_n(\mathrm{F}, r)$$

where $\mathcal{E}_n(\mathrm{F}, r)$ the set of of F-endo-classes of degree dividing n and depth r, which we claim to be surjective. Indeed, given $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F}, r)$, let us choose $\pi \in \mathrm{Irr}(\mathrm{G}, \mathbf{C})$ of endo-class $\boldsymbol{\Theta}$, and let σ be its Jacquet–Langlands transfer to H. Then the endo-class of σ is an antecedent of $\boldsymbol{\Theta}$ by $\boldsymbol{\pi}_1$. Since $\mathcal{E}_n(\mathrm{F}, r)$ is a finite set and $\boldsymbol{\pi}_{1,r}$ is surjective, it follows that $\boldsymbol{\pi}_{1,r}$ is bijective for all r, thus the map $\boldsymbol{\pi}_1$ is bijective. □

As a corollary, we have the following refinement of Proposition 6.1.

Corollary 6.3. — *Let $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ be ℓ-adic discrete series representations of* G*, and write $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ for their Jacquet–Langlands transfers to* H*, respectively. Then $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ are in the same ℓ-block of* H *if and only if $\widetilde{\pi}_1$, $\widetilde{\pi}_2$ are in the same ℓ-block of* G*.*

Allowing ℓ to vary, we deduce

Theorem 6.4. — *Two complex discrete series representations of* G *are linked if and only if their transfers to* H *are linked.*

Recall that the parametric degree of a cuspidal representation of G has been defined in 3.1.

Proposition 6.5. — *For every complex discrete series representation of* G*, there is a cuspidal complex representation of* G *with the same endo-class and with parametric degree n.*

Proof. — Let π be a complex discrete series representation of G with endo-class $\boldsymbol{\Theta}$. To find a complex cuspidal representation with same endo-class and parametric degree n, we need to find a $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{d})$-regular complex character $\alpha \in \mathrm{X}$ which is also $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{e})$-regular. The latter implies the former, so let us find a $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{e})$-regular character $\alpha \in \mathrm{X}$. For this, it is enough to choose for α a generator of the cyclic group X. □

As an immediate consequence, we get the following.

Theorem 6.6. — *Given an endo-class $\boldsymbol{\Theta}$ in $\mathcal{E}_n(\mathrm{F})$, if there is a complex cuspidal representation ρ of* G *with endo-class $\boldsymbol{\Theta}$ and parametric degree n such that $\boldsymbol{\pi}(\rho)$ has endo-class $\boldsymbol{\Theta}$, then $\boldsymbol{\pi}_1(\boldsymbol{\Theta})$ is equal to $\boldsymbol{\Theta}$.*

Corollary 6.7. — *Assume that $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F})$ is essentially tame, that is, its ramification order is prime to p. Then, for all complex discrete series representation π of* G *with endo-class $\boldsymbol{\Theta}$, the representations π and ${}_{\mathrm{JL}}\pi$ have the same endo-class.*

Proof. — This follows from Theorem 6.6 together with the main result of [**12**]. □

Remark 6.8. — Let $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F})$ and write $\boldsymbol{\Theta}' = \boldsymbol{\pi}_1(\boldsymbol{\Theta}) \in \mathcal{E}_n(\mathrm{F})$. Let ρ be a cuspidal irreducible representation of G with endo-class $\boldsymbol{\Theta}$ and parametric degree n. Its Jacquet–Langlands transfer to $\mathrm{GL}_n(\mathrm{F})$ is a cuspidal representation denoted σ. Then, for any $a \geqslant 1$, the discrete series representation $\mathrm{L}(\rho, a)$ of $\mathrm{GL}_{am}(\mathrm{D})$ has endo-class $\boldsymbol{\Theta}$, and its transfer $\mathrm{L}(\sigma, a)$ to $\mathrm{GL}_{an}(\mathrm{F})$ has endo-class $\boldsymbol{\Theta}'$. Thus $\boldsymbol{\pi}_1(\boldsymbol{\Theta})$ does not depend on the choice of the integer $n \geqslant 1$ such that $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F})$.

7. Reduction to the maximal totally ramified case

We continue with the previous notation, so that $\mathrm{G} = \mathrm{GL}_m(\mathrm{D})$ and $\mathrm{H} = \mathrm{GL}_n(\mathrm{F})$. In this section, we closely follow the ideas of [**12**, Section 6] to make a further reduction to the maximal totally ramified case (see Paragraph 1.1). All representations in this section are complex.

7.1.

Let π be a cuspidal (complex) representation of G with parametric degree n. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be an extended maximal simple type of G contained in π [**26**, §3.1 and Théorème 3.11], attached to a simple stratum $[\mathfrak{a}, \beta]$ and a simple character θ. We write B for the centralizer of β in $\mathrm{M}_m(\mathrm{D})$, so that $\mathrm{B} \simeq \mathrm{M}_{m'}(\mathrm{D}')$, for some integer $m' \geqslant 1$ and $\mathrm{F}[\beta]$-division algebra D'. We fix a maximal unramified extension L of $\mathrm{F}[\beta]$ in B, and write K for the maximal unramified subextension of L over F.

We fix a root of unity $\zeta \in \mathrm{K}$ of order relatively prime to p such that $\mathrm{K} = \mathrm{F}[\zeta]$. Write G_K for the centralizer of K in G. Let u be a pro-unipotent, elliptic regular element of G_K in the sense of [**12**, Paragraph 1.6]. The element $h = \zeta u$ then lies in the set $\mathrm{G}_{\mathrm{reg}}^{\mathrm{ell}}$ of elliptic regular elements of G, so we have

$$\mathrm{tr}\ \pi(h) = \sum_{x \in \mathrm{G}/\mathbf{J}} \mathrm{tr}\ \boldsymbol{\lambda}(x^{-1}hx)$$

as in [**12**, (6.3.1)]. Write $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta) = \mathbf{J} \cap \mathfrak{a}^\times$. A coset $x\mathbf{J}$ can only contribute to the sum if we have $x^{-1}hx \in \mathbf{J}$ or, equivalently, $x^{-1}hx \in \mathrm{J}$.

Write Ψ for the Galois group of K/F and Γ for that of $\mathrm{L}/\mathrm{F}[\beta]$. Restriction of operators identifies Γ with a subgroup of Ψ. Write Ψ_t for the unique subgroup of Γ (thus of Ψ) of order m'. As in [**12**, (6.3.2)], by using [**12**, 6.3 Proposition] we have

$$\mathrm{tr}\ \pi(\zeta u) = \sum_{\alpha \in \Psi/\Psi_\mathrm{t}} \sum_{y \in \mathrm{G}_\mathrm{K}/\mathbf{J}_\mathrm{K}} \mathrm{tr}\ \boldsymbol{\lambda}(y^{-1}\zeta^\alpha u^\alpha y)$$

where $\mathbf{J}_\mathrm{K} = \mathbf{J} \cap \mathrm{G}_\mathrm{K}$.

Let us fix a uniformizer ϖ_F of F. We choose an irreducible representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ such that:

(1) the restriction of $\boldsymbol{\kappa}$ to J is a β-extension of θ;

(2) the character $\det(\boldsymbol{\kappa})$ has order a power of p;

(3) the automorphism $\boldsymbol{\kappa}(\varpi_{\mathrm{F}})$ is the identity.

Note that such a representation is not unique. We now write

$$\boldsymbol{\sigma} = \mathrm{Hom}_{\mathrm{J}^1}(\boldsymbol{\kappa}, \boldsymbol{\lambda})$$

which carries an action of $\mathbf{J}$ given by $g \cdot f = \boldsymbol{\lambda}(g) \circ f \circ \boldsymbol{\kappa}(g)^{-1}$ for $g \in \mathbf{J}$ and $f \in \boldsymbol{\sigma}$. This representation is irreducible and trivial on $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$, and we have the decomposition $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\sigma}$. As in [**12**, (6.4.1)] this gives us

$$\mathrm{tr}\ \pi(\zeta u) = \sum_{\alpha \in \Psi/\Psi_{\mathrm{t}}} \mathrm{tr}\ \boldsymbol{\sigma}(\zeta^{\alpha}) \sum_{y \in \mathrm{G}_{\mathrm{K}}/\mathbf{J}_{\mathrm{K}}} \mathrm{tr}\ \boldsymbol{\kappa}(y^{-1}\zeta^{\alpha}u^{\alpha}y).$$

We are now going to interpret the sum over $\mathrm{G}_{\mathrm{K}}/\mathbf{J}_{\mathrm{K}}$ as the trace of a cuspidal irreducible representation of G_{K}.

7.2.

Write θ_{K} for the restriction of θ to $\mathrm{H}^1(\mathfrak{a}, \beta) \cap \mathrm{G}_{\mathrm{K}}$, which is the interior K/F-lift of the simple character θ in the sense of [**6**, Section 5]. The group $\mathbf{J}_{\mathrm{K}}$ is also the normalizer of θ_{K} in G_{K}. We choose an irreducible representation $\boldsymbol{\kappa}_{\mathrm{K}}$ of $\mathbf{J}_{\mathrm{K}}$ such that:

(1) the restriction of $\boldsymbol{\kappa}_{\mathrm{K}}$ to J_{K} is a β-extension of θ_{K};
(2) the character $\det(\boldsymbol{\kappa}_{\mathrm{K}})$ has order a power of p;
(3) the automorphism $\boldsymbol{\kappa}_{\mathrm{K}}(\varpi_{\mathrm{F}})$ is the identity.

Again, such a choice may not be unique. The pair $(\mathbf{J}_{\mathrm{K}}, \boldsymbol{\kappa}_{\mathrm{K}})$ is an extended maximal simple type in G_{K}. It thus defines a cuspidal irreducible representation ρ of G_{K}. By [**11**, (3.4.3) and (5.6.2)], there is a sign $\epsilon \in \{-1, +1\}$ such that

$$\mathrm{tr}\ \boldsymbol{\kappa}(y^{-1}\zeta^{\alpha}u^{\alpha}y) = \epsilon \cdot \mathrm{tr}\ \eta_{\mathrm{K}}(y^{-1}\zeta^{\alpha}u^{\alpha}y)$$

where η_{K}, which denotes the restriction of $\boldsymbol{\kappa}_{\mathrm{K}}$ to the pro-p-subgroup $\mathrm{J}^1 \cap \mathrm{G}_{\mathrm{K}}$, is the Heisenberg representation of θ_{K}. As in [**12**, (6.4.2)] this gives us

$$\mathrm{tr}\ \pi(\zeta u) = \epsilon \sum_{\alpha \in \Psi/\Psi_{\mathrm{t}}} \mathrm{tr}\ \boldsymbol{\sigma}(\zeta^{\alpha})\ \mathrm{tr}\ \rho^{\alpha^{-1}}(u). \tag{7.1}$$

We do not know whether a result similar to [**12**, 6.5 Lemma] holds, that is, we do not know whether the Ψ-stabilizers of ρ and of its inertial class are both equal to Γ. However, let Ψ_0 denote the stabilizer in Ψ of the inertial class of ρ and let X_0 be a set of representatives for Ψ mod Ψ_0. For $\gamma \in \Psi_0$ there is an unramified character χ_{γ} of G_{K} such that $\rho^{\gamma^{-1}} \simeq \rho\chi_{\gamma}$. Since u is pro-unipotent (thus compact) we have $\chi_{\gamma}^{\alpha^{-1}}(u) = 1$, for all $\alpha \in \Psi/\Psi_{\mathrm{t}}$. Therefore (7.1) can be rewritten as

$$\mathrm{tr}\ \pi(\zeta u) = \epsilon \sum_{\alpha \in \mathrm{X}_0} \mathrm{tr}\ \rho^{\alpha^{-1}}(u) \sum_{\gamma \in \Psi_0/\Psi_{\mathrm{t}}} \mathrm{tr}\ \boldsymbol{\sigma}(\zeta^{\alpha\gamma}) \tag{7.2}$$

Note that the map

$$w : \zeta \mapsto \sum_{\gamma \in \Psi_0/\Psi_{\mathrm{t}}} \mathrm{tr}\ \boldsymbol{\sigma}(\zeta^{\gamma}) \tag{7.3}$$

is not identically zero on the set of K/F-regular roots of unity, by [**36**, Theorem 1.1(ii)]. Thus there is an $\alpha \in \mathrm{X}_0$ such that the coefficient $w(\zeta^{\alpha})$ in (7.2) is nonzero.

7.3.

Now write π' for the Jacquet–Langlands transfer of π to H. Since π has parametric degree n, the torsion number $t(\pi)$ is equal to the degree of K over F. We now do for π' what we did for π.

Let $(\mathbf{J}', \boldsymbol{\lambda}')$ be an extended maximal simple type of H contained in π', attached to a simple stratum $[\mathfrak{a}', \beta']$. Write B$'$ for the centralizer of β' in $\mathrm{M}_n(\mathrm{F})$, fix a maximal unramified extension L$'$ of $\mathrm{F}[\beta']$ in B$'$ and write K$'$ for the maximal unramified subextension of L$'$ over F. The relation $t(\pi) = t(\pi')$, together with the fact that π' also has parametric degree n, implies that K$'$ and K have the same degree over F. Therefore, we may identify the maximal unramified subextension of L$'$/F with K.

We have an analogue $\boldsymbol{\sigma}'$ of $\boldsymbol{\sigma}$ and an analogue ρ' of ρ in the argument of the previous paragraph so that we get

$$\mathrm{tr}\ \pi'(\zeta u') = \epsilon' \sum_{\alpha' \in \mathrm{X}'_0} \mathrm{tr}\ \rho'^{\alpha'^{-1}}(u') \sum_{\gamma' \in \Psi'_0/\Psi'_{\mathrm{t}}} \mathrm{tr}\ \boldsymbol{\sigma}'(\zeta^{\alpha'\gamma'})$$

where $\zeta \in \mathrm{K}$ is as above, u' is a pro-unipotent elliptic regular element of the centralizer H_{K} of K in H, $\epsilon' \in \{-1, +1\}$ is a sign and the subgroups Ψ'_{t}, Ψ'_0 and X'_0 are defined as in the previous paragraph. If $\zeta u'$ is chosen to have the same reduced characteristic polynomial over F as ζu, this is equal to $(-1)^{n-m} \cdot \mathrm{tr}\ \pi(\zeta u)$, by the trace relation characterizing the Jacquet–Langlands correspondence. We thus get:

$$\epsilon' \sum_{\alpha' \in \mathrm{X}'_0} w'(\zeta^{\alpha'})\ \mathrm{tr}\ \rho'^{\alpha'^{-1}}(u') = (-1)^{n-m} \cdot \epsilon \sum_{\alpha \in \mathrm{X}_0} w(\zeta^{\alpha})\ \mathrm{tr}\ \rho^{\alpha^{-1}}(u)$$

where the function w and its analogue w' are defined by (7.3).

We apply [**12**, 6.6 Lemma] (note that ρ has maximal parametric degree since L/K is maximal). The $\rho'^{\alpha'^{-1}}$, $\alpha' \in \mathrm{X}'_0$, are not unramified twists of each other, and the same holds for the Jacquet–Langlands transfers to H_{K} of the $\rho^{\alpha^{-1}}$, $\alpha \in \mathrm{X}_0$. Thanks to linear independence of characters, it follows that there is a $\alpha \in \Psi$ such that

$$\boldsymbol{\pi}_{\mathrm{K}}(\rho^{\alpha^{-1}}) = \rho'\chi$$

for some unramified character χ of H_{K}, where $\boldsymbol{\pi}_{\mathrm{K}}$ is the local Jacquet–Langlands correspondence from G_{K} to H_{K}.

Assume now that $\boldsymbol{\pi}_{\mathrm{K}}$ preserves K-endo-classes in the maximal totally ramified case. Then the representations $\rho^{\alpha^{-1}}$ and ρ' have the same K-endo-class. But the K-endo-class of $\rho^{\alpha^{-1}}$ (respectively, of ρ') is a K/F-lift of the F-endo-class of π (respectively, of π') in the sense of [**8**]. Applying the restriction map from $\mathcal{E}(\mathrm{K})$ onto $\mathcal{E}(\mathrm{F})$, we get that π and π' have the same F-endo-class. Thus we have proved Theorem A of the introduction:

Theorem 7.1. — *Assume that, for all* F *and* n*, and all maximal totally ramified, cuspidal irreducible complex representations* ρ *of* G*, the representations* ρ *and* $\boldsymbol{\pi}(\rho)$ *have the same endo-class. Then the map* $\boldsymbol{\pi}_1$ *is the identity.*

8. Explicit Jacquet–Langlands correspondence up to unramified twist

Now let us fix an endo-class $\boldsymbol{\Theta} \in \mathcal{E}_n(\mathrm{F})$ and suppose that $\boldsymbol{\pi}_1(\boldsymbol{\Theta}) = \boldsymbol{\Theta}$. Write $\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta})$ for the set of inertial classes of discrete series representations of G with endo-class $\boldsymbol{\Theta}$. The local Jacquet-Langlands correspondence (6.1) thus induces a bijective map

$$\boldsymbol{\pi}_0 : \mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta}) \to \mathcal{D}_0(\mathrm{H}, \boldsymbol{\Theta}).$$

The cuspidal support induces a bijection between $\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta})$ and the set of inertial classes of simple supercuspidal pairs of G with endo-class $\boldsymbol{\Theta}$. Thanks to (3.7) the sets $\mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta})$ and $\mathcal{D}_0(\mathrm{H}, \boldsymbol{\Theta})$ are parametrized by the set $\Gamma\backslash\mathrm{X}$ of Γ-orbits of characters of $\boldsymbol{k}^\times$. The bijection $\boldsymbol{\pi}_0$ thus turns into a permutation Υ of $\Gamma\backslash\mathrm{X}$, that we would like to describe. The purpose of Proposition 8.8 below is to show that, in a certain sense, by considering various $m \geqslant 1$ such that md is divisible by the degree of $\boldsymbol{\Theta}$, one can reduce the computation of $\Upsilon([\alpha])$ to the case where α is suitably regular.

8.1.

We fix a simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_m(\mathrm{D})$ such that $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is maximal in B, together with a simple character $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ with endo-class $\boldsymbol{\Theta}$, and a β-extension κ of θ. The integer m' coming from (3.3) is $m' = m(d, g)/g$, where g denotes the degree of $\boldsymbol{\Theta}$. Write X for the group of complex characters of $\boldsymbol{k}^\times$. Thanks to Proposition 3.4 (see also (3.7)) we have a bijective map

$$\begin{array}{ccc} \Gamma\backslash\mathrm{X} & \to & \mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta}) \\ [\alpha] & \mapsto & \Omega(\kappa, \alpha) \end{array} \tag{8.1}$$

where $\Omega(\kappa, \alpha)$ is the inertial class of discrete series representations of G that contain the simple type $\boldsymbol{\lambda}(\alpha)$.

Similarly, we choose a maximal simple character $\theta' \in \mathcal{C}(\mathfrak{a}', \beta')$ in H with endo-class $\boldsymbol{\Theta}$, and a maximal β-extension κ' of θ'. We get a bijection $[\alpha] \mapsto \Omega(\kappa', \alpha)$ between $\Gamma\backslash\mathrm{X}$ and $\mathcal{D}_0(\mathrm{H}, \boldsymbol{\Theta})$.

Let Υ be the unique bijective map such that the diagram

$$\begin{array}{ccc} \Gamma\backslash\mathrm{X} & \xrightarrow{\ \Upsilon\ } & \Gamma\backslash\mathrm{X} \\ \downarrow & & \downarrow \\ \mathcal{D}_0(\mathrm{G}, \boldsymbol{\Theta}) & \xrightarrow[\boldsymbol{\pi}_0]{} & \mathcal{D}_0(\mathrm{H}, \boldsymbol{\Theta}) \end{array}$$

is commutative, where the vertical maps are given by (8.1) and its analogue for H. It depends on the choice of the maximal β-extensions κ and κ'. By Proposition 5.5 and Corollary 6.3, we have the following fact.

Proposition 8.1. — *For any prime number ℓ, the bijection Υ is compatible with taking ℓ-regular parts. More precisely, the Γ-orbits of $\alpha, \beta \in \mathrm{X}$ have the same ℓ-regular part if and only if $\Upsilon([\alpha])$ and $\Upsilon([\beta])$ have the same ℓ-regular part.*

Proposition 8.1 suggests that, with a suitable choice of ℓ, it may be possible to deduce $\Upsilon([\alpha])$ from the knowledge of $\Upsilon([\beta])$. We will illustrate this idea in Proposition 8.8 below.

8.2.

We first give another property of the map Υ. Set $n' = n/g = m'd'$. Given $\alpha \in \mathrm{X}$, let f be the cardinality of its Γ-orbit, and write

$$s(\alpha) = s_{\mathrm{D}}([\alpha]) = \frac{d'}{(f, d')}. \tag{8.2}$$

Recall that d' is the degree of $\boldsymbol{d}$ over $\boldsymbol{e}$ (the residue field of $\mathrm{F}[\beta]$), thus we have $d' = d/(d, g)$. Note that the cardinality of its $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{d})$-orbit is equal to $f/(f, d')$, which was denoted by u in paragraph 3.3.

Definition 8.2. — We call the integer f the *parametric degree* of $\alpha \in \mathrm{X}$.

This is related to the notion of parametric degree for a discrete series representation as follows: any discrete series representation in $\Omega(\kappa, \alpha)$ has parametric degree fg.

Since the local Jacquet Langlands correspondence preserves the parametric degree [**12**] we have the following result.

Lemma 8.3. — *For all $\alpha \in \mathrm{X}$, the parametric degrees of $[\alpha]$ and $\Upsilon([\alpha])$ are equal.*

Note that the set $\Omega(\kappa, \alpha)$ is made of cuspidal representations with cuspidal Jacquet–Langlands transfers if and only if $f = n'$, that is, if and only if α is $\boldsymbol{e}$-regular. Indeed, from [**12**], a discrete series representation of G is cuspidal with cuspidal Jacquet–Langlands transfer if and only if its parametric degree is n.

8.3.

Let $a \geqslant 1$ be a positive integer. We consider the simple stratum $[\mathfrak{a}^*, \beta]$ in $\mathrm{M}_{am}(\mathrm{D})$, where $\mathfrak{a}^*$ is the hereditary order $\mathrm{M}_a(\mathfrak{a})$, and write $\theta^* \in \mathcal{C}(\mathfrak{a}^*, \beta)$ for the transfer of θ. Associated with κ, there is a coherent choice of a maximal β-extension κ^* of the simple character θ^* ([**26**, Remarque 5.17]). We fix a finite extension $\boldsymbol{k}^*$ of $\boldsymbol{k}$ of degree a and write X^* for the group of complex characters of $\boldsymbol{k}^{*\times}$. Repeating the arguments of paragraph 8.1 with $\mathrm{GL}_{am}(\mathrm{D})$ and $\mathrm{GL}_{an}(\mathrm{F})$, we get a bijective map $\Upsilon^* : \Gamma\backslash\mathrm{X}^* \to \Gamma\backslash\mathrm{X}^*$. We have the following straightforward result.

Lemma 8.4. — *Let $[\alpha] \in \Gamma\backslash\mathrm{X}$, and let $\mathrm{L}(\rho, r)$ be in the inertial class $\Omega(\kappa, \alpha)$, for some integer r dividing m and cuspidal representation ρ of $\mathrm{GL}_{m/r}(\mathrm{D})$. Then $\mathrm{L}(\rho, ar)$ is in the inertial class $\Omega(\kappa^*, \alpha^*)$, where α^* is the character $\alpha \circ \mathrm{N}_{\boldsymbol{k}^*/\boldsymbol{k}}$ of $\boldsymbol{k}^{*\times}$.*

Proof. — With the notation of paragraph 3.3 and writing M for the Levi subgroup $\mathrm{G} \times \cdots \times \mathrm{G} \subseteq \mathrm{GL}_{am}(\mathrm{D})$ and U for the unipotent radical of the parabolic subgroup made of upper $a \times a$ block triangular matrices of $\mathrm{GL}_{am}(\mathrm{D})$, this follows from the fact that the representation of $\mathrm{J}(\mathfrak{a}^*_r, \beta) \cap \mathrm{M}$ on the $\mathrm{J}(\mathfrak{a}^*_r, \beta) \cap \mathrm{U}$-invariant subspace of the transfer κ^*_{ar} of κ^* to $\mathrm{J}(\mathfrak{a}^*_{ar}, \beta)$ is $\kappa \otimes \cdots \otimes \kappa$. ☐

For $\alpha \in \mathrm{X}$, the orbit $[\alpha^*]$ depends only on $[\alpha]$, and we denote it $[\alpha]^*$. By Lemma 8.4 we thus have

$$\Upsilon^*([\alpha^*]) = \Upsilon(\alpha)^*$$

for any character $\alpha \in \mathrm{X}$.

Given $\alpha \in \mathrm{X}$, we write f for its parametric degree, and $\boldsymbol{e}[\alpha]$ for the subfield of $\boldsymbol{k}$ of degree f over $\boldsymbol{e}$.

Lemma 8.5. — *Let $\alpha \in \mathrm{X}$. There are an integer $a \geqslant 1$, a prime number $\ell \neq p$ not dividing the order of $\boldsymbol{e}[\alpha]^\times$ and an $\boldsymbol{e}$-regular character $\beta \in \mathrm{X}^*$ such that $\beta \equiv \alpha^*$ mod ℓ.*

Proof. — First recall the following result, known as Zsigmondy's Theorem [**43**].

Lemma 8.6. — *Let $b, r \geqslant 2$ be integers. There exists a prime number ℓ which divides $b^r - 1$ but not $b^i - 1$ for any $i \in \{1, \dots, r-1\}$, except when $r = 6$ and $b = 2$, and when $r = 2$ and $b = 2^k - 1$ for some $k \geqslant 1$.*

Let us write Q for the cardinality of $\boldsymbol{e}$, and let us fix an $a \geqslant 1$ such that $an' > 6f$. Applying Lemma 8.6 with $b = \mathrm{Q}^f$ and $r = an'/f$, we obtain a prime number ℓ dividing $b^r - 1$ but not dividing $b^i - 1$ for any $i \in \{1, \dots, r-1\}$. It follows that b has order r in the group $(\mathbf{Z}/\ell\mathbf{Z})^\times$.

Let ξ be a nontrivial character of $\boldsymbol{k}^{*\times}$ of order ℓ. Then the character $\beta = \xi\alpha^*$ is congruent to α^* mod ℓ. Since the order of α is prime to ℓ (for it divides $b - 1$), the cardinality of the Γ-orbit of β is the least common multiple of f and the order of Q in $(\mathbf{Z}/\ell\mathbf{Z})^\times$. This cardinality is equal to $fr = an'$, thus β is $\boldsymbol{e}$-regular. □

Remark 8.7. — (1) It is not always possible to choose $a = 1$. For instance, if α is trivial, $n' = 2$ and $\boldsymbol{e}$ has 7 elements, then no prime number ℓ satisfies the required condition. We thank Guy Henniart for a suggestion that brought us to introduce the process described here.

(2) The proof of Lemma 8.5 shows that, for any character $\alpha \in \mathrm{X}$, we can choose a to be any integer $\geqslant 7$. Moreover, the choice of a and ℓ only depend on the parametric degree f, not on α.

(3) Note that ℓ cannot be 2. Indeed we have $\ell \neq p$ and, if p is odd, then the fact that ℓ does not divide $\mathrm{Q}^f - 1$ (the order of $\boldsymbol{e}[\alpha]^\times$) implies that $\ell \neq 2$.

With the notation of Lemma 8.5, we get the following result.

Proposition 8.8. — *Assume that $\Upsilon^*([\beta]) = [\beta\mu]$ for some character $\mu \in \mathrm{X}^*$. Then $\mu_\ell = \nu^*$ for some character $\nu \in \mathrm{X}$ and we have $\Upsilon([\alpha]) = [\alpha\nu]$.*

Proof. — Let us write $\Upsilon([\alpha]) = [\alpha']$ for some $\alpha' \in \mathrm{X}$. Then $[\alpha'^*] \equiv [\beta\mu]$ mod ℓ. By Lemma 8.3, the parametric degree of α' is f, thus $\boldsymbol{e}[\alpha'] = \boldsymbol{e}[\alpha]$. It follows that ℓ does not divide the order of α'. Write $\beta = \xi\alpha^*$ for some character ξ whose order is a power of ℓ. Taking ℓ-regular parts, we get $[\alpha'^*]_\ell = [\alpha'^*] = [\alpha^*\mu_\ell]$. Changing α' in its Γ-orbit, we may assume that $\alpha'^* = \alpha^*\mu_\ell$. Thus $\mu_\ell = \nu^*$ for some $\nu \in \mathrm{X}$. Since $\mathrm{N}_{\boldsymbol{k}^*/\boldsymbol{k}}$ is surjective, we get $\Upsilon([\alpha]) = [\alpha\nu]$. □

9. The essentially tame case

The purpose of this section is to illustrate Proposition 8.8 in the essentially tame case. Assume that $\boldsymbol{\Theta}$ is essentially tame. We thus have $\boldsymbol{\pi}_1(\boldsymbol{\Theta}) = \boldsymbol{\Theta}$ (see Corollary 6.7). With the notation of paragraph 8.1, we let κ be the unique β-extension of θ such that $\det(\kappa)$ has order a power of p, and similarly for κ'. We thus get a canonical permutation Υ of the finite set $\Gamma\backslash\mathrm{X}$.

Our purpose is to use [**12**, Theorem A] together with Proposition 8.8 in order to get a formula for $\Upsilon([\alpha])$ for *all* characters $\alpha \in \mathrm{X}$, and not only for those α giving rise to cuspidal representations of H, that is, $\boldsymbol{e}$-regular ones. The main result of this section is the following theorem.

Theorem 9.1. — *There is a canonically determined character μ of $\boldsymbol{k}^\times$, depending only on m, d and $\boldsymbol{\Theta}$, such that $\mu^2 = 1$ and*

$$\Upsilon([\alpha]) = [\alpha\mu] \tag{9.1}$$

for all characters $\alpha \in \mathrm{X}$.

More precisely, we will see that the character μ is the "rectifier" given by Bushnell–Henniart's First Comparison Theorem [**12**, 6.1] together with [**12**, Corollary 6.9 and (6.7.4)].

Let us write $\mathrm{X}_{\mathrm{reg}}$ for the set of $\boldsymbol{e}$-regular characters in X. We will first prove that (9.1) holds for all characters $\alpha \in \mathrm{X}_{\mathrm{reg}}$, using [**12**, Theorem A] (see Paragraph 9.3). Since this theorem is formulated in terms of admissible pairs, we will have to translate it in terms of our α-parameters.

9.1.

We first recall the definition of admissible pairs [**22, 12**], and basic facts about them.

Definition 9.2. — An *admissible pair* is a pair $(\mathrm{L}/\mathrm{F}, \xi)$ made of a finite, tamely ramified field extension L/F and a character ξ of $\mathrm{L}^\times$ such that:

(1) ξ does not factor through $\mathrm{N}_{\mathrm{L}/\mathrm{K}}$ for any field K such that $\mathrm{F} \subseteq \mathrm{K} \subsetneq \mathrm{L}$;

(2) if the restriction of ξ to the 1-units $1+\mathfrak{p}_{\mathrm{L}}$ factors through $\mathrm{N}_{\mathrm{L}/\mathrm{K}}$ for some field K such that $\mathrm{F} \subseteq \mathrm{K} \subseteq \mathrm{L}$, then L/K is unramified.

Two admissible pairs $(\mathrm{L}_i/\mathrm{F}, \xi_i)$, $i = 1, 2$, are said to be *isomorphic* if there is an F-isomorphism $\phi : \mathrm{L}_2 \to \mathrm{L}_1$ such that $\xi_2 = \xi_1 \circ \phi$. The *degree* of an admissible pair $(\mathrm{L}/\mathrm{F}, \xi)$ is $[\mathrm{L} : \mathrm{F}]$. We also introduce the following definition, which will be convenient to us.

Definition 9.3. — Two admissible pairs $(\mathrm{L}_i/\mathrm{F}, \xi_i)$ for $i = 1, 2$, are said to be *inertially equivalent* if there are an unramified character χ of $\mathrm{L}_2^\times$ and an isomorphism $\phi : \mathrm{L}_2 \to \mathrm{L}_1$ of extensions of F such that $\chi\xi_2 = \xi_1 \circ \phi$. We will write $[\mathrm{L}_1/\mathrm{F}, \xi_1]$ for the inertial class of $(\mathrm{L}_1/\mathrm{F}, \xi_1)$.

Let $(\mathrm{L}/\mathrm{F}, \xi)$ be an admissible pair. By [**12**, 4.1 Lemma], there is a unique sub-extension P/F of L/F such that $\xi \mid 1+\mathfrak{p}_{\mathrm{L}}$ factors through the norm $\mathrm{N}_{\mathrm{L}/\mathrm{P}}$ and which is minimal for this property. It is called the *parameter field* of the admissible pair. Then L/P is unramified and, if we write $\xi \mid 1+\mathfrak{p}_{\mathrm{L}} = \xi_1 \circ \mathrm{N}_{\mathrm{L}/\mathrm{P}}$ for some character ξ_1 of $1+\mathfrak{p}_{\mathrm{P}}$, then $(\mathrm{P}/\mathrm{F}, \xi_1)$ is an admissible 1-pair in the sense of [**12**, 3.3]. As in [**11**, 4.3], the admissible 1-pair $(\mathrm{P}/\mathrm{F}, \xi_1)$ determines an endo-class. This endo-class is essentially tame with parameter field P/F, in the sense of [**14**, 2.4]. The parameter field P and the endo-class only depend on the inertial class $[\mathrm{L}/\mathrm{F}, \xi]$.

Remark 9.4. — Suppose that $(\mathrm{L}/\mathrm{F}, \xi)$ has associated endo-class $\boldsymbol{\Theta}$, the one that we have fixed at the beginning of this section. Then P is F-isomorphic to E. Moreover, by changing $(\mathrm{L}/\mathrm{F}, \xi)$ in its isomorphism class [**12**, 4.1], we may assume that P is equal to E.

9.2.

Now let $(\mathrm{L}/\mathrm{F},\xi)$ be an admissible pair with associated endo-class $\boldsymbol{\Theta}$ and degree t dividing n. By [**12**, 4.3 Lemma 1], there is a unique character ξ_{w} of the group $\mathcal{O}_{\mathrm{L}}^{\times}$ of units of $\mathcal{O}_{\mathrm{L}}$ such that:

(1) the characters ξ_{w} and ξ coincide on the principal unit subgroup $1+\mathfrak{p}_{\mathrm{L}}$;
(2) the order of ξ_{w} is a power of p.

The character $\xi\cdot\xi_{\mathrm{w}}^{-1}$ of $\mathcal{O}_{\mathrm{L}}^{\times}$ is tamely ramified, thus induces a character of $\boldsymbol{l}^{\times}$, denoted ξ_{t}, where $\boldsymbol{l}$ is the residue field of L. This character ξ_{t} depends only on the inertial class of $(\mathrm{L}/\mathrm{F},\xi)$. Moreover, since $(\mathrm{L}/\mathrm{F},\xi)$ is an admissible pair, ξ_{t} is $\boldsymbol{e}$-regular. Let us fix an $\boldsymbol{e}$-embedding of $\boldsymbol{l}$ in $\boldsymbol{k}$ and write α_{ξ} for the character $\xi_{\mathrm{t}}\circ\mathrm{N}_{\boldsymbol{k}/\boldsymbol{l}}$ of $\boldsymbol{k}^{\times}$. Its Γ-conjugacy class does not depend on the choice of the embedding of $\boldsymbol{l}$ in $\boldsymbol{k}$, and its parametric degree f is equal to $[\mathrm{L}:\mathrm{E}]$. We thus have $t=fg$.

We write $\mathcal{P}_n(\boldsymbol{\Theta})$ for the set of isomorphism classes of admissible pairs with endo-class $\boldsymbol{\Theta}$ and degree dividing n.

Lemma 9.5. — (1) *The character α_{ξ} is $\boldsymbol{e}$-regular if and only if $[\mathrm{L}:\mathrm{F}]=n$.*
(2) *The map*

$$[\mathrm{L}/\mathrm{F},\xi]\mapsto[\alpha_{\xi}] \tag{9.2}$$

induces a bijection between the set of inertial classes of admissible pairs in $\mathcal{P}_n(\boldsymbol{\Theta})$ and $\Gamma\backslash\mathrm{X}$.

Remark 9.6. — Note that the map (9.2) is well-defined, thanks to Remark 9.4.

Proof. — The character α_{ξ} is $\boldsymbol{e}$-regular if and only if $f=n'$. Multiplying by g, this is equivalent to $t=n$. This gives us the first part of the lemma.

Given $\alpha\in\mathrm{X}$, there is a uniquely determined field $\boldsymbol{l}$ such that $\boldsymbol{e}\subseteq\boldsymbol{l}\subseteq\boldsymbol{k}$ and α factors through the norm $\mathrm{N}_{\boldsymbol{k}/\boldsymbol{l}}$, and which is minimal for this property. Write $\alpha=\beta\circ\mathrm{N}_{\boldsymbol{k}/\boldsymbol{l}}$ for some character β of $\boldsymbol{l}^{\times}$, which is $\boldsymbol{e}$-regular by minimality of $\boldsymbol{l}$. Let L be an unramified extension of E with residue field $\boldsymbol{l}$. Then β inflates to a tamely ramified character of the units subgroup of L, still denoted β. Now let ξ be any character of $\mathrm{L}^{\times}$ extending $\xi_{\mathrm{w}}\beta$. Since the character β is $\boldsymbol{e}$-regular, it follows that the pair $(\mathrm{L}/\mathrm{F},\xi)$ is admissible. The Γ-orbit $[\alpha_{\xi}]$ associated with its inertial class is equal to $[\alpha]$. The map (9.2) is thus surjective.

We now assume that we have two admissible pairs $(\mathrm{L}_i/\mathrm{F},\xi_i)$ for $i=1,2$, with same image $[\alpha]$ in $\Gamma\backslash\mathrm{X}$. For each i, we may assume that the parameter field of $(\mathrm{L}_i/\mathrm{F},\xi_i)$ is E by Remark 9.4. The character $\xi_i\mid 1+\mathfrak{p}_{\mathrm{L}_i}$ thus factors through $\mathrm{N}_{\mathrm{L}_i/\mathrm{E}}$ and E is minimal for this property. We have an $\boldsymbol{e}$-regular character $\xi_{i,\mathrm{t}}$ of $\boldsymbol{l}_i^{\times}$, where $\boldsymbol{l}_i$ is the residue field of L_i. Since the Γ-orbits $[\alpha_{\xi_1}]$ and $[\alpha_{\xi_2}]$ are equal, they have the same cardinality f. The fields $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ thus have the same degree over $\boldsymbol{e}$, and L_1, L_2 have the same degree f over E. We thus may assume (by changing the pair $(\mathrm{L}_2/\mathrm{F},\xi_2)$ in its isomorphism class) that $\mathrm{L}_1=\mathrm{L}_2$, denoted L. We now have two characters $\xi_{1,\mathrm{t}}$ and $\xi_{2,\mathrm{t}}$ of $\boldsymbol{l}^{\times}$, which are conjugate under $\mathrm{Gal}(\boldsymbol{l}/\boldsymbol{e})$. Changing again $(\mathrm{L}_2/\mathrm{F},\xi_2)$ in its isomorphism class, we may assume that they are equal. Thus the admissible pairs $(\mathrm{L}_i/\mathrm{F},\xi_i)$, for $i=1,2$, are inertially equivalent. □

9.3.

The Parametrization Theorem [**12**, 6.1] gives us a bijective map

$$(\mathrm{L}/\mathrm{F}, \xi) \mapsto \Pi(\mathrm{G}, \xi) \tag{9.3}$$

between F-isomorphism classes of admissible pairs of degree n and isomorphism classes of essentially tame irreducible cuspidal representations of G, that is, cuspidal representations with essentially tame endo-class and parametric degree n.

More precisely, by examining [**12**, 4.2 and 4.3], we see that the cuspidal representation $\Pi(\mathrm{G}, \xi)$ associated with an admissible pair $(\mathrm{L}/\mathrm{F}, \xi)$ of degree n and endo-class $\boldsymbol{\Theta}$ contains the maximal simple type $\kappa \otimes \sigma$, where κ is the maximal β-extension fixed at the beginning of this section and σ is the irreducible cuspidal representation of $\mathcal{G}$ with Green parameter ξ_{t}.

Lemma 9.7. — (1) *Given an admissible pair* $(\mathrm{L}/\mathrm{F}, \xi)$ *of degree* n *and endo-class* $\boldsymbol{\Theta}$, *the irreducible cuspidal representation* $\Pi(\mathrm{G}, \xi)$ *belongs to the inertial class* $\Omega(\kappa, \alpha_{\xi})$.

(2) *The bijection* (9.3) *induces a bijection between inertial classes of admissible pairs of degree* n *and inertial classes of essentially tame cuspidal representations of* G.

Proof. — Comparing with the construction in paragraph 3.3, the maximal simple type $\kappa \otimes \sigma$ above is $\boldsymbol{\lambda}(\alpha_{\xi})$. Note that α_{ξ} is equal to ξ_{t} since L/F has degree n. This gives us the first part of the lemma.

An inertial class of essentially tame cuspidal representations of G with endo-class $\boldsymbol{\Theta}$ has the form $\Omega(\kappa, \alpha)$ for some $\alpha \in \mathrm{X}_{\mathrm{reg}}$. The second part of the lemma thus follows from Lemma 9.5. □

We now prove Theorem 9.1 for $\boldsymbol{e}$-regular characters.

Proposition 9.8. — (1) *There is a canonically determined character* $\mu \in \mathrm{X}$, *depending only on* m, d *and* $\boldsymbol{\Theta}$, *such that* $\mu^2 = 1$ *and*

$$\Upsilon([\alpha]) = [\alpha\mu]$$

for all characters $\alpha \in \mathrm{X}_{\mathrm{reg}}$.

(2) *The character* μ *is non-trivial if and only if* $p \neq 2$ *and the integer*

$$y(\boldsymbol{\Theta}, m, d) = m(d-1) + m'(d'-1) + u(v-1)$$

is odd, where the integers $u, v \geqslant 1$ *are defined by* $uv = n/w$, $v = d/(d, w)$ *with* $w = n/e(\mathrm{E}/\mathrm{F})$.

Proof. — Let $\alpha \in \mathrm{X}_{\mathrm{reg}}$ and let $(\mathrm{L}/\mathrm{F}, \xi)$ be an admissible pair of degree n and endo-class $\boldsymbol{\Theta}$ whose inertial class is associated with $[\alpha]$. By [**12**, Theorem A], there is a tamely ramified character ν of $\mathrm{L}^{\times}$ such that $(\mathrm{L}/\mathrm{F}, \xi\nu)$ is admissible, ν^2 is trivial and the Jacquet–Langlands transfer of $\Pi(\mathrm{G}, \xi)$ is $\Pi(\mathrm{H}, \xi\nu)$. Since L/F has degree n and L is unramified over E, the residue field of L identifies with $\boldsymbol{k}$. Write μ for the character of $\boldsymbol{k}^{\times}$ induced by the restriction of ν to the units subgroup of L. This character is entirely described by [**12**, Corollary 6.9], which gives us Assertion (2).

Taking inertial classes and using Lemma 9.7, the Jacquet–Langlands correspondence matches together the inertial class $\Omega(\kappa, \alpha)$ of $\Pi(\mathrm{G}, \xi)$ with that of $\Pi(\mathrm{H}, \xi\nu)$, and the latter can be written $\Omega(\kappa', \alpha')$ for $[\alpha'] = [\alpha_{\xi\nu}] = [\alpha\mu]$. The result follows. □

9.4.

We now prove Theorem 9.1. Let us fix an *odd* integer $a \geqslant 7$ (see Remark 8.7(2)). We will see below why it is convenient to choose a odd. We use the notation introduced in paragraph 8.3. In particular, we have β-extensions κ^*, κ'^* and a permutation Υ^* of $\Gamma\backslash \mathrm{X}^*$. We must pay attention to the fact that the determinants of κ^* and κ'^* have orders which may not be powers of p, thus Proposition 9.8 may not apply to Υ^* directly.

Let us write $\varkappa^*$ for the β-extension on $\mathrm{J}(\mathfrak{a}^*, \beta)$ whose determinant has order a power of p. By Remark 3.7 there is a character ξ of $\mathrm{J}(\mathfrak{a}^*, \beta)$ trivial on $\mathrm{J}^1(\mathfrak{a}^*, \beta)$ such that $\varkappa^* = \kappa^*\xi$. It induces a character of $\mathcal{G}$ of the form $\chi \circ \mathrm{N}_{\boldsymbol{d}/\boldsymbol{e}} \circ \det$ for some character χ of $\boldsymbol{e}^\times$.

Similarly, we have a β-extension $\varkappa'^*$ whose determinant has order a power of p, and characters ξ', χ' such that $\varkappa'^* = \kappa'^*\xi'$ and ξ' induces the character $\chi' \circ \mathrm{N}_{\boldsymbol{d}/\boldsymbol{e}} \circ \det$ of $\mathcal{G}$. We write Φ for the permutation of $\Gamma\backslash \mathrm{X}^*$ corresponding to the β-extensions $\varkappa^*$ and $\varkappa'^*$. We write δ for the character $(\chi'\chi^{-1}) \circ \mathrm{N}_{\boldsymbol{k}^*/\boldsymbol{e}} \in \mathrm{X}^*$.

Lemma 9.9. — *The character δ is trivial.*

Proof. — Let $\beta \in \mathrm{X}^*$ be $\boldsymbol{e}$-regular. Applying Proposition 9.8 to Φ gives us $\Phi([\beta]) = [\beta\lambda]$, where $\lambda \in \mathrm{X}^*$ is the rectifying character corresponding to am, d and $\boldsymbol{\Theta}$. It has order at most 2, and it is non-trivial if and only if $p \neq 2$ and the integer

$$y(\boldsymbol{\Theta}, am, d) = a \cdot y(\boldsymbol{\Theta}, m, d)$$

is odd. Since a is odd, it follows that λ is trivial if and only if μ is, that is $\lambda = \mu^*$. Now let ε be the character $\chi \circ \mathrm{N}_{\boldsymbol{k}^*/\boldsymbol{e}}$, and define ε' similarly. Comparing Φ and Υ^* thanks to Remark 3.7, we get $\Upsilon^*([\beta\varepsilon]) = [\beta\mu^*\varepsilon']$ for all $\boldsymbol{e}$-regular $\beta \in \mathrm{X}^*$. Since $\beta\varepsilon^{-1}$ is $\boldsymbol{e}$-regular if and only if β is, this gives us

$$\Upsilon^*([\beta]) = [\beta\delta\mu^*] \tag{9.4}$$

for all $\boldsymbol{e}$-regular $\beta \in \mathrm{X}^*$.

Now let $\alpha \in \mathrm{X}_{\mathrm{reg}}$. By Lemma 8.5 there are a prime number $\ell \neq p$ not dividing the order of $\boldsymbol{k}^\times$ and an $\boldsymbol{e}$-regular character $\beta \in \mathrm{X}^*$ such that $\beta \equiv \alpha^* \bmod \ell$. By (9.4) and Proposition 8.8 we get $\Upsilon([\alpha]) = [\alpha\nu]$ for some $\nu \in \mathrm{X}$ such that ν^* is the ℓ-regular part of $\delta\mu^*$. Since α is $\boldsymbol{e}$-regular, Proposition 9.8 applied to Υ gives us $\Upsilon([\alpha]) = [\alpha\mu]$. Putting these equalities together, we get

$$[\alpha^*\mu^*] \equiv [\alpha^*\mu^*\delta] \bmod \ell.$$

The character δ can thus be written $\xi(\alpha^*\mu^*)^{\mathrm{Q}^i-1}$ for some integer $i \in \{0, \dots, n'-1\}$ and some $\xi \in \mathrm{X}^*$ whose order is a power of ℓ. (Recall that Q is the cardinality of $\boldsymbol{e}$.) Since μ has order at most 2, we get $\delta = \xi(\alpha^*)^{\mathrm{Q}^i-1}$. Since the order of δ divides $\mathrm{Q}-1$, we have

$$\alpha^{*(\mathrm{Q}^i-1)(\mathrm{Q}-1)} = \xi^{1-\mathrm{Q}}.$$

Since both $\mathrm{Q}-1$ and the order of α are prime to ℓ, we get $\xi = 1$. Thus the order of α, that we may assume to be $\mathrm{Q}^{n'}-1$ by choosing for α a generator of X, divides $(\mathrm{Q}^i-1)(\mathrm{Q}-1)$. This implies $i = 0$, thus δ is trivial as expected. □

Remark 9.10. — If a were chosen to be even, the same proof would apply (with $\lambda = 1$) and we would get $\delta = \mu^*$.

Now let $\alpha \in X$ be arbitrary. By Lemma 8.5 there are a prime number $\ell \neq p$ not dividing the order of $\boldsymbol{e}[\alpha]^\times$ and an $\boldsymbol{e}$-regular character $\beta \in X^*$ such that β is congruent to α^* mod ℓ. Since δ is trivial, (9.4) gives us $\Upsilon^*([\beta]) = [\beta\mu^*]$. By Proposition 8.8, we have $\Upsilon([\alpha]) = [\alpha\nu]$ for some character $\nu \in X$ such that ν^* is the ℓ-regular part of μ^*. Thus $\nu^* = \mu^*$, which implies $\nu = \mu$. This completes the proof of Theorem 9.1.

9.5.

We now translate Theorem 9.1 in terms of admissible pairs. Let $(L/F, \xi)$ be an admissible pair with degree dividing n and endo-class $\boldsymbol{\Theta}$. The character class $[\alpha_\xi] \in \Gamma\backslash X$ given by the map (9.2) corresponds to an inertial class of discrete series representations $\Omega(\kappa, \alpha_\xi)$. Let us write $\Pi_0(G, \xi)$ for this inertial class. The map

$$[L/F, \xi] \mapsto \Pi_0(G, \xi)$$

is a bijection between $\mathcal{P}_n(\boldsymbol{\Theta})$ and $\mathcal{D}_0(G, \boldsymbol{\Theta})$.

Theorem 9.11. — *Let* $(L/F, \xi)$ *be an admissible pair with degree dividing* n*. There is a canonically determined tamely ramified character* μ *of the units subgroup of* L *such that* $\mu^2 = 1$ *and*

$$\boldsymbol{\pi}_0(\Pi_0(G, \xi)) = \Pi_0(H, \xi\mu).$$

It depends only on m*,* d *and the restriction of* ξ *to the principal units* $1 + \mathfrak{p}_L$.

Note that by $\Pi_0(H, \xi\mu)$ we mean the inertial class corresponding to the pair $[L/F, \xi\hat{\mu}]$ for any choice of extension $\hat{\mu}$ of μ to $L^\times$; this is independent of the choice of $\hat{\mu}$.

Remark 9.12. — Let t be the degree of L/F and write s for the integer $s(\alpha_\xi)$ defined by (8.2). The parametric degree $f = [L : E]$ of α_ξ divides $m'd'$. Hence $u = f/(f, d')$ divides $m's$, thus m'. Let us define an integer $r \geqslant 1$ by $m' = ur$, or equivalently by $n = rst$. Any discrete series representation in $\Pi_0(G, \xi)$ has the form $L(\rho, r)$ for some cuspidal representation ρ of $GL_{m/r}(D)$ with parametric degree t.

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